Structural extensions of display calculi: a general recipe

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Abstract. We present a systematic procedure for constructing cut-free display calculi for axiomatic extensions of a logic via structural rule extensions. The sufficient conditions for the procedure are given in terms of (purely syntactic) abstract properties of the display calculus and thus the method applies to large classes of calculi and logics. As a case study, we present cut-free calculi for extensions of well-known logics including Bi-intuitionistic and tense logic.

1 Introduction

Driven by the rising demand of researchers and practitioners, the last decades have witnessed a tremendous growth in research on logics different from classical logic and also the definition of many new logics. The usefulness of these logics and the key to their application often lies in the existence of *analytic calculi*, that is, calculi in which proofs proceed by stepwise decomposition of the formulae to be proved. Indeed, analytic calculi have been widely applied to establish fundamental properties of the logics and are themselves the focus of much research.

Since its introduction by Gentzen [8], the sequent calculus has been the favourite framework for defining analytic calculi. However, this framework is not powerful enough to formalise all interesting logics. For this reason a huge range of extensions of the sequent calculus have been introduced, in many cases for the sole purpose of obtaining analytic calculi for particular logics. General and well-known formalisms include hypersequents [1], bunched calculi [13], labelled deductive systems [7, 16] and the calculus of structures [12]. Since the construction of an analytic calculus is often tailored to the specific logic under consideration, a large numbers of papers in the literature deal with this topic and yet many interesting logics still lack an analytic calculus. The display calculus [2] is a powerful and semantic-independent formalism that can be used to capture a variety of different logics ranging from resource-oriented logics [4] to substructural [9] and temporal logics [14]. The beauty of the display calculus lies in a general cut-elimination theorem for all calculi obeying eight easily verifiable syntactic conditions [2, 17]; this makes the display calculus a good candidate for capturing large classes of logics in a unified way, irrespective of their semantics or connectives.

Nonclassical logics are often introduced by adding properties — expressed as Hilbert axioms — to known systems. Systematic procedures to automate the construction of new analytic calculi from such Hilbert axioms are highly desirable. In this direction e.g. [5, 6, 16, 14, 15, 11] introduce methods to extract rules out of suitable Hilbert axioms. More precisely [5, 6] generate sequent and hypersequent rules, [11] nested sequent rules, [15] sequent rules for certain modal axioms, and [16] labelled rules; finally

[14] transforms suitable modal and tense axioms (called primitive tense axioms) into structural rules for the display calculus. [14] also provides a characterisation as it is shown that each such rule added to the base system is equivalent to the extension of the logic by primitive tense axioms.

All the above results start with a *specific logic* and introduce calculi for (some of) its axiomatic extensions, e.g., Full Lambek calculus with exchange FLe for [5, 6], or the tense logic Kt for [14, 11]. This paper proposes instead a recipe that utilises the more common Hilbert axioms to construct analytic calculi[1]. In particular, given a suitable base calculus for a logic, we identify a hierarchy of axiom classes — computed as a function of the invertible (logical) introduction rules of the base calculus — and show how to translate axioms from suitable classes into equivalent structural rules. More invertible rules in the base calculus lead to larger sets of axioms in each suitable class, and then to the construction of cut-free calculi for more logics. In the case of intermediate logics, for example, we capture more logics than the hypersequent calculi in [5].

The emphasis is not to define such calculi for specific families of logics but to provide a *methodology* to construct them in a uniform and systematic way starting from a display calculus satisfying general conditions. Since the conditions are given in terms of (purely syntactic) abstract properties of the display calculus, the method applies to large classes of calculi and logics. As a case study, we present analytic calculi for axiomatic extensions of propositional (Bi-)intuitionistic logic, bunched, modal and tense logics. This allows for the automated introduction of (infinitely many) analytic display calculi for logics.

2 Display calculi in a nutshell

Given a language $\mathcal{L}$, we write $\mathbf{For}\mathcal{L}$ to denote the formulae of $\mathcal{L}$. We identify a *logic* with the set of theorems in its Hilbert calculus.

Belnap's Display Calculus [2] — introduced under the name Display Logic — generalises Gentzen's sequent calculus by supplementing the structural connective (comma) with new structural connectives. A (display) sequent $X \vdash Y$ is a tuple (X, Y) where X and Y are *structures* which are built from formulae and structure constants using the structural connectives of the calculus. Structure X (resp. Y) is called the antecedent (succedent) of the sequent. A display calculus consists of initial sequents and rules and includes the cut-rule. The rules of the calculus are usually presented as rule schemata. Concrete instances of a rule are obtained by substitution of a formula (resp. structure) for each schematic formula (structure) variable. Following standard practice, we do not explicitly distinguish between a rule and a rule schema. A derivation in the display calculus is defined in the usual way. In this paper we use $A, B, C, D, \ldots$ (possibly with subscripts) to denote formulae and $X, Y, U, V, \ldots$ to denote structures.

A *structural* rule in the display calculus is constructed from structure variables using structural connectives and structure constants. The *logical* rules usually introduce exactly one logical connective, as the primary connective in a formula that is the whole

[1] In the direction of general results, [10] shows how to extract display calculi starting from the algebraic Gaggle-theoretic semantics of a logic.

of the antecedent or succedent of the conclusion. The cut-rule has the following form, where X, Y are structures and A is a formula:

$$\frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\, cut$$

The calculus obtained by the addition of structural rules is a *structural rule extension*. A rule is *admissible* in $\mathcal{C}$ if the conclusion is derivable when the premises are derivable. A rule is *invertible* in $\mathcal{C}$ if the premises are derivable when the conclusion is derivable.

Definition 1 (equivalent rules). *Let $\mathcal{R}_0$ and $\mathcal{R}_1$ be sets of rules. We say that $\mathcal{R}_0$ and $\mathcal{R}_1$ are* equivalent *wrt $\mathcal{C}$ if each rule in $\mathcal{R}_i$ is admissible in $\mathcal{C} + \mathcal{R}_{1-i}$ for $i = 0, 1$.*

By viewing a sequent $X \vdash Y$ as the zero-premise rule with conclusion $X \vdash Y$, we can define in the obvious way what it means for two sequents to be equivalent, and for a sequent to be equivalent to a rule.

Let Z be a structure. Any structure that occurs in Z is called a *substructure* of Z. Trivially, Z is a substructure of itself. The defining feature of a display calculus is that it satisfies the display property.

Definition 2 (display property; a-part, s-part). *Let Z be an occurrence of a substructure occurring in a sequent $X \vdash Y$. Using the invertible structural rules (the 'display rules') a sequent of the form $Z \vdash U$ or $U \vdash Z$ can be derived for suitable U. In the former (resp. latter) case, the occurrence Z is said to be* displayed *as an* a-part *(*s-part*) structure.*

Since a formula is itself a structure, the display property applies to a formula occurring in a sequent but not to its proper subformulae.

A calculus is said to be *cut-eliminable* if it is possible to eliminate all occurrences of the cut-rule from a given derivation in order to obtain a *cut-free* derivation of the same sequent. A display calculus has the *subformula property* if every formula that occurs in a cut-free derivation appears as a subformula of the final sequent. An important feature of the display calculus are Belnap's conditions C1–C8 on the rules of the calculus.

- (C1) Each (schematic) formula variable occurring in a premise of a rule $\rho \neq cut$ is a subformula of some formula in the conclusion of ρ.
- (C2) *Congruent parameters* is a relation between parameters of the identical structure variable occurring in the premise and conclusion sequents of a rule.
- (C3) Each parameter is congruent to at most one structure variable in the conclusion. Ie. no two structure variables in the conclusion are congruent to each other.
- (C4) Congruent parameters are all either a-part or s-part structures.
- (C5) A formula variable in the conclusion of a rule ρ is either the entire antecedent or the entire succedent. This formula is called a *principal formula* of ρ.
- (C6/7) Each rule is closed under simultaneous substitution of arbitrary structures for congruent parameters.
- (C8) If there are rules ρ and σ with respective conclusions $X \vdash A$ and $A \vdash Y$ with formula A principal in both inferences (in the sense of C5) and if cut is applied to yield $X \vdash Y$, then either $X \vdash Y$ is identical to either $X \vdash A$ or $A \vdash Y$; or it is possible to pass from the premises of ρ and σ to $X \vdash Y$ by means of inferences falling under cut where the cut-formula always is a proper subformula of A.

Belnap's general cut-elimination theorem states that C2–C8 constitute sufficient conditions for a calculus to be cut-eliminable and C1 is the subformula property. Only condition C8 is non-trivial to check. However, C8 is not relevant for structural rules. This further motivates the interest in structural rule extensions of the display calculus.

Definition 3. *Let $\mathcal{C}$ be a display calculus and let L be a logic in the language $\mathcal{L}$. We say that* $\mathcal{C}$ is a calculus for L *to mean that for every $A \in \mathbf{For}\mathcal{L}$: $\mathcal{C}$ derives A iff $A \in L$.*

Given a display calculus $\mathcal{C}$, we denote by $\mathcal{L}_{\mathcal{C}}$ the language determined by the connectives introduced by its logical rules. We do not exclude the possibility that a display calculus $\mathcal{C}$ for a logic in the language $\mathcal{L}$ derives B for some $B \notin \mathbf{For}\mathcal{L}$. This can occur only when the subset relation $\mathcal{L} \subset \mathcal{L}_{\mathcal{C}}$ is strict.

3 The recipe

Suppose that $\mathcal{C}$ is a display calculus for a logic L in the language $\mathcal{L}$ satisfying C1–C8. We show how to define structural rules $r_1, \ldots, r_m$ so that $\mathcal{C} + \{r_1, \ldots, r_m\}$ is a cut-eliminable calculus for the axiomatic extension $L + \{A_1, \ldots, A_n\}$ ($A_i \in \mathbf{For}\mathcal{L}$). Our method is constructive and works whenever the base calculus $\mathcal{C}$ is 'expressive enough' (i.e., it is *amenable*), and the axioms A_i have a certain syntactic form.

Definition 4 (amenable calculus). *Let $\mathcal{C}$ be a display calculus satisfying C1–C8. Assume that we have two functions l and r mapping structures into $\mathbf{For}\mathcal{L}_{\mathcal{C}}$ such that $l(A) = r(A) = A$ when $A \in \mathbf{For}\mathcal{L}_{\mathcal{C}}$, and for an arbitrary structure X*

(i) $X \vdash l(X)$ and $r(X) \vdash X$ are derivable in $\mathcal{C}$.
(ii) $X \vdash Y$ derivable implies $l(X) \vdash r(Y)$ is derivable C.

Let there be a structure constant $\mathbf{I}$, and let the following rules be admissible in $\mathcal{C}$ for arbitrary structures X, Y such that the premise and conclusion are well-defined in $\mathcal{C}$.

$$\frac{\mathbf{I} \vdash X}{Y \vdash X}\, l\mathbf{I} \qquad\qquad \frac{X \vdash \mathbf{I}}{X \vdash Y}\, r\mathbf{I}$$

Let there be binary logical connectives $\vee, \wedge \in \mathcal{L}_{\mathcal{C}}$ such that $\cdot \in \{\vee, \wedge\}$ is associative *in $\mathcal{C}$ — $A \cdot (B \cdot C) \vdash (A \cdot B) \cdot C$ and $(A \cdot B) \cdot C \vdash A \cdot (B \cdot C)$ are derivable — and* commutative *in $\mathcal{C}$ — $A \cdot B \vdash B \cdot A$ is derivable. Also, for $A, B \in \mathbf{For}\mathcal{L}_{\mathcal{C}}$:*

$(a)_\vee$ $A \vdash X$ and $B \vdash X$ implies $\vee(A, B) \vdash X$
$(b)_\vee$ $X \vdash A$ implies $X \vdash \vee(A, B)$ for any formula B.
$(a)_\wedge$ $X \vdash A$ and $X \vdash B$ implies $X \vdash \wedge(A, B)$
$(b)_\wedge$ $A \vdash X$ implies $\wedge(A, B) \vdash X$ for any formula B.

A display calculus satisfying the above conditions is said to be amenable.

Requiring that $l\mathbf{I}$ and $r\mathbf{I}$ are admissible in $\mathcal{C}$ is weaker than requiring that $\mathcal{C}$ contains weakening rules. Indeed, the rules $l\mathbf{I}$ and $r\mathbf{I}$ are admissible in the bi-Lambek calculus [9]. The function l (resp. r) 'interprets' the structural connectives in the antecedent (resp. succedent). Above, we use the notation $\wedge$ and $\vee$ to reflect that in a classical calculus, the connectives conjunction and disjunction satisfy the respective properties.

Our recipe abstracts and reformulates for display calculi the procedure in [5, 6], defined for (hyper)sequent calculi and substructural logics. To transform axioms into structural rules we use: (1) the invertible logical rules of $\mathcal{C}$ and (2) the display calculus formulation, below, of the so-called Ackermann's lemma that allows a formula in a rule to switch sides of the sequent moving from conclusion to premises.

Lemma 1. *The following rules are pairwise equivalent in an amenable calculus where $A \in \mathbf{For}\mathcal{L}$, $\mathcal{S}$ is a set of sequents and $Z (\neq X)$ is a structure variable not in $\mathcal{S}$.*

$$\boxed{\frac{\mathcal{S}}{X \vdash A}\rho_1 \quad \frac{\mathcal{S} \quad A \vdash Z}{X \vdash Z}\rho_2} \qquad \boxed{\frac{\mathcal{S}}{A \vdash X}\delta_1 \quad \frac{\mathcal{S} \quad Z \vdash A}{Z \vdash X}\delta_2}$$

Proof. Suppose that we have concrete derivations of the premises $\mathcal{S} \cup \{A \vdash Z\}$ of ρ_2. Applying ρ_1 to $\mathcal{S}$ we get $X \vdash A$. Applying cut with $A \vdash Z$ we get $X \vdash Z$ and thus it follows that ρ_2 is admissible in a calculus containing ρ_1.

Now suppose we have concrete derivations of the premises $\mathcal{S}$ of ρ_1. Observe that $r(A) \vdash A$ is derivable. Applying ρ_2 to $\mathcal{S} \cup \{r(A) \vdash A\}$ we get $X \vdash A$ as required. The proof that δ_1 and δ_2 are equivalent is analogous.

We now give an abstract description of the axioms that we can handle. The description is based on the invertible rules of the chosen display calculus $\mathcal{C}$ and is inspired by the classification in [5] of formulae of FLe. We identify three classes of formulae in the language of $\mathcal{L}$ from which the logical connectives can be removed using the invertible rules of $\mathcal{C}$ at various levels. The class $\mathcal{I}_0$ consists of formulae with no logical connective (so there is no need for the invertible rules). Logical connectives in formulae in $\mathcal{I}_1$ can be eliminated by repeatedly applying the invertible rules starting with sequents (thus obtaining sets of sequents). Logical connectives in formulae in $\mathcal{I}_2$ can be eliminated by repeatedly applying the invertible rules to formulae, sequents and to the premises of rules obtained via Lemma 1 (thus obtaining sets of structural rules).

Definition 5 (inv). *The function inv takes a sequent $X \vdash Y$ and applies all the invertible logical rules in $\mathcal{C}$ that are possible and returns the (necessarily finite) set $\{X_i \vdash Y_i\}_{i \in \Omega}$ of sequents for some index set Ω.*

Definition 6 (soluble). *A formula $A \in \mathbf{For}\mathcal{L}$ is* a-soluble *(resp.* s-soluble*) if the sequents in inv($A \vdash$) (resp. inv($\vdash A$)) do not contain any logical connectives.*

Definition 7. *Let $\mathcal{C}$ be an amenable calculus for L. The classes $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2$ of formulae of $\mathbf{For}\mathcal{L}$ are defined in the following way: $A \in \mathbf{For}\mathcal{L}$ with inv($\vdash A$) $= \{U_i \vdash V_i\}_{i \in \Omega}$ for some finite Ω belongs to*

- $\mathcal{I}_0$ *if A contains no logical connectives*
- $\mathcal{I}_1$ *if each a-part formula in $U_i \vdash V_i$ is a-soluble and each s-part formula in $U_i \vdash V_i$ is s-soluble*
- $\mathcal{I}_2$ *if each a-part formula in $U_i \vdash V_i$ is s-soluble and each s-part formula in $U_i \vdash V_i$ is a-soluble*

A propositional variable is both a-soluble and s-soluble so $\mathcal{I}_0 \subseteq \mathcal{I}_1$ and $\mathcal{I}_0 \subseteq \mathcal{I}_2$. Note that every a-part (resp. s-part) formula B occuring in a sequent in inv($\vdash A$) that is a-soluble (s-soluble) must be a propositional variable and thus B is s-soluble (a-soluble) in that sequent. It follows that $\mathcal{I}_1 \subseteq \mathcal{I}_2$.

Remark 1. The above classes are a function of the invertible rules of the base calculus. In particular, these coincide with the classes in the hierarchy of [5] that can be handled by structural sequent rules, when the base calculus has the same invertible rules. More invertible rules lead to larger classes of formulae in $\mathcal{I}_1$ and $\mathcal{I}_2$ (see Section 4.2).

Henceforth a rule whose conclusion is constructed from structure variables and structure constants using structural connectives, and whose premises might additionally contain propostional variables will be called a *semi-structural* rule.

Proposition 1 *Let $\mathcal{C}$ be an amenable calculus for L. Suppose $A \in$ $\mathbf{For}\mathcal{L}$ with inv($\vdash A$) $= \{U_i \vdash V_i\}_{i\in\Omega}$. If $A \in \mathcal{I}_2$ then there are equivalent semi-structural rules $\{\rho_i\}_{i\in\Omega}$ so that $\mathcal{C} + \{\rho_i\}_{i\in\Omega}$ is a cut-eliminable calculus for $L + A$.*

Proof. Clearly $\vdash A$ is equivalent to $\{U_i \vdash V_i\}_{i\in\Omega}$ in $\mathcal{C}$. We show how to construct a semi-structural rule equivalent to each $U_i \vdash V_i$. Suppose that $U_i \vdash V_i$ consists of a-part formulae $C_1, \ldots, C_n$ and s-part formulae $D_1, \ldots, D_m$. Starting with $U_i \vdash V_i$, display each C_i (as $C_i \vdash W_i$ for suitable W_i) and apply Lemma 1 in turn, to obtain an equivalent rule of the form below left. Start with this rule and display in the conclusion each D_i (as $W_{n+i} \vdash D_i$ for suitable W_{n+i}) and apply Lemma 1 in turn, to obtain an equivalent rule of the form below right:

$$\frac{Z_1 \vdash C_1 \quad \ldots \quad Z_n \vdash C_n}{Z_n \vdash W_n} \qquad \frac{Z_1 \vdash C_1 \ \ldots\ Z_n \vdash C_n \quad D_1 \vdash Z_{n+1} \ \ldots\ D_m \vdash Z_{n+m}}{W_{n+m} \vdash Z_{n+m}}$$

Observe that $W_{n+m} \vdash Z_{n+m}$ is constructed only from structure variables and structure constants using structural connectives. Since $A \in \mathcal{I}_2$, every C_i (resp. D_i) formula is s-soluble (a-soluble) and so the following is a semi-structural rule equivalent to $U_i \vdash V_i$:

$$\frac{\text{inv}(Z_1 \vdash C_1) \ \ldots\ \text{inv}(Z_n \vdash C_n) \quad \text{inv}(D_1 \vdash Z_{n+1}) \ \ldots\ \text{inv}(D_m \vdash Z_{n+m})}{W_{n+m} \vdash Z_{n+m}}\ \rho_i$$

By inspection it may be verified that ρ_i satisfies Belnap's conditions with the possible exception of C1 and C4 since the same propositional variable might appear in the premises as an a-part and s-part formula. However, since no propositional variable occurs in the conclusion of ρ_i, cut-elimination for $\mathcal{C}+\{\rho_i\}_{i\in\Omega}$ proceeds without difficulty.

Notice that the calculus $\mathcal{C} + \{\rho_i\}_{i\in\Omega}$ in the above result has cut-elimination but not in general the subformula property. If we restrict our attention to a subclass of $\mathcal{I}_2$ satisfying the additional condition of *acyclicity* then the propositional variables appearing in each ρ_i can be suitably removed. In this way we obtain structural rules satisfying C1–C8 so the resulting calculus is cut-eliminable and has the subformula property.

Definition 8 (proper structural rules; extensions). *A proper structural rule (extension) is a structural rule (extension) that satisfies C1–C8.*

Our transformation of semi-structural rules into proper structural rules mirrors the 'completion' procedure in [6] and amounts to applying the cut-rule in all possible ways to the premises of the former rules. Below we present formally the transformation.

A (possibly empty) set $\mathcal{S}$ of sequents is said to *respect multiplicities wrt* p for some propositional variable p if it can be written in one of the forms below:

$$\{p \vdash U \,|\, p \notin U\} \cup \{V \vdash p \,|\, \text{every } p \text{ in } V \vdash p \text{ is s-part}\} \cup \{S \,|\, p \notin S\} \tag{1}$$

$$\{U \vdash p \,|\, p \notin U\} \cup \{p \vdash V \,|\, \text{every } p \text{ in } p \vdash V \text{ is a-part}\} \cup \{S \,|\, p \notin S\} \tag{2}$$

An alternative definition is that (i) no $S \in \mathcal{S}$ contains both an a-part and s-part occurrence of p — eg. $p \vdash p$ cannot be in $\mathcal{S}$, and (ii) there do not exist $S_1, S_2 \in \mathcal{S}$ such that S_1 contains multiple a-part occurrences of p and S_2 contains multiple s-part occurrences of p. Eg. if both occurrences of p in $p \otimes p \vdash X$ (resp. $Y \vdash p \otimes p$) are a-part (s-part) for a structural connective $\otimes$, then it cannot be that $p \otimes p \vdash X \in \mathcal{S}$ and $Y \vdash p \otimes p \in \mathcal{S}$.

Let $\mathcal{S}$ be a set of sequents respecting multiplicities wrt p. If it is not the case that $p \vdash U \in \mathcal{S}$ *and* $V \vdash p \in \mathcal{S}$ (upto display equivalence) then define $\mathcal{S}_p$ as $\{S \in \mathcal{S} \,|\, p \notin S\}$. Otherwise, depending on the form of $\mathcal{S}$ as (1) or (2), define respectively $\mathcal{S}_p$ as follows:

$$\{S \,|\, S \text{ is a subst. instance of } V \vdash p \in \mathcal{S} \text{ s.t. each occ. } p \mapsto U \text{ for some } p \vdash U \in \mathcal{S}\} \cup \{S \,|\, p \notin S\}$$

$$\{S \,|\, S \text{ is a subst. instance of } p \vdash V \in \mathcal{S} \text{ s.t. each occ. } p \mapsto U \text{ for some } U \vdash p \in \mathcal{S}\} \cup \{S \,|\, p \notin S\}$$

In the above, notice that distinct occurrences of p in $V \vdash p$ (resp. $p \vdash V$) may be substituted for distinct U_i so long as $p \vdash U_i$ ($U_i \vdash p$) is in $\mathcal{S}$. Also observe that the substitution instance S contains no occurrence of p since $p \notin U$.

Intuitively, if $\mathcal{S}$ contains sequents of the form $p \vdash U$ *and* $V \vdash p$ then $\mathcal{S}_p$ is obtained by (i) applying the cut-rule in all possible ways on p (using the sequents in $\mathcal{S}$ as the premises) and then keeping only those conclusion sequents of the cut-rule that do not contain p, and (ii) retaining $\{S \in \mathcal{S} \,|\, p \notin S\}$.

Lemma 2. *If $\mathcal{S}$ respects multiplicities wrt p, then p does not occur in $\mathcal{S}_p$.*

Proof. Follows immediately from the form of $\mathcal{S}$ and the definition of $\mathcal{S}_p$.

Let $\mathbb{V}(\mathcal{S})$ be the set of propositional variables occurring in a set $\mathcal{S}$ of sequents.

Definition 9. *A finite set $\mathcal{S}$ of sequents is* acyclic *if $\mathbb{V}(\mathcal{S}) = \emptyset$ or for every $p \in \mathbb{V}(\mathcal{S})$: (i) $\mathcal{S}$ respects multiplicities wrt p, and (ii) $\mathcal{S}_p$ is acyclic.*

Definition 10. *Suppose that $A \in \mathcal{I}_2$ and let $\{\rho_i\}_{i \in \Omega}$ be the equivalent semi-structural rules obtained using Prop. 1. We say that A is* acyclic *if the set of premises of each rule in $\{\rho_i\}_{i \in \Omega}$ is acyclic.*

Remark 2. Every axiom in $\mathcal{I}_1$ is acyclic. This follows from the observation that for every $A \in \mathcal{I}_1$, the premise of each semi-structural rule obtained using Prop. 1 has the form $p \vdash L$ or $L \vdash p$ where L is a schematic structure variable.

Lemma 3. *Let $\mathcal{S}$ be an acyclic set of sequents and $p \in \mathbb{V}(\mathcal{S})$. Then the semi-structural rule ρ with premises $\mathcal{S}$ and the semi-structural rule ρ_p with premises $\mathcal{S}_p$ are equivalent w.r.t. an amenable calculus $\mathcal{C}$.*

Proof. Let $\mathcal{S}$ be an acyclic set of sequents. Suppose that $\mathcal{S}$ does not contain sequents of the form $p \vdash U$ *and* $V \vdash p$. Then $\mathcal{S}$ has one of the following forms

$$\{V_1 \vdash p, \ldots, V_{n+1} \vdash p\} \cup \{S \mid p \notin S\} \qquad \{p \vdash V_1, \ldots, p \vdash V_{n+1}\} \cup \{S \mid p \notin S\}$$

and $\mathcal{S}_p$ is $\{S \in \mathcal{S} \mid p \notin S\}$. Suppose the case above left (the other case is similar). One direction is immediate, and to show that ρ_p is admissible in $\mathcal{C} + \rho$ it is enough to apply ρ using the derivable sequents $\{r(\mathbf{I}) \vdash V_i[p \mapsto r(\mathbf{I})]\}_{1 \leq i \leq n+1}$ — obtained from the derivation of $r(\mathbf{I}) \vdash \mathbf{I}$ using the rule $r\mathbf{I}$ — for the missing premises.

Now suppose that $\mathcal{S}$ contains sequents of the form $p \vdash U$ *and* $V \vdash p$. Clearly ρ is admissible in $\mathcal{C} + \rho_p$ — it suffices to apply the cut-rule to concrete premises of ρ and then apply ρ_p. For the other direction, assume, to fix ideas that the premises $\mathcal{S}$ of ρ have the form (1) (the other case is similar, use (a)$_\vee$ and (b)$_\vee$ instead of (a)$_\wedge$ and (b)$_\wedge$), i.e.,

$$\{p \vdash U_i \mid p \notin U_i; 1 \leq i \leq n\} \cup \{V \vdash p \mid \text{every } p \text{ in } V \vdash p \text{ is s-part}\} \cup \{S \mid p \notin S\}$$

Then the premises $\mathcal{S}_p$ of ρ_p have the following form:

$$\{S \mid S \text{ is a subst. instance of } V \vdash p \in \mathcal{S} \text{ s.t. each occ. } p \mapsto U_i \text{ for some } 1 \leq i \leq n)\ \} \cup \{S \mid p \notin S\}$$

Suppose that we are given concrete instances of the premises of ρ_p. Repeatedly using (a)$_\wedge$ and the display rules, obtain the set $\mathcal{S}_p^*$:

$$\{S \mid S \text{ is a subst. instance of } V \vdash p \in \mathcal{S} \text{ s.t. each occ. } p \mapsto \wedge_{1 \leq i \leq n} \mathrm{r}(U_i)\ \} \cup \{S \mid p \notin S\}$$

Making use of (b)$_\wedge$, derive the set $\{\wedge_{1 \leq i \leq n} \mathrm{r}(V_i) \vdash V_i\}$ of sequents. By inspection, this set together with $\mathcal{S}_p^*$ yield concrete instances of the premises of ρ (in particular, p has been instantiated with $\wedge_{1 \leq i \leq n} \mathrm{r}(V_i)$). Applying ρ to these and noting that ρ and ρ_p have the same conclusion, we have that ρ_p is admissible in $\mathcal{C} + \rho$.

Theorem 2. *Let $\mathcal{C}$ be an amenable calculus for L and suppose that $A \in \mathcal{I}_2$ is acyclic. Then there is a proper structural rule extension for $L + A$.*

Proof. Let $\{\rho_i\}_{i \in \Omega}$ be the semi-structural rules computed from A in Prop. 1. Notice that each ρ_i might violate (only) Belnap's conditions C1 and C4 due to the presence of propositional variables in the set $\mathcal{S}$ of sequents that are its premises. Let $\mathbb{V}(\mathcal{S}) = \{p_1, p_2, \ldots, p_n\}$ be such variables and ρ_i' be the rule with premises $((\ldots(\mathcal{S}_{p_1})_{p_2} \ldots)_{p_{n-1}})_{p_n}$. By inspection of the construction of ρ_i' from ρ follows that ρ_i' is a proper structural rule (in particular, observe that any structure variable that appears only as an a-part (resp. s-part) structure in every sequent in $\mathcal{S}$ has the same property in $((\ldots(\mathcal{S}_{p_1})_{p_2} \ldots)_{p_{n-1}})_{p_n}$). Since A is acyclic, so is $\mathcal{S}$ and hence, by (repeteadly applying) Lemma 2 it follows that ρ_i' is equivalent to ρ_i.

By repeating this process to all $\{\rho_i\}_{i \in \Omega}$ we obtain a new set of rules $\{\rho_i'\}_{i \in \Omega}$ such that $\mathcal{C} + \{\rho_i'\}_{i \in \Omega}$ is a proper structural rule extension of $L + A$.

4 Case studies

We apply the recipe in Section 3 to obtain many existing results *uniformly*, and to show that new calculi can be defined in an automated way. When dealing with a concrete base calculus we can provide an explicit description (grammar-like) of the class $\mathcal{I}_2$ of axioms that can be transformed into equivalent structural rules to obtain cut-eliminable display calculi. We present this grammar for the case of intermediate logics to compare our results with those in [5].

4.1 Bi-intuitionistic logic

Bi-intuitionistic logic (also known as Heyting-Brouwer logic) is the logic which results when the dual $\rightarrow_d$ of implication (alias coimplication) is added to the language of intuitionistic logic. Here we show how to construct cut-free display calculi for infinitely many axiomatic extensions of this logic in a uniform way.

The language $\mathcal{L}_{\mathrm{HB}}$ of Heyting-Brouwer logic HB is obtained from the language $\mathcal{L}_{\mathrm{Ip}}$ of intuitionistic propositional logic Ip by the addition of $\rightarrow_d$. To simplify the language, we abbreviate $\neg p := p \rightarrow \bot$ and $\neg_d p := \top \rightarrow_d p$. Wansing [18] give a proper display calculus δHB for HB. Our presentation here differs in that we use the invertible forms for $\wedge r$ and $\vee l$. Equivalence with the original rules can be shown using the structural rules of contraction and weakening in δHB.

The set of structures $\mathfrak{Str}(\mathcal{L}_{\mathrm{HB}})$ generated from $\mathcal{L}_{\mathrm{HB}}$ has the following grammar:

$$X ::= A \in \mathbf{For}\mathcal{L}_{\mathrm{HB}} \,|\, \mathbf{I} \,|\, (X \circ X) \,|\, (X \bullet X)$$

The initial sequents of δHB are of the form $p \vdash p$ for any propositional variable p, and $\mathbf{I} \vdash \top$ and $\bot \vdash \mathbf{I}$. Now we present the structural rules. In the first row, below, we use a double line to separate the premises from the conclusion to indicate that a rule is invertible and also 'combine' two rules into a single one for the sake of brevity. The first two columns (counting from the left) in the first row are the *display rules* of δHB.

$$\begin{array}{c} Y \vdash X \circ Z \\ \hline\hline X \circ Y \vdash Z \\ \hline\hline X \vdash Y \circ Z \end{array} \qquad \begin{array}{c} X \bullet Y \vdash Z \\ \hline\hline X \vdash Y \bullet Z \\ \hline\hline X \bullet Z \vdash Y \end{array} \qquad \begin{array}{c} \mathbf{I} \circ X \vdash Y \\ \hline\hline X \vdash Y \\ \hline\hline X \circ \mathbf{I} \vdash Y \end{array} \qquad \begin{array}{c} X \vdash Y \bullet \mathbf{I} \\ \hline\hline X \vdash Y \\ \hline\hline X \vdash \mathbf{I} \bullet Y \end{array}$$

$$\frac{X \vdash Y}{X \vdash Y \bullet Z} \qquad \frac{X \vdash Y}{X \circ Z \vdash Y} \qquad \frac{X \vdash Y \bullet Z}{X \vdash Z \bullet Y} \qquad \frac{X \circ Z \vdash Y}{Z \circ X \vdash Y}$$

$$\frac{X \vdash Y \bullet Y}{X \vdash Y} \qquad \frac{X \circ X \vdash Y}{X \vdash Y} \qquad \frac{X \vdash (Y \bullet Z) \bullet U}{X \vdash Y \bullet (Z \bullet U)} \qquad \frac{(X \circ Y) \circ Z \vdash U}{X \circ (Y \circ Z) \vdash U}$$

The logical rules of δHB are given below:

$$\frac{\mathbf{I} \vdash X}{\top \vdash X}\ \top l \qquad \frac{X \vdash \mathbf{I}}{X \vdash \bot}\ \bot r \qquad \frac{A \circ B \vdash X}{A \wedge B \vdash X}\ \wedge l$$

$$\frac{X \vdash A \qquad X \vdash B}{X \vdash A \wedge B}\ \wedge r \qquad \frac{A \vdash X \qquad B \vdash X}{A \vee B \vdash X}\ \vee l \qquad \frac{X \vdash A \bullet B}{X \vdash A \vee B}\ \vee r$$

$$\frac{X \vdash A \qquad Y \vdash B}{A \rightarrow B \vdash X \circ Y} \rightarrow l \qquad \frac{X \vdash A \circ B}{X \vdash A \rightarrow B} \rightarrow r \qquad \frac{B \bullet A \vdash X}{B \rightarrow_d A \vdash X} \rightarrow_d l$$

$$\frac{X \vdash B \qquad Y \vdash A}{X \bullet Y \vdash B \rightarrow_d A} \rightarrow_d r$$

Define the functions l and r from $\mathfrak{Str}(\mathcal{L}_{\text{HB}})$ into $\mathbf{For}\mathcal{L}_{\text{HB}}$:

$$\begin{aligned} l(A) &= A & r(A) &= A \\ l(\mathbf{I}) &= \top & r(\mathbf{I}) &= \bot \\ l(X \circ Y) &= l(X) \wedge l(Y) & r(X \circ Y) &= l(X) \to r(Y) \\ l(X \bullet Y) &= l(X) \to_d l(Y) & r(X \bullet Y) &= r(X) \vee r(Y) \end{aligned}$$

It is easy to check that δHB is amenable. (Note that $\wedge$ and $\vee$ are associative and commutative connectives in δHB). We have the following result.

Proposition 3 *Every logical rule except* $\to l$ *and* $\to_d r$ *is invertible.*

Theorem 4. *Let A be any (acyclic) axiom within $\mathcal{I}_2$. Then there is a (proper) structural rule extension of δHB for HB $+ A$.*

The following examples contain analytic display calculi for two axiomatic extensions of HB introduced in [19].

Example 1. Let A_1 be the axiom $(p \to q) \vee (q \to p)$. Then inv($\vdash A_1$) is the sequent $\vdash (p \circ q) \bullet (q \circ p)$. Since each formula in that sequent is a propositional variable, it is a,s-soluble. Thus $A_1 \in \mathcal{I}_1$. From Prop. 1 we obtain the equivalent semi-structural rule (below left). The set $\mathcal{S}$ of premises of this rule can be written $\{X \vdash p\} \cup \{p \vdash V\} \cup \{Z \vdash q, q \vdash Y\}$. Then $\mathcal{S}_p = \{X \vdash V, Z \vdash q, q \vdash Y\}$. Hence $\mathcal{S}_{pq} = \{X \vdash V, Z \vdash Y\}$. So $\mathcal{S}$ is acyclic and is equivalent to the proper structural rule below right:

$$\frac{X \vdash p \quad q \vdash Y \quad Z \vdash q \quad p \vdash V}{\mathbf{I} \vdash (X \circ Y) \bullet (Z \circ V)}\,\rho_1 \qquad \frac{X \vdash V \qquad Z \vdash Y}{\mathbf{I} \vdash (X \circ Y) \bullet (Z \circ V)}\,\rho_1'$$

Thus δHB $+ \rho_1'$ is a cut-eliminable display calculus for HB $+ A_1$ with subformula property. In practice, ρ_1' can be obtained from ρ_1 on sight, by applying the cut-rule to the premises in 'all possible ways'.

Example 2. Let A_2 be $\neg((p \to_d q) \wedge (q \to_d p))$. $A_2 \in \mathcal{I}_1$. Applying our recipe we get the equivalent rule ρ_2 such that δHB $+ \rho_2$ is a proper display calculus for HB $+ A_2$.

$$\frac{X \vdash Z \qquad U \vdash Y}{(X \bullet Y) \circ (U \bullet Z) \vdash \mathbf{I}}\,\rho_2$$

4.2 Intuitionistic logic

We discuss intermediate logics and compare our algorithm for display logic with the algorithm in [5] that works for hypersequent calculus – a simple generalization of Gentzen calculus [1] whose basic objects are multisets of sequents.

The calculus δHB^- obtained by deleting the logical rules for $\to_d$ is a display calculus for Ip — soundness of HB^- relies on the fact that Ip is a conservative extension of HB, and completeness follows from cut-elimination for HB. Observe that in δHB^- — unlike in Gentzen's calculus LJ — the $\vee r$ rule is also invertible. Following the idea of the classification in [5], which is sketched below for the connectives of Ip (= FLe with weakening and contraction), we can define $\mathcal{I}_2$ axioms for Ip and δHB$^-$ as follows

$$\mathcal{I}_0 ::= \text{prop. variables} \quad \mathcal{I}_{n+1} ::= \bot \mid \top \mid \mathcal{I}_n \to \mathcal{I}_{n+1} \mid \mathcal{I}_{n+1} \wedge \mathcal{I}_{n+1} \mid \mathcal{I}_{n+1} \vee \mathcal{I}_{n+1}$$

The class $\mathcal{I}_2$ is larger than the class of axioms that can be captured by structural hypersequent rules over LJ (see [5]). The latter consists of all axioms within the class $\mathcal{P}_3$ defined by the following grammar: $\mathcal{N}_0, \mathcal{P}_0$ contains the set of atomic formulae, and

$$\mathcal{P}_{n+1} ::= \bot \mid \top \mid \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1}$$
$$\mathcal{N}_{n+1} ::= \bot \mid \top \mid P_n \mid \mathcal{P}_{n+1} \to \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1}$$

(the classes $\mathcal{P}_n$ and $\mathcal{N}_n$ stand for axioms with leading positive and negative connective, i.e. having left (resp. right) logical rule invertible). It is easy to see that $\mathcal{P}_3 \subseteq I_2$.

By applying our recipe and making use of the weakening, commutativity and contraction rules in δHB$^-$ we can show the following:

Proposition 5 *There is a proper structural rule extension of δHB$^-$ for Ip + $\mathcal{A}$, for any set $\mathcal{A}$ of axioms in $\mathcal{P}_3$.*

Proof. By [5, Lemma 3.4] any $\mathcal{P}_3$ formula can be written as a conjunction of formulae $(*)$ $\vee_{1 \leq i \leq N} (\alpha_1^i \wedge \ldots \wedge \alpha_{n_i}^i \to \beta^i)$ where each β^i has the form $\overline{q}_1^i \vee \ldots \vee \overline{q}_{m_i}^i$ ($\overline{q}_j^i$ is a conjunction of propositional variables or $\bot$). Hence inv($\mathbf{I} \vdash A$) consists of sequents of the following form, where q_j^i is some propositional variable occurring in $\overline{q}_j^i$.

$$\mathbf{I} \vdash \big((\alpha_1^1 \circ \ldots \circ \alpha_{n_1}^1) \circ (q_1^1 \bullet \ldots \bullet q_{m_1}^1)\big) \bullet \ldots \bullet \big((\alpha_1^N \circ \ldots \circ \alpha_{n_N}^N) \circ (q_1^N \bullet \ldots \bullet q_{m_N}^N)\big)$$

Note that each α_j^i is an a-part formula and each q_j^i is an s-part formula. Now, following Prop. 1 we apply Lemma 1 and obtain that A is equivalent to the following semi-structural rule, where L_j^i is a structure variable (corresponding to α_j^i) and Q_j^i is a structure variable (corresponding to q_j^i).

$$\frac{\{\text{inv}(L_j^i \vdash \alpha_j^i)\}_{1 \leq i \leq N; 1 \leq j \leq n_i} \qquad \{q_j^i \vdash Q_j^i\}_{1 \leq i \leq N; 1 \leq j \leq m_i}}{\mathbf{I} \vdash \big((L_1^1 \circ \ldots \circ L_{n_1}^1) \circ (Q_1^1 \bullet \ldots \bullet Q_{m_1}^1)\big) \bullet \ldots \bullet \big((L_1^N \circ \ldots \circ L_{n_N}^N) \circ (Q_1^N \bullet \ldots \bullet Q_{m_N}^N)\big)}\ \rho'$$

where the structure variables $L_1^1, \ldots, L_{n_N}^N, Q_1^1, \ldots, Q_{m_N}^N$ are distinct. By [5, Lemma 3.4], each α_j^i in $(*)$ has the form $\wedge_{1 \leq k \leq a_j} (U_{jk}^i \to v_{jk}^i)$ where U_{jk}^i is $\top$ or a conjunction of propositional variables. Hence each set inv($L_j^i \vdash \alpha_j^i$) consists of sequents of the form

$$L_j^i \vdash (u_{j1}^i \circ \ldots \circ u_{ja_j^i}^i) \circ v_j^i$$

Observe that each propositional variable u_{jk}^i is an a-part formula and each propositional variable v_j^i is an s-part formula. Since the calculus has contraction on $\circ$ in the antecedent, we may assume without loss of generality that the u_{jk}^i are distinct for fixed i. If some $u_{jk}^i = v_j^i$ then the sequent is derivable as follows by repeated use of the weakening, commutativity and display rules for $\circ$:

$$\dfrac{\dfrac{\dfrac{v^i_j \vdash v^i_j}{(u^i_{j1} \circ \ldots \circ u^i_{ja_{ij}}) \vdash v^i_j}}{L^i_j \circ (u^i_{j1} \circ \ldots \circ u^i_{ja_{ij}}) \vdash v^i_j}}{L^i_j \vdash (u^i_{j1} \circ \ldots \circ u^i_{ja_{ij}}) \circ v^i_j}$$

Thus we can delete those premises of ρ' such that $u^i_{jk} = v^i_j$ to obtain an equivalent rule ρ. The premises $\mathcal{S}$ of ρ have the following form:

$$\{U \vdash p \,|\, p \notin U\} \cup \{p \vdash V \,|\, p \notin V\} \cup \{S \,|\, p \notin S\}$$

Let $\mathcal{S}'_p$ be the set $\{U \vdash V \,|\, p \notin U, p \notin V\} \cup \{S \,|\, p \notin S\}$. Arguing as in Lemma 3 we can show that ρ is equivalent to the rule ρ'_p obtained by replacing the premises $\mathcal{S}$ with $\mathcal{S}'_p$. While $\mathcal{S}'_p$ does not contain p, it may contain a sequent with (i) multiple a-part occurrences of some propositional variable or, (ii) an a-part and s-part occurrence of the same propositional variable. Obtain the rule ρ_p from ρ'_p by contracting multiplicities and deleting sequents witnessing (ii). Denote the premises of ρ_p by $\mathcal{S}_p$. Repeat for all propositional variables in $\mathcal{S}$ to obtain ultimately an equivalent proper structural rule.

Hence we can get proper structural rule extensions of δHB$^-$ for all intermediate logics that can be formalized by hypersequent calculi using the algorithm in [5]. But we can do more. Consider the axioms (Bd_k) $(k \geq 1)$, defining intermediate logics semantically characterized by Kripke models of depth $\leq k$, belong to the classes $\mathcal{P}_{2k}$ in the classification in [5]; these axioms are recursively defined as follows:

$$(Bd_1) \quad p_1 \vee \neg p_1 \qquad (Bd_{i+1}) \quad p_{i+1} \vee (p_{i+1} \rightarrow (Bd_i))$$

For $k \geq 2$, no axiom within $\mathcal{P}_3$ is known to be equivalent, yet these all belong to $\mathcal{I}_1$.

Example 3. The proper structural rule equivalent to the axiom (Bd_2) is

$$\dfrac{Y \vdash X \qquad V \vdash U}{\mathbf{I} \vdash X \bullet (Y \circ (U \bullet (V \circ \mathbf{I})))}\ \rho$$

In contrast *no* equivalent hypersequent structural rule is known.

Although our algorithm is inspired by that in [5], the key point is that the expressive power of the display calculus permits a base calculus for Ip in which the $\vee r$ rule is also invertible, leading to cut-eliminable structural rule extensions for more logics (see Remark 1). This justifies the use of the more complex machinery of the display calculus.

Example 4. δHB$^-$+ρ'_1 (cf. Example 1) is a cut-free calculus for Ip+A_1 (= Gödel logic) with subformula property. Classical propositional logic Cp is obtained as Ip + $p \vee \neg p$. Since $p \vee \neg p \in I_1$ we can define a proper structural rule extension of δHB$^-$ for Cp.

4.3 Bunched logics

Bunched logics [13] provide a powerful framework to reason about resources. They are obtained by combining an additive propositional logic with a multiplicative linear logic [4]. The combination led to the definition of four systems: BI, BBI (Boolean BI), dMBI (de Morgan BI) and CBI (classical BI). Brotherston [4] obtains display calculi for these logics by freely combining a calculus DL_{IL} (resp. DL_{CL}) for intutionistic (classical) propositional logic with a calculus DL_{LM} (resp. DL_{dMM}) for multiplicative intuitionistic linear logic (multiplicative classical linear logic).

Using the calculus δHB$^-$ for intuitionistic logic instead of DL_{IL}, new calculi for BI and BBI can be obtained. These calculi can be extended with the structural rule for classical logic (see Example 4) to obtain new calculi for dMBI and CBI that are structural extensions of the calculi for BI and BBI.

More generally, our algorithm yields proper structural rule extensions over δHB$^-$ for a large class of intermediate logics. Taking the free combination of such calculi with $\{DL_{LM}, DL_{dMM}\}$ yield cut-eliminable calculi for new bunched logics (intermediate between BI and CBI) which may express interesting properties on resources.

4.4 Modal and tense logics

The modal language $\mathcal{L}_K$ is obtained from the propositional classical language by the addition of the modal operators $\Diamond$ and $\Box$. The tense language $\mathcal{L}_{Kt}$ is obtained from $\mathcal{L}_K$ by the addition of the tense operators $\blacklozenge$ and $\blacksquare$. The normal basic modal logic K and tense logic Kt are conservative extensions of classical propositional logic Cp, obtained by the addition of the usual axioms (see [3]).

The display calculus δKt [14] for Kt is well-known. Here we use the invertible form of the rules for $\wedge r$, $\vee l$ and $\rightarrow l$. The set of structures $\mathfrak{Str}(\mathcal{L}_{Kt})$ generated from $\mathcal{L}_{Kt}$ has the following grammar:

$$X ::= A \in \mathbf{For}\mathcal{L}_{Kt} \,|\, \mathbf{I} \,|\, (X \circ X) \,|\, \bullet X \,|\, {*X}$$

The initial sequents of δKt are of the form $p \vdash p$ for any propositional variable p, and $\mathbf{I} \vdash \top$ and $\bot \vdash \mathbf{I}$. In the following we use a double line to separate the premises from the conclusion to indicate that a rule is invertible. The *display rules* of δKt are:

$$\frac{X \circ Y \vdash Z}{\overline{X \vdash Z \circ *Y}} \qquad \frac{X \circ Y \vdash Z}{\overline{Y \vdash *X \circ Z}} \qquad \frac{X \vdash Y \circ Z}{\overline{X \circ *Z \vdash Y}}$$

$$\frac{X \vdash Y \circ Z}{\overline{*Y \circ X \vdash Z}} \qquad \frac{*X \vdash Y}{\overline{*Y \vdash X}} \qquad \frac{X \vdash *Y}{\overline{Y \vdash *X}}$$

$$\frac{**X \vdash Y}{\overline{X \vdash Y}} \qquad \frac{X \vdash **Y}{\overline{X \vdash Y}} \qquad \frac{X \vdash \bullet Y}{\overline{\bullet X \vdash Y}}$$

The remaining structural rules of δKt are given below.

$$\frac{X \vdash Z}{\overline{\mathbf{I} \circ X \vdash Z}} \qquad \frac{X \vdash Z}{\overline{X \vdash \mathbf{I} \circ Z}} \qquad \frac{\mathbf{I} \vdash Y}{\overline{*\mathbf{I} \vdash Y}} \qquad \frac{X \vdash \mathbf{I}}{\overline{X \vdash *\mathbf{I}}}$$

$$\frac{X \vdash Z}{Y \circ X \vdash Z} \qquad \frac{X \vdash Z}{X \circ Y \vdash Z} \qquad \frac{\mathbf{I} \vdash Y}{\bullet\mathbf{I} \vdash Y} \qquad \frac{X \vdash \mathbf{I}}{X \vdash \bullet\mathbf{I}}$$

$$\frac{X \circ Y \vdash Z}{Y \circ X \vdash Z} \qquad \frac{Z \vdash X \circ Y}{Z \vdash Y \circ X} \qquad \frac{X \circ X \vdash Z}{X \vdash Z} \qquad \frac{Z \vdash X \circ X}{Z \vdash X}$$

$$\frac{X_1 \circ (X_2 \circ X_3) \vdash Z}{\overline{(X_1 \circ X_2) \circ X_3 \vdash Z}} \qquad \frac{Z \vdash X_1 \circ (X_2 \circ X_3)}{\overline{Z \vdash (X_1 \circ X_2) \circ X_3}}$$

Name	Axiom	Rule	Name	Axiom	Rule
D	$\Box A \to \Diamond A$	$(* \bullet *)\bullet X \vdash Y / X \vdash Y$	B	$A \to \Box\Diamond A$	$* \bullet * X \vdash Y / \bullet X \vdash Y$
	$\Diamond\Box A \to \Box\Diamond A$	$\bullet X \vdash * \bullet * Y / * \bullet * X \vdash \bullet Y$	4	$\Box A \to \Box\Box A$	$\bullet X \vdash Y / \bullet \bullet X \vdash Y$
5	$\Diamond A \to \Box\Diamond A$	$* \bullet * X \vdash Y / * \bullet * X \vdash \bullet Y$	T	$\Box A \to A$	$\bullet X \vdash Y / X \vdash Y$

Fig. 1. Some $\mathcal{I}_2$ axioms and corresponding proper structural rules

The logical rules of δKt are given below.

$$\frac{\mathbf{I} \vdash X}{\top \vdash X}\ \top l \qquad \frac{X \vdash \mathbf{I}}{X \vdash \bot}\ \bot r \qquad \frac{*A \vdash X}{\neg A \vdash X}\ \neg l$$

$$\frac{X \vdash *A}{X \vdash \neg A}\ \neg r \qquad \frac{A \circ B \vdash X}{A \wedge B \vdash X}\ \wedge l \qquad \frac{X \vdash A \qquad X \vdash B}{X \vdash A \wedge B}\ \wedge r$$

$$\frac{A \vdash X \qquad B \vdash X}{A \vee B \vdash X}\ \vee l \qquad \frac{X \vdash A \circ B}{X \vdash A \vee B}\ \vee r \qquad \frac{X \circ *Y \vdash A \qquad B \vdash *X \circ Y}{A \to B \vdash *X \circ Y}\ {\to} l$$

$$\frac{X \circ A \vdash B}{X \vdash A \to B}\ {\to} r \qquad \frac{A \vdash X}{\Box A \vdash \bullet X}\ \Box l \qquad \frac{X \vdash \bullet A}{X \vdash \Box A}\ \Box r$$

$$\frac{* \bullet * A \vdash X}{\Diamond A \vdash X}\ \Diamond l \qquad \frac{X \vdash A}{* \bullet * X \vdash \Diamond A}\ \Diamond r \qquad \frac{\bullet A \vdash X}{\blacklozenge A \vdash X}\ \blacklozenge l$$

$$\frac{X \vdash A}{\bullet X \vdash \blacklozenge A}\ \blacklozenge r \qquad \frac{A \vdash X}{\blacksquare A \vdash * \bullet * X}\ \blacksquare l \qquad \frac{X \vdash * \bullet * A}{X \vdash \blacksquare A}\ \blacksquare r$$

Define the functions l and r from $\mathfrak{Str}(\mathcal{L}_{Kt})$ into $\mathbf{For}\mathcal{L}_{Kt}$.

$$\begin{aligned} l(A) &= A & r(A) &= A \\ l(\mathbf{I}) &= \top & r(\mathbf{I}) &= \bot \\ l(*X) &= \neg r(X) & r(*X) &= \neg l(X) \\ l(X \circ Y) &= l(X) \wedge l(Y) & r(X \circ Y) &= r(X) \vee r(Y) \\ l(\bullet X) &= \blacklozenge l(X) & r(\bullet X) &= \Box r(X) \end{aligned}$$

It is easy to check that δKt is amenable.

Proposition 6 *Every logical rule with the exception of* $\Box l$*,* $\Diamond r$*,* $\blacklozenge r$ *and* $\blacksquare l$ *is invertible.*

Theorem 7. *There is a proper structural rule extension of* δKt *for axiomatic extension of* Kt *with acyclic* $\mathcal{I}_2$ *axioms.*

A procedure to define proper structural display logic rules for primitive axiomatic extensions of K and Kt was introduced by Kracht's [14]. A primitive tense axiom has the form $A \to B$ where both A and B are constructed from propositional variables and $\top$ using $\{\wedge, \vee, \Diamond, \blacklozenge\}$ and A contains each propositional variable at most once.

Kracht's method to extract structural rules is very different from our method, and relies on being able to transform the axiom into a primitive tense formula. Eg. the axiom $\Box A \to A$ must be rewritten as the primitive tense formula $A \to \Diamond A$. [17] rewrites the familiar B axiom $A \to \Box\Diamond A$ in the primitive tense form as $(A \wedge \Diamond B) \to \Diamond(B \wedge \Diamond A)$.

Example 5. Fig. 1 contains some examples of $\mathcal{I}_2$ axioms (see Table IV in [17]) and corresponding rules generated using our procedure. Contrast our strucutural rule for the B axiom with the rule generated by Kracht's method (see [17]):

$$\frac{*\bullet*(X \circ *\bullet*Y) \vdash Z}{Y \circ *\bullet*X \vdash Z}$$

In contrast with out method, Kracht's result provides a characterisation (a necessary and sufficient condition). Indeed

Theorem 8 (Kracht). *Let L be a tense logic. Then L is an axiomatic extension of Kt by primitive tense axioms iff there is a proper structural rule extension of δKt for L.*

It follows that every acyclic $\mathcal{I}_2$ axiom is equivalent to a primitive tense axiom.

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