Cut-elimination for provability logics and some results in display logic

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To my parents

Declaration

The work in this thesis is my own except where otherwise stated.

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Abstract

A syntactic proof of cut-elimination yields a procedure to eliminate every instance of the cut-rule from a derivation in the sequent calculus thus leading to a cut-free derivation. This is a central result in the proof-theory of a logic. In 1983, Valentini [71] presented a syntactic proof of cut-elimination for the provability logic *GL* for a traditional Gentzen sequent calculus built from sets, as opposed to multisets, thus avoiding an explicit rule of contraction. From a syntactic point of view it is more satisfying and formal to explicitly identify the applications of the contraction rule that are hidden in such calculi. Recently it has been claimed that the cut-elimination procedure does not terminate when applied to the corresponding sequent calculus built from multisets. Here we show how to resolve this issue in order to obtain a syntactic proof of cut-elimination for *GL*. The logics *Grz* and *Go* have a syntactically similar axiomatisation to *GL* which suggests that it might be possible to extend the proof to these logics. This is borne out by an existing proof for *Grz*. However, no proof has been presented for *Go*. We fill this gap in the literature by presenting a proof of syntactic cut-elimination for this logic. The transformations for *Go* require a deeper analysis of the derivation structures than the proofs for the other logics.

Next we examine Kracht's syntactic characterisation for the class of logics that can be presented via cutfree modal and tense display calculi. Recently it has been shown that the characterisation for modal display calculi is incorrect. In this work we significantly extend the class of modal logics that can be presented using cutfree modal display calculi. We utilise this result to give a proof theory for a syntactically-specified class of superintuitionistic logics. Then we take a semantic approach and show how to construct display calculi for superintuitionistic logics specified by suitable frame conditions.

Finally, we introduce the natural maps between tree-hypersequent and nested sequent calculi, and a proper subclass of labelled sequent calculi that we call labelled tree sequent calculi. Then we show how to embed certain labelled sequent

calculi into the corresponding labelled tree sequent systems. Using the existing soundness and completeness and cut-admissibility results for the labelled sequent calculus $G3GL$ we then obtain the corresponding results for Poggiolesi's [58] tree-hypersequent calculus $CSGL$, thus alleviating the need for independent proofs and answering a question posed in that paper. Next, we generalise a scheme for obtaining labelled tree sequent rules corresponding to modal axioms and investigate the possibility of using this method to obtain modular cutfree nested sequent systems for a large class of modal logics. Although the general result remains to be proved, we consider some concrete cases and show that the scheme leads to cutfree systems.

Contents

Chapter 1

Introduction

David Hilbert [47] initiated the branch of Logic called proof theory as a part of his investigation into the foundation of mathematics. Hilbert's original intention was to prove the consistency of axiomatisations for mathematics by elementary means. This agenda called for a complete *formalisation* of various parts of mathematics. In particular, mathematical proofs as well as the steps of reasoning employed in a logical argument were to be formalised using a *proof-system* called the Hilbert calculus [16]. The Hilbert calculus consists of a number of axioms or 'truths' and some number of inference rules which specify how to obtain new 'truths' from existing ones. The point is that a proof *in* the Hilbert calculus (let us call this a *derivation*) is a well-defined object, as opposed to the informal and imprecise notion employed in ordinary discourse. Indeed, most real-world mathematical proofs fall into the latter category, for instance, due to the omission of 'obvious' steps and trivial cases. By precisely formalising the notion of proof, it became possible to undertake a serious study of proofs, or *proof theory*. Although Hilbert's original aim of an elementary proof of consistency for mathematical theories was shown to be unattainable due to Gödel's [27] famous incompleteness theorems, variations of this programme have been a driving force for the development of proof theory [70]. Furthermore, the results and techniques of proof theory have been fruitfully employed in areas such as automated reasoning [34] and linguistics [14].

A drawback with the Hilbert calculus is the fact that derivations in this system seem to lack a discernible structure, making them difficult to analyse. Moreover, derivations in the Hilbert calculus are very different to the proofs written by mathematicians in terms of the style of reasoning that is used. In response, Gerhard Gentzen [25] introduced the system of *natural deduction*. As the name suggests, the formal inference rules in this system mimic (formalise) the sort of deductive reasoning that is employed in practice. In order to study the properties of this system, Gentzen then constructed yet another proof-system called the *sequent calculus*. Gentzen's *Hauptsatz* or main theorem for the sequent calculus is the cut-elimination theorem which shows how to obtain a standard form for derivations using a special type of sequent calculus called a *cutfree* sequent calculus. Derivations in a cutfree sequent calculus have a particularly nice structure making cutfree calculi an excellent tool for proof-theoretical study. In addition, the cut-elimination theorem often leads to simple proofs of consistency and interpolation [70]. This work established the branch of study called *structural proof theory* which is concerned with the structure and properties of proofs and proof-systems. Structural proof theory focusses on syntactic structures, which means that it is concerned primarily with the syntax of formal systems as opposed to the interpretation or meaning that is attached to them.

Broadly speaking, since Logic is concerned with systems of reasoning ('logics'), a major objective of proof theory is to construct proof-systems that formalise these logics. Cutfree sequent calculi have been presented for a large class of logics. Nevertheless, there are logics that have defied treatment as a cutfree (Gentzen) sequent calculus, for example the modal logic $S5$ [66, 20, 36, 56]. In part, this is due to the fact that the proofs of cut-elimination tend to be highly sensitive to minor alterations in the form of the inference rules — this is an important consideration because new proof-systems are most easily constructed by the addition or alteration of existing inference rules. Many variants/refinement proof-systems have been proposed in an effort to address the shortcomings in Gentzen's sequent calculus, although it should be noted that each of these systems has its own shortcomings. Examples of proof-systems that we will encounter in this thesis include the display calculus [5] and labelled sequent calculi [24, 52]. Structural proof-theory encompasses the study of these proof-systems as well as the logic-specific results that can be obtained through their analysis.

The work in this thesis broadly concerns the following areas:

(i) cut-elimination for some sequent calculi,

(ii) display calculi for modal and superintuitionistic logics,

(iii) importing results from labelled sequent calculi into other proof-systems.

The common theme is that all these areas pertain to structural proof-theory, although it goes without saying that the solutions draw from techniques in other fields as well.

In the following we provide a brief introduction to each problem and present the contributions of this thesis. A historical introduction to each problem is given in its respective chapter.

1.1 Cut-elimination for sequent calculi

Gentzen [25] introduced the *sequent calculus* as a tool for studying his system of natural deduction for classical and intuitionistic logic. The sequent calculus is built from ordered pairs (X, Y) (these ordered pairs are called *sequents* and are written $X \Rightarrow Y$) where X and Y are sets or multisets (Gentzen originally used ordered lists) of logical formulae. The logical formulae are constructed using variables and logical symbols such as $\neg$ ("not"), $\vee$ ("disjunction"), $\wedge$ ("conjunction") and $\supset$ ("implication"). The sequent $X \Rightarrow Y$ is intended to correspond to the formula $\bigwedge X \supset \bigvee Y$ where the notation $\bigwedge X$ (resp. $\bigvee Y$) denotes the conjunction (disjunction) of all formulae in X (Y). The sequent calculus typically consists of (i) a set of ('initial') sequents and (ii) inference rules. Each inference rule specifies how to obtain a new sequent from an appropriate set of sequents. A *derivation* is obtained by repeatedly applying the inference rules beginning with the initial sequents. A sequent is said to be *derivable* if there is a derivation concluding with that sequent. It is usual to view a logic, under a suitable interpretation for the variables and logical symbols, in terms of some *set* L of logical formulae. Then a sequent calculus is said to present the logic L if the derivable sequents in the calculus correspond exactly to the formulae in L.

One sequent calculus inference rule that receives special attention is the *cut-rule*. In the following we use the standard convention of writing X, A to mean the set (resp. multiset) $X \cup \{A\}$ for some set (multiset) X and formula A. The cut-rule states that if $X \Rightarrow Y, A$ and $A, U \Rightarrow V$ are derivable sequents then so is the sequent $X, U \Rightarrow Y, V$. In the usual notation for sequent calculi, the cut-rule is presented as follows:

$$\frac{X \Rightarrow Y, A \qquad A, U \Rightarrow V}{X, U \Rightarrow Y, V}\ \mathit{cut}$$

The sequents above the line are called the *premises*, the sequent below the line is called the *conclusion* of the rule, and A is called the *cut-formula*. The name "cut-rule" can be motivated by noting that in the above, the formula A has been 'cut' away from the premises to obtain the conclusion of the rule. The cut-rule is analogous to the *modus ponens* rule in Hilbert calculi. Not surprisingly, the presence of this rule is helpful in establishing completeness results between a sequent calculus and the corresponding Hilbert calculus. At the same time, the presence of the rule in the sequent calculus is undesirable as well. One reason is that the cut-rule causes a formula (when viewed downwards from the premises) to disappear, violating the so-called *subformula property* which states that every formula appearing in the premise of an inference rule appears as a subformula in the conclusion. If every inference rule in the sequent calculus obeys the subformula property, then every formula appearing in a derivation appears as a subformula in the final sequent. This facilitates the possibility of constructing a derivation (should it exist) for a given sequent by beginning with this sequent and exhaustively applying the inference rules backwards, from conclusion sequent to premise sequents — this approach is often called *backward proof search*. The presence of the cut-rule makes it difficult to conduct backward proof search because we would need to deduce the cut-formula first, in order to apply the cut-rule backwards, and it is usually not clear how to do this.

One way to have our cake and eat it too is to show that it is always possible to eliminate instances of the cut-rule from a given derivation, transforming the given derivation into a *cutfree* derivation of the same sequent. A syntactic proof of cut-elimination, or *syntactic cut-elimination* for short, is a proof that it is always possible via constructive transformation to eliminate the cut-rules in a given derivation in order to obtain a cutfree derivation of the same sequent. This is one of the most important results in the proof theory of a logic and the existence of such a transformation is a highly desirable property for a sequent calculus. The first such proof was given by Gentzen [25] who recognised the importance of a constructive procedure in his celebrated *Hauptsatz* where syntactic cut-elimination is presented for the classical and intuitionistic sequent calculi LK and LJ respectively. From the onset it was noted that the cut-elimination result is highly dependent on the form and structure of the rules in the sequent calculus. Moreover, Gentzen observed that the use of sequents built from multisets rather than sets can create further complications — in his proof of the *Hauptsatz* he introduced a new multicut rule to handle the ensuing cases.

Let us expand on some of the consequences of the cut-elimination result. A logic is inconsistent if it contains a false or 'absurd' statement. If some sequent calculus presents this logic, then there must be a derivation of the absurd statement. The cut-elimination theorem tells us that if there is a derivation of the absurd, then there must be a cutfree derivation of the absurd. In practice, for most sequent calculi, it is easy to check that there cannot be a cutfree proof of the absurd, thus establishing consistency. Under these circumstances, the cut-elimination result is at least as strong as consistency of the logic.

Gentzen's original use of the cut-elimination result was to prove the normalisation result for the natural deduction system for intuitionistic logic. The correspondence between cut-elimination and normalisation has been extended to many other logics. We have already noted that suitable cutfree sequent calculi can be used for backward proof search.

If we interpret each formula in a sequent as a mathematical statement, then an instance of the cut-rule in a derivation can be viewed as an occurrence of a lemma within a mathematical proof. The syntactic cut-elimination theorem says that whenever lemmata are employed in the proof of some statement there is a constructive procedure for rewriting the proof in order to obtain a new proof, containing no lemmata, of the same statement.

1.1.1 Cut-elimination for GL

We begin by looking at syntactic cut-elimination for provability logic GL. The logic GL is obtained by the addition of Löb's axiom $\Box(\Box A \supset A) \supset \Box A$ to the basic normal modal logic K. The history of syntactic cut-elimination for GL is rather convoluted. In 1981, the first proof was presented by Leivant [42]. Valentini [71] found an error in that proof and presented a new proof of cut-elimination. These proofs were presented for sequent calculi where the sequents were built from sets as opposed to multisets. Subsequently Moen [51] claimed that Valentini's arguments break down when lifted to a sequent calculus for sequents built from multisets. Because Moen used a non-constructive argument, it was not possible to determine if the problem was with Valentini's original proof, the proof for a sequent calculus built from multisets, or if Moen's claim was incorrect. The resulting confusion and continued interest in the problem is reflected by the fact that it has motivated several new solutions for a variety of different proof-systems (see [8, 65, 52, 49, 58]), although none of these works address Moen's claim.

Here, we successfully translate Valentini's original set-based arguments for cut-elimination to a sequent calculus built from multisets. A new transformation is required to handle the *contraction rules* that need to be included in the calculus to handle the transition from sets to multisets. Moreover, the transformation needs to be stated precisely in order to prove that the induction measure justifying the proof of cut-elimination is well-founded. Under these conditions, we show that

Valentini's argument can be applied to a sequent built from multisets. Finally, we have identified a specific error in Moen's work. Thus this work lays to rest the controversy surrounding Valentini's cut-elimination for GL.

Aside from the cut-elimination, we describe also how to use the decision and countermodel procedure for GL (Sambin and Valentini [64]) to obtain a new decision procedure for intuitionistic propositional logic [16]. A novelty in this work is the usage of proof-theoretic methods to prove that the resulting countermodel has the persistence property of intuitionistic models.

1.1.2 Extending the cut-elimination result to *Go*

The logics Grz (Grzegorczyk's logic [46]) and Go [44] can be obtained by the addition to the basic modal logic K [7] of the axiom

$$\Box(\Box(A \supset \Box A) \supset A) \supset \Box A \tag{1.1}$$

as well as the reflexivity axiom $\Box A \supset A$ (for Grz) or the transitivity axiom $\Box A \supset \Box\Box A$ (for Go) Notice the similarity in form between Löb's axiom and the formula (1.1). In fact, the inference rules for sequent calculi for these logics are very similar as well. This raises the obvious question — can we exploit the similarity of the inference rules to obtain cut-elimination procedures for Grz and Go? For Grz there is an existing proof of syntactic cut-elimination due to Borga and Gentilini [9]. Indeed, the transformations for that proof bear a similarity to the proof of cut-elimination for GL. However, no proof has been presented for Go. In this work, we will fill this gap in the literature by presenting a proof of cut-elimination for Go. We observe that the proof for Go is significantly more complicated than the proofs for GL and Grz, and the transformations require a greater generality and a deeper analysis of the derivation structures than the proofs for the other logics.

1.2 Display calculi for modal and superintuitionistic logics

Belnap's [5] Display Calculus is a proof-system that is capable of capturing a large class of logics. Roughly speaking, the display calculus can be obtained from the Gentzen sequent calculus by augmenting the logical connectives with a set of metalevel connectives ('structural connectives'). Inference rules specify the behaviour of these structural connectives as well as the logical connectives. Typically, the subformula property is enforced for inference rules introducing logical connectives but not for inference rules dealing with structural connectives. This lattitude with respect to structural connectives permits nice properties such as the so-called *display property*, and the general cut-elimination theorem, which applies whenever the rules of the calculus satisfy some well-defined criteria. Indeed, if we focus on extensions of a given display calculus via structural rules (these are inference rules that do not contain logical connectives) then these criteria can be verified directly. A logic that can be presented using suitable structural rule extensions of the display calculus is said to be *properly displayable*.

Kracht [39] presented an elegant result characterising the axiomatic extensions of the basic tense logic Kt that are properly displayable. Kracht also claimed an analogous characterisation for axiomatic extensions of the basic modal logic K. A counterexample to this claim has been suggested [80]. However, the validity of the counterexample rests on the statement that the logic in question cannot be expressed using a certain axiomatisation, and we are not aware of any existing proof of this statement. Here, we show that the counterexample is indeed valid by proving this non-trivial result. Next, we propose a new characterisation of axiomatic extensions over K that are properly displayable. Although the complete characterisation for properly displayable modal logics rests on a conjecture that has yet to be proved[1], even without this conjecture our work extends significantly the class of modal logics that are properly displayable.

The logics between intuitionistic propositional logic and classical propositional logic are called superintuitionistic logics [16]. It is well-known that the Gödel translation [27] induces a map between the class of superintuitionistic logics and modal logics. We apply our results to construct display calculi for superintuitionistic logics that are axiomatised by formule of a certain syntactic form. We note that this method is limited in scope due to the difficulty of expressing a given superintuitionistic logic in the required syntactic form. Next, we utilise a semantic characterisation of properly displayable modal logics and show how to construct display calculi for superintuitionistic logics specified by suitable semantic frame conditions. Using this technique we are able to properly display a large class of superintuitionistic logics.

[1]M. Kracht has given a ‘proof’ for the conjecture, but we have shown that his proof is incomplete. He completely agrees with our analysis regarding this problem and concedes that it is not clear how to obtain the result: personal correspondence by email dated 13/Dec/2010.

1.3 Importing results from labelled sequent calculi

Labelled sequent calculi [24, 52], tree-hypersequent calculi [57] and nested sequent calculi [37, 11] are examples of variant proof-systems of the Gentzen sequent calculus that have been studied in recent years. Here, we identify a subclass of labelled sequent calculi called *labelled tree sequent calculi* and show how to construct maps between labelled tree sequent, tree-hypersequent and nested sequent calculi. These maps allow us to translate derivations faithfully between these systems. We can exploit this translation to answer a question posed as future work by Poggiolesi [58] concerning the relationship between the tree-hypersequent calculus $CSGL$ for provability logic GL and the labelled sequent calculus $G3GL$ [52] for GL. In particular, this allows us to import results such as soundness and completeness for GL, and cut-admissibility from $G3GL$ to $CSGL$.

Next we generalise a scheme by Hein [35] for constructing labelled tree sequent calculi for modal logics axiomatised over K using 3/4 Lemmon-Scott formulae. Hein has conjectured that the resulting calculi have cut-elimination but does not present a proof. Here we show that cut-elimination holds for these calculi for some concrete modal logics, by utilising existing results [12] for nested sequent calculi. Although we do not yet have a general proof of cut-elimination, the work here indicates how this problem can be phrased in terms of importing results from suitable labelled sequent calculi into labelled tree sequent calculi.

1.4 Organisation of material

The thesis is organised as follows. In Chapter 2 we present cut-elimination for GL. In Chapter 3 we present cut-elimination for Go. In Chapter 4 we introduce some preliminary results in correspondence theory and introduce the display calculus. In Chapter 5 we present the characterisation for display calculi for modal and tense logics. In Chapter 6 we discuss how to construct display calculi for superintuitionistic logics. Finally, in Chapter 7 we study how to import results for labelled sequent calculi into labelled tree sequent calculi and other notational variant systems. In Chapter 8 we present the Conclusion. The chapters have been organised into Parts I, II and III. Each part is self-contained. Finally, to maintain the flow of the text, certain proofs and results have been placed in Appendix A.

Part I

Cut-elimination for sequent calculi

Chapter 2

Cut-elimination for provability logic GL resolved

In this chapter, we first introduce the provability logic GL and discuss some problems with existing proofs of cut-elimination for GL (Section 2.1). In Section 2.2 we formally introduce the sequent calculus GLS together with some basic definitions and terminology. Next, we introduce the technical device we call the stub-derivation (Section 2.3.1). Informally, a stub-derivation is obtained from a derivation τ by replacing one or more subderivations in τ with a stub ('hole'). The stub-derivation will help us to model the changing derivation under the cut-elimination transformations. We then prove invertibility results for the logical rules of GLS (Section 2.3.2). In Section 2.4 we present the new proof of cut-elimination.

In the final three sections we discuss extensions of this work. In Section 2.7 we show that the cut-elimination argument can be adapted to handle a multiplicative $L\supset$ rule instead of the additive $L\supset$ rule in GLS. Section 2.8 extends our work in a different direction. We obtain a decision/countermodel construction procedure for intuitionistic logic, building on a decision/countermodel construction procedure for GL described in [64]. Finally, in Section 2.9 we discuss how the cut-elimination procedure can be adapted to sequent calculi for some other logics bearing a similar axiomatisation to GL. The final section serves as a lead-in to Chapter 3 where a proof of cut-elimination for Go is presented.

2.1 Problems with cut-elimination for GL

The provability logic GL is obtained by adding Löb's axiom $\Box(\Box A \supset A) \supset \Box A$ to the standard Hilbert calculus for propositional normal modal logic K [64]. Interpreting the modal operator $\Box A$ as the provability predicate "A is provable in Peano arithmetic", it can be shown that GL is complete with respect to the formal provability interpretation in Peano arithmetic (see [68]). For an introduction to provability logic see [69].

In 1981, Leivant [42] proposed a syntactic proof of cut-elimination for a sequent calculus for GL. Valentini [71] soon described a counter-example to this proof, proposing a more complicated proof for the sequent calculus GLS_V for GL. The calculus GLS_V is a sequent calculus for classical propositional logic together with a single modal rule GLR. Valentini's proof appears to be the first proof of cut-elimination for a sequent calculus for GL and relies on a complicated transformation justified by a Gentzen-style induction on the degree of the cut-formula and the cut-height, as well as a new induction parameter — the width of a cut-formula. Roughly speaking, the width of a cut-instance is the number of GLR rule instances above that cut which contain a parametric ancestor of the cut-formula in their conclusion. However, Valentini's proof is very brief, informal and difficult to check. For example, he only considers a cut-instance where the cut-formula is left and right principal by the GLR rule (the Sambin Normal Form), noting that "the presence of the new parameter [width] does not affect the [remainder of the standard cut-elimination proof]" [71]. While it is true that the standard transformations appropriately reduce the degree and/or cut-height, he fails to observe that these transformations can sometimes increase the width of lower cuts, casting doubt on the validity of the induction. A careful study of the proposed transformation is required to confirm that the proof is not affected (see Section 2.5).

Several other solutions for cut-elimination have been proposed. Borga [8] presented one solution, while Sasaki [65] described a proof for a sequent calculus very similar to GLS_V, relying on cut-elimination for $K4$. Note that only Leivant and Valentini used traditional Gentzen-style methods involving an induction over the degree of the cut-formula and the cut-height.

All four authors used sequents $X \Rightarrow Y$ where X and Y are *sets*, so these calculi did not require a rule of contraction as there is no notion of a set containing an element multiple times (unlike a multiset where the number of occurrences is important). Thus the following instance of the $L\wedge$ rule is legal in GLS_V even though it ‘hides’ a contraction on $A \wedge B$:

$$\frac{A \wedge B, A, X \Rightarrow Y}{A \wedge B, X \Rightarrow Y}\ L\wedge$$

From a syntactic viewpoint, it is more satisfying and formal to explicitly identify the contractions that are ‘hidden’ in these set-based proofs of cut-elimination. The appropriate formalisation to understand the reliance on contraction is to use multisets. Gentzen [25] in his original proof of the *Hauptsatz* for the classical sequent calculus LK, introduced a ‘multicut’ rule to deal with a complication in the case of contractions above cut. Recent investigations into cut-elimination (especially in *structural proof-theory*) recognise the fact that the multicut rule combines the structural rules of contraction and *cut*. This is undesirable as it hinders our ability to analyse the independent ‘effect’ of each rule. Syntactic proofs of cut-elimination for classical logic without the use of additional rules such as the multicut rule have appeared in the literature, for example see [77, 10, 6]. This is despite the existence of numerous sequent calculi built from sets for classical logic. Since a syntactic proof of cut-elimination for a calculus built from sets can be used to induce cutfree derivations in the corresponding calculus, these works indicate the independent proof-theoretical interest in *how* the cutfree derivations are obtained. In particular, with syntactic cut-elimination, the interest is in direct proofs of cut-elimination. Inducing a proof for calculi built from multisets from a calculus built from sets is essentially equivalent to the use of the multicut rule and hence this is considered unsatisfactory as a method of syntactic cut-elimination. More broadly, we should note here that proof theory is concerned not just with the logic, but also with specific proof calculi for the logic. Hence questions such as the data structure of the sequent (lists, sets, multisets, for example), the form of its rules (are the rules invertible? do they have the subformula property?), and its properties (such as cut-elimination) are of great interest to the proof-theorist.

In the case of GL, it turns out that additional complications also arise when formulating Valentini’s arguments in a multiset setting, for example, due to the interplay between weakening and contraction rules (see Remark 2.16). Thus the move to a proof of cut-elimination for sequents built from multisets is not straightforward. Moen [51] attempted to lift Valentini’s set-based arguments to obtain a proof for sequents built from multisets before concluding that this was not possible. Specifically, he presents a concrete derivation ϵ containing cut, and claims that a multiset formulation of Valentini’s argument does not terminate when applied to ϵ. This claim has ignited the search for new proofs of purely syntactic cut-elimination in a Gentzen-style multiset setting for GL.

In response, Negri [52] and Mints [49] proposed two different solutions. Negri uses a non-standard multiset sequent calculus in which sequents are built from multisets of labelled formulae of the form $x : A$, where A is a traditional formula and x is an explicit name for a *state* in the frame semantics [16]. She shows that contraction is height-preserving admissible in this calculus and uses this to handle contractions above cut in her cut-elimination argument. In our view, the use of semantic information in the calculus deviates from a purely proof theoretic approach. Mints [49] solves the problem using a sequent calculus similar to the multiset-formulation of GLS_V, but does not specify how to handle the case of contractions above cut.

Recently, Poggiolesi [58] presented a proof of syntactic cut-elimination for a *tree-hypersequent* calculus for GL. A tree-hypersequent is built using some number of (Gentzen) sequents that are ordered and nested using new meta-level symbols '/' and ';', and yields a structure that can be read in terms of a tree. In particular, the order of the sequents (as determined by the '/', ';' symbols) is important. This contrasts with hypersequents [3] which are built from some number of traditional sequents separated by '/' where the order of the sequents is *not* important. Poggiolesi claims that the tree-hypersequent calculus for GL has all the advantages of Negri's calculus and in addition does not make use of any semantic elements. However, it appears that tree-hypersequents 'hide' the labelling for frame states by making use of the ordering and nesting induced by '/' and ';'. In other words, the reliance on the ordering of the sequents means that tree-hypersequents contain similarly explicit semantic information, although disguised through the use of less suggestive symbols.

So there are two issues to consider:

1. formalise "width" more precisely to clarify Valentini's original definition, and check whether it is a suitable induction measure, and

2. determine whether Valentini's arguments can be used to obtain a purely syntactic proof of cut-elimination in a *multiset* setting.

Our contribution is as follows: we have successfully translated Valentini's set-based arguments for cut-elimination to the corresponding sequent calculus built from multisets. To this end, we have formalised the notion of parametric ancestor and width to correspond intuitively with Valentini's original definition. With this formalisation we show that Valentini's arguments can be applied in the multiset setting, noting that although certain transformations may increase the width of lower cuts, this does not affect the proof. In the case where the last rule in either premise derivation of the cut-rule is a contraction on the cut-formula, we avoid the multicut rule by using von Plato's arguments [77] when the cut-formula is not boxed, and a new argument for the case when the cut-formula is boxed. Thus we obtain a purely syntactic proof of cut-elimination in a multiset setting. We also believe that we have identified an error in Moen's claim that Valentini's arguments (in a multiset setting) do not terminate. It appears that Moen has not used a faithful representation of Valentini's arguments for the inductive case, but instead a transformation he titles Val-II(core) that is already known to be insufficient [64]. We discuss this further in Section 2.6. Of course, the incorrectness of Moen's claim does not imply the correctness of Valentini's arguments in a multiset setting. Indeed, the whole point is that complications do arise in the multiset setting, and that these have to be dealt with carefully.

We remind the reader that it is trivial to show that the cut-rule is redundant for both set and multiset sequent calculus formulations for GL by proving that the calculus without the cut-rule is sound and complete for the frame semantics of GL. However, our purpose here is to resolve the claim about the failure of *syntactic* cut-elimination based on Valentini's arguments for the corresponding sequent calculus built with multisets. A formalisation of the cut-elimination result based on the proof presented here and using the proof assistant Isabelle appears in Dawson and Goré [19].

The proof of cut-elimination presented in this chapter is based on our work reported in [32].

2.2 Basic definitions and notation

We use the letters $p, q, \ldots$ to denote propositional variables. Formulae are defined in the usual way [16] in terms of propositional variables, the logical constant $\bot$ and the logical connectives $\land$ (conjunction), $\lor$ (disjunction), $\supset$ (implication) and $\Box$ (necessity, or in this context, provability). Formulae are denoted by $A, B, \ldots$. Multisets of formulae are denoted by X, Y, U, V, W, G and also as a list of comma-separated formula enclosed in "$\langle$" and "$\rangle$". A formula A is said to be *boxed* if it is of the form $\Box B$ for some formula B and is *not boxed* otherwise. The relation '$\equiv$' is used to denote syntactic equality. Let X be the multiset $\langle A_1, \ldots, A_n\rangle$. By a slight abuse of notation, X, B denotes the multiset $\langle A_1, \ldots, A_n, B\rangle$ Also define the multiset $\Box X$ to be $\langle \Box A_1, \ldots, \Box A_n\rangle$. Furthermore $B \in X$ iff $B \equiv A_i$ for some

Initial sequents: $A \Rightarrow A$ for each formula A

Logical rules:

$$\frac{X \Rightarrow Y, A}{X, \neg A \Rightarrow Y}\, L\neg \qquad \frac{A, X \Rightarrow Y}{X \Rightarrow Y, \neg A}\, R\neg$$

$$\frac{A_i, X \Rightarrow Y}{A_1 \wedge A_2, X \Rightarrow Y}\, L\wedge \qquad \frac{X \Rightarrow Y, A_1 \qquad X \Rightarrow Y, A_2}{X \Rightarrow Y, A_1 \wedge A_2}\, R\wedge$$

$$\frac{A_1, X \Rightarrow Y \qquad A_2, X \Rightarrow Y}{A_1 \vee A_2, X \Rightarrow Y}\, L\vee \qquad \frac{X \Rightarrow Y, A_i}{X \Rightarrow Y, A_1 \vee A_2}\, R\vee$$

$$\frac{X \Rightarrow Y, A \qquad B, X \Rightarrow Y}{A \supset B, X \Rightarrow Y}\, L\supset \qquad \frac{A, X \Rightarrow Y, B}{X \Rightarrow Y, A \supset B}\, R\supset$$

Modal rule:

$$\frac{\Box X, X, \Box A \Rightarrow A}{\Box X \Rightarrow \Box A}\, GLR$$

Structural rules:

$$\frac{X \Rightarrow Y}{A, X \Rightarrow Y}\, LW \qquad \frac{X \Rightarrow Y}{X \Rightarrow Y, A}\, RW$$

$$\frac{A, A, X \Rightarrow Y}{A, X \Rightarrow Y}\, LC \qquad \frac{X \Rightarrow Y, A, A}{X \Rightarrow Y, A}\, RC$$

Cut-rule:

$$\frac{X \Rightarrow Y, A \qquad A, U \Rightarrow W}{X, U \Rightarrow Y, W}\, cut$$

Table 2.1: The sequent calculus GLS. Note: $i \in \{1, 2\}$ in the rules $L\wedge$ and $R\vee$.

$1 \leq i \leq n$. The negation of $B \in X$ is denoted by $B \notin X$. The notation $(A)^m$ or A^m denotes m comma-separated occurrence of A.

A *sequent* is a tuple (X, Y) of multisets X and Y of formulae and is written $X \Rightarrow Y$. Sequents are denoted using $\mathcal{S}, \mathcal{S}'$. The multiset X (resp. Y) is called the antecedent (succedent). The multiset sequent calculus we use here is called GLS (Table 2.1). For the logical and structural rules in GLS, the multisets X and Y are called the *context*. In the conclusion of each of these rules, the formula occurrence not in the context is called the *principal formula*. This follows standard practice (see [70]). For the GLR rule, each formula in $\Box X, X, \Box A, A$ is called a principal formula. The $\Box A$ in the succedent of the conclusion of the GLR rule is called the *diagonal formula* (and is of course boxed). In the cut-rule, the formula A is called the *cut-formula*. A rule with one premise (resp. two premises) is called a unary (binary) rule.

A binary rule where the context in both premises is required to be identical is called an *additive* binary rule (eg: $L\vee, R\wedge$). A binary rule where the context in each premise need not be identical is called a *multiplicative* binary rule (eg: *cut*). The term context-sharing (context-independent) is also used to refer to an additive (multiplicative) rule (see [70]).

Note, we have deleted the initial sequent $\bot \Rightarrow \bot$ and the $\bot$-rule that appears in GLS_V. As observed in [69], it is not necessary to include the symbol $\bot$ although its presence can be convenient from a semantic viewpoint. Since our interest here is proof-theoretic we shall not require it. We have also replaced the multiplicative $L\supset$ in GLS_V with an additive version. As all the other logical rules in GLS are additive, it seems appropriate to use an additive $L\supset$. In every other respect, the inference rules in GLS have the identical form to the rules in the calculus GLS_V. We observe that the definitions and proofs in this paper apply, with slight amendment, to a sequent calculus built from multisets that is obtained directly from GLS_V.

A *derivation* (in GLS) is defined recursively with reference to Table 2.1 as:

(i) an initial sequent $A \Rightarrow A$ for any formula A is a derivation, and

(ii) an application of a logical, modal, structural or cut-rule to derivations concluding its premise(s) is a derivation.

This is the standard definition. Viewing a derivation as a tree, we call the root of the tree the *end-sequent* of the derivation. If there is a derivation with end-sequent $X \Rightarrow Y$ we say that $X \Rightarrow Y$ is *derivable* in GLS. Let $\bigwedge X$ ($\bigvee Y$) denote the conjunction (disjunction) of all formula occurrences in X (Y). Interpreting the sequent $X \Rightarrow Y$ as the formula $\bigwedge X \supset \bigvee Y$, from [64] we see that derivability in GLS is sound and complete wrt GL.

We write $\{\pi\}_1^r/^\rho X \Rightarrow Y$ to denote the following derivation, where ρ is a rule with r premises:

$$\frac{\pi_1 \qquad \ldots \qquad \pi_r}{X \Rightarrow Y}\,\rho$$

Intuitively, the above reads "from π_1 to π_r obtain $X \Rightarrow Y$ via rule ρ". We refer to $\pi_1, \ldots, \pi_r$ as the *premise derivations* of ρ. If ρ is unary (binary) then $r = 1$ ($r = 2$). Rather than $\{\pi\}_1^1$ and $\{\pi\}_1^2$, we write, respectively, "π_1" and "$\pi_1 \quad \pi_2$".

Let ρ be some rule-occurrence in a derivation τ. Then $\rho(A)$ indicates that the principal formula is A, while $\rho^*(X)$ denotes some number (≥ 0) of applications of ρ that make each formula occurrence (including multiple formula occurrences) in the multiset X a principal formula. To identify a rule-occurrence in the text we occasionally use subscripts, eg: GLR_1, cut_0.

A derivation τ is *cut-free* if τ contains no instances of the cut-rule. A cut-instance is said to be *topmost* if its premise derivations are cut-free.

Definition 2.1 (n-ary GLR rule) *Given a derivation τ, an instance ρ of the GLR rule appearing in τ is n-ary if there are exactly $n-1$ GLR rule instances on the path between ρ and the end-sequent of τ.*

Let $GLR(n,\tau)$ denote the number of n-ary GLR rules in τ. Next we define the height, cut-height, and degree of a formula in the standard manner.

Definition 2.2 (height, cut-height, degree) *The height $s(\tau)$ of a derivation τ is the greatest number of successive applications of rules in it plus one. The cut-height h of an instance of the cut-rule with premise derivations τ_1 and τ_2 is $s(\tau_1)+s(\tau_2)$. The degree $deg(A)$ of a formula A is defined as the number of symbol occurrences in A from $\{\Box, \neg, \wedge, \vee, \supset\}$ plus one.*

2.3 Technical devices and basic results

2.3.1 Generalising the notion of derivation

To formalise the notion of width we need a more general structure than a derivation. The structure we have in mind can be obtained from a derivation τ by deleting a proper subderivation τ' in τ. We call this structure a *stub-derivation.* To emphasise the point of deletion we use the annotation stub.

Formally a stub-derivation (in GLS) is defined recursively with reference to Table 2.1 as follows:

(i) an initial sequent $A \Rightarrow A$ for any formula A is a stub-derivation, and

(ii) for any sequent $\mathcal{S}$ and stub-derivation π, each of

$$\text{(a) } \frac{\mathsf{stub}}{\mathcal{S}} \qquad \text{(b) } \frac{\mathsf{stub} \quad \pi}{\mathcal{S}} \qquad \text{(c) } \frac{\pi \quad \mathsf{stub}}{\mathcal{S}}$$

is a stub-derivation, and

(iii) an application of a logical, modal, structural or cut-rule to stub-derivations concluding its premise(s) is a stub-derivation.

Viewing a stub-derivation τ as a tree, we call the root of the tree the *end-sequent* of the stub-derivation (denoted $ES(\tau)$). The leaves of the tree are called the *top-sequents*. Clearly a derivation is a stub-derivation in which every top-sequent is an initial sequent. Thus a stub-derivation generalises the notion of a derivation.

We use the term 'stub-instance' to refer to an occurrence of either $\mathsf{stub}/\mathcal{S}$ or $\mathsf{stub} \quad \pi/\mathcal{S}$ or $\pi \quad \mathsf{stub}/\mathcal{S}$. An *sstub-derivation* (read: single stub-derivation) is a stub-derivation containing exactly one stub-instance. We write $d[\mathsf{stub}]$ instead of d, to remind the reader that the structure contains exactly one stub-instance.

Let d' be a derivation with end-sequent $\mathcal{S}'$, let $d[\mathsf{stub}]$ be an sstub-derivation with an occurrence of one of the following:

$$\mathsf{stub}/\mathcal{S} \qquad\qquad \mathsf{stub} \quad \pi/\mathcal{S} \qquad\qquad \pi \quad \mathsf{stub}/\mathcal{S}$$

and suppose that

$$\mathcal{S}'/^{\rho}\mathcal{S} \qquad\qquad \mathcal{S}' \quad ES(\pi)/\mathcal{S} \qquad\qquad ES(\pi) \quad \mathcal{S}'/\mathcal{S}$$

respectively is a legal instance of some logical or structural rule ρ. We say that $d[\mathsf{stub}]$ and d' are *compatible* and write $d[\mathsf{stub}] \hookleftarrow d'$ to denote, respectively

$$\frac{d'}{\mathcal{S}}\,\rho \qquad\qquad \frac{d' \qquad \pi}{\mathcal{S}}\,\rho \qquad\qquad \frac{\pi \qquad d'}{\mathcal{S}}\,\rho$$

obtained by "attaching" the tree d' to the tree $d[\mathsf{stub}]$ at the node stub under rule ρ. We refer to ρ as a *binding rule* for $d[\mathsf{stub}]$ and d'.

By permitting formula occurrences in a (stub-)derivation to contain $*$ or $\circ$ decorations, we define an *annotated (stub-)derivation*. The forgetful map $\lfloor\cdot\rfloor$ maps an annotated stub-derivation to the stub-derivation obtained by erasing all $*$ and $\circ$ decorations. Clearly $\lfloor\cdot\rfloor$ maps an annotated derivation to a derivation. A *transformed (stub-)derivation* τ' is a (stub-)derivation that is obtained from some existing (stub-)derivation τ by syntactic transformation. We write $A^{\circ n}$ or A^{*n} to mean n occurrences of the formula A° or A^{*} respectively.

Formally a stub-derivation and an annotated stub-derivation are different structures. Because these structures are very similar, for economy of space we will introduce definitions and prove results for stub-derivations alone and note, whenever applicable, that the definitions and results can be extended to annotated stub-derivations.

Example 2.3 *Let us denote the sstub-derivation at below left by* $d[\mathsf{stub}]$ *and the derivation at below right by* d'*:*

$$\dfrac{\dfrac{\mathsf{stub}}{B \Rightarrow A \supset B} \qquad A \supset B \Rightarrow A \supset B}{B \vee (A \supset B) \Rightarrow A \supset B}\,L\vee \qquad\qquad \dfrac{B \Rightarrow B}{A, B \Rightarrow B}\,LW$$

Observe that $d[\mathsf{stub}]$ *has a stub-instance of type* $\mathsf{stub}/\mathcal{S}$*, with* $\mathcal{S} \equiv B \Rightarrow A \supset B$*, and* d' *has endsequent* $\mathcal{S}' \equiv A, B \Rightarrow B$*. Because* $\mathcal{S}'/\mathcal{S}$ *is an instance of* $R{\supset}$*, the structures* $d[\mathsf{stub}]$ *and* d' *are compatible. The derivation* $d[\mathsf{stub}] \hookleftarrow d'$ *is:*

$$\dfrac{\dfrac{\dfrac{B \Rightarrow B}{A, B \Rightarrow B}\,LW}{B \Rightarrow A \supset B}\,R{\supset} \qquad A \supset B \Rightarrow A \supset B}{B \vee (A \supset B) \Rightarrow A \supset B}\,L\vee$$

and the binding rule is $R{\supset}$.

Example 2.4 *Let us denote the sstub-derivation at below left by* $d[\mathsf{stub}]$ *and the derivation at below right by* d'*:*

$$\dfrac{\mathsf{stub} \qquad A \supset B \Rightarrow A \supset B}{B \vee (A \supset B) \Rightarrow A \supset B} \qquad\qquad \dfrac{\dfrac{B \Rightarrow B}{A, B \Rightarrow B}\,LW}{B \Rightarrow A \supset B}\,R{\supset}$$

Observe that $d[\mathsf{stub}]$ *has a stub-instance of type* $\mathsf{stub}\quad \tau/\mathcal{S}$*, with* $\mathcal{S} \equiv B \vee (A \supset B) \Rightarrow A \supset B$*, and* d' *has endsequent* $\mathcal{S}' \equiv B \Rightarrow A \supset B$*.*

Since $\mathcal{S}' \quad A \supset B \Rightarrow A \supset B/\mathcal{S}$ *is an instance of* $L\vee$*, the structures* $d[\mathsf{stub}]$ *and* d' *are compatible. The derivation* $d[\mathsf{stub}] \hookleftarrow d'$ *is identical to that obtained in Example 2.3, although here the binding rule is* $L\vee$*.*

Definition 2.5 *Let* τ *be a stub-derivation and* G *a formula multiset. The antecedent operator* $\oplus$: *stub-derivation* $\times$ *formula multiset* $\mapsto$ *stub-derivation is defined as follows:*

Case $G = \langle\rangle$*: let* $\tau \oplus G = \tau$

Case $G \neq \langle\rangle$*: define* $\tau \oplus G$ *recursively on* τ *as follows*

1. *initial sequent:* $(A \Rightarrow A) \oplus G = (A \Rightarrow A/^{LW^*(G)}A, G \Rightarrow A)$

2. *stub-instance:*

   (a) $(\mathsf{stub}/X \Rightarrow Y) \oplus G = (\mathsf{stub}/X, G \Rightarrow Y)$

   (b) $(\mathsf{stub}\quad \pi/X \Rightarrow Y) \oplus G = (\mathsf{stub}\quad \pi \oplus G/X, G \Rightarrow Y)$

   (c) $(\pi\quad \mathsf{stub}/X \Rightarrow Y) \oplus G = (\pi \oplus G\quad \mathsf{stub}/X, G \Rightarrow Y)$

3. *unary non-GLR rule:* $(\pi/X \Rightarrow Y) \oplus G = (\pi \oplus G/X, G \Rightarrow Y)$

4. *GLR rule:* $(\pi/^{GLR}X \Rightarrow Y) \oplus G = (\pi/^{GLR}X \Rightarrow Y)/^{LW^*(G)}X, G \Rightarrow Y$

5. *binary additive rule:* $(\pi_1\ \pi_2/X \Rightarrow Y) \oplus G = (\pi_1 \oplus G\ \pi_2 \oplus G/X, G \Rightarrow Y)$

6. *cut-rule:* $(\pi_1 \quad \pi_2/^{cut} X \Rightarrow Y) \oplus G = (\pi_1 \oplus G \quad \pi_2/^{cut} X, G \Rightarrow Y)$.

That $\oplus$ maps into the set of stub-derivations follows by inspection of the definition. Notice that the recursion terminates at an initial sequent, stub-instance or a GLR rule. The operator $\oplus$ will bind stronger that $\hookleftarrow$.

Lemma 2.6 *If d is a stub-derivation and G is a formula multiset, then $d \oplus G$ is a stub-derivation. Furthermore, if d is in fact an sstub-derivation $d[\mathsf{stub}]$, then $d[\mathsf{stub}] \oplus G$ is an sstub-derivation.*

Proof. The result follows immediately from Definition 2.5. Q.E.D.

Example 2.7 *Refer to the sstub-derivation $d[\mathsf{stub}]$ in Example 2.3. If G is a non-empty formula multiset, then $d[\mathsf{stub}] \oplus G$ is the stub-derivation:*

$$\dfrac{\dfrac{\mathsf{stub}}{B, G \Rightarrow A \supset B} \qquad \dfrac{A \supset B \Rightarrow A \supset B}{A \supset B, G \Rightarrow A \supset B}\, LW^*(G)}{B \vee (A \supset B), G \Rightarrow A \supset B}\, L\vee$$

By observation, we can confirm that $d[\mathsf{stub}] \oplus G$ is a sstub-derivation as predicted by Lemma 2.6. Notice that $d[\mathsf{stub}] \oplus G$ and d' (from Example 2.3) are not compatible, because there is no logical or structural inference rule that can take us from the premise sequent $A, B \Rightarrow B$ to the conclusion sequent $B, G \Rightarrow A \supset B$.

Definition 2.5 can be extended in the obvious way to apply to annotated stub-derivations. It is easy to verify that Lemma 2.6 holds under the uniform substitution of the term "annotated (s)stub-derivation" for "(s)stub-derivation" in the statement of the lemma.

Cut-elimination often involves tracing the "parametric ancestors" of the cut-formula. The following definition uses the symbols $\circ$ and $*$ as annotations to help trace the parametric ancestors.

Definition 2.8 *($f_C[\cdot]$: annotated derivation wrt C).*

Let τ be a cut-free derivation with endsequent $X \Rightarrow Y$, and C a formula.

1. *if C is not boxed then let $f_C[\tau] = \tau$.*

2. *if C is boxed ($C \equiv \Box B$) and $\Box B \notin X$ then let $f_{\Box B}[\tau] = \tau$.*

3. *if C is boxed ($C \equiv \Box B$) and $\Box B \in X$. Then τ must be a derivation of the form $\Box B \Rightarrow \Box B$ or $\{\pi\}_1^r/^\rho X', \Box B \Rightarrow Y$.*

   Set $f_{\Box B}[\tau]$ as $\Phi_{\Box B}[(\Box B)^* \Rightarrow \Box B)]$ or $\Phi_{\Box B}[\{\pi\}_1^r/X', (\Box B)^* \Rightarrow Y]$ respectively, where $\Phi_{\Box B}$ (see Table 2.2 page 53) is defined on the class of cut-free annotated derivations.

Observe that the annotation operator $f_C[\cdot]$ is a total function mapping derivations to annotated derivations.

Remark 2.9 *Let τ be a derivation with endsequent $X \Rightarrow Y$. If $\Box B \in X$ then the formula occurrences $(\Box B)^\circ$ and $(\Box B)^*$ in $f_{\Box B}[\tau]$ are each called a parametric ancestor of the formula occurrence $\Box B \in X$ in the endsequent. Intuitively, the annotation $\circ$ denotes the final parametric ancestor when tracing ancestors upwards. That is, the $\Box B$ is introduced at that point.*

Definition 2.10 *Define $\partial^\circ(B, \tau)$ for a formula B and an annotated derivation τ, as the number of occurrences of the GLR rule in τ whose conclusion contains an occurrence of the annotated formula B° in the antecedent.*

Lemma 2.11 *Let $d[\mathsf{stub}]$ be an annotated sstub-derivation and G a formula multiset. Then*

(a) $\partial^\circ(B, d[\mathsf{stub}] \oplus G) = \partial^\circ(B, d[\mathsf{stub}])$

(b) Let d' be a derivation such that $d[\mathsf{stub}]$ and d' are compatible. Then

$$\partial^\circ(B, d[\mathsf{stub}] \hookleftarrow d') = \partial^\circ(B, d[\mathsf{stub}]) + \partial^\circ(B, d')$$

Proof.

(a) Because $\partial^\circ(B, \cdot)$ counts the number of instances of the GLR rule with conclusion sequents containing the formula occurrence B°, the result is an immediate consequence of the fact that $\oplus$ does not introduce formulae into the conclusion sequent of an instance of the GLR rule (see Definition 2.5(4)).

(b) By the definition of compatibility, the binding rule for $d[\mathsf{stub}]$ and d' cannot be GLR. Thus if an instance ρ of the GLR rule appears in $d[\mathsf{stub}] \hookleftarrow d'$, then ρ must appear in one of $d[\mathsf{stub}]$ or d'. Also, if an instance ρ of the GLR rule appears in either $d[\mathsf{stub}]$ or d', then ρ must appear in $d[\mathsf{stub}] \hookleftarrow d'$. The result follows immediately.

Q.E.D.

Remark 2.12 *Lemma 2.11(a) holds even if G contains decorated formulae.*

Definition 2.13 (width) *Let cut_0 be a topmost cut as shown below:*

$$\dfrac{\dfrac{\{\pi\}_1^r}{X \Rightarrow Y, B}\,\rho \qquad \dfrac{\{\sigma\}_1^s}{B, U \Rightarrow W}}{X, U \Rightarrow Y, W}\,cut_0$$

Then, the width of cut_0 is defined as:

$$width(cut_0) = \begin{cases} \partial^\circ(B, f_B[\pi_1]) & \textit{if } \rho = GLR \textit{ (so } \{\pi\}_1^r = \pi_1\textit{)} \\ GLR(2, \{\pi\}_1^r / X \Rightarrow Y, B) & \textit{otherwise} \end{cases}$$

Remark 2.14 *(i) The width has been defined only for a topmost cut as this context is sufficient for our purposes.*

(ii) $width(cut_0)$ is independent of the right premise derivation of cut_0.

Example 2.15 *Let us calculate $width(cut_0)$ in the following derivation:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\{\pi\}_1^r}{\Box C, C, \Box\Box B, \Box B, \Box B \Rightarrow B}}{\Box C, \Box\Box B \Rightarrow \Box B}\,GLR \qquad \dfrac{\dfrac{\{\sigma\}_1^s}{\Box D \Rightarrow \Box B}}{\Box D, \Box\Box B \Rightarrow \Box B}\,LW}{\Box C \vee \Box D, \Box\Box B \Rightarrow \Box B}\,L\vee}{\Box(\Box C \vee \Box D), \Box C \vee \Box D, \Box\Box B \Rightarrow \Box B}\,LW}{\Box(\Box C \vee \Box D) \Rightarrow \Box\Box B}\,GLR \qquad \Box\Box B, U \Rightarrow W}{\Box(\Box C \vee \Box D), U \Rightarrow W}\,cut_0$$

Writing the left premise derivation of cut_0 as $\mu / \Box(\Box C \vee \Box D) \Rightarrow \Box\Box B$, we get $width(cut_0) = \partial^\circ(\Box\Box B, f_{\Box\Box B}[\mu])$ where $f_{\Box\Box B}[\mu]$ is

$$\dfrac{\dfrac{\dfrac{\dfrac{\{\pi\}_1^r}{\Box C, C, \Box\Box B, \Box B, \Box B \Rightarrow B}}{\Box C, (\Box\Box B)^\circ \Rightarrow \Box B}\,GLR \qquad \dfrac{\dfrac{\{\sigma\}_1^s}{\Box D \Rightarrow \Box B}}{\Box D, (\Box\Box B)^\circ \Rightarrow \Box B}\,LW}{\Box C \vee \Box D, (\Box\Box B)^* \Rightarrow \Box B}\,L\vee}{\Box(\Box C \vee \Box D), \Box C \vee \Box D, (\Box\Box B)^* \Rightarrow \Box B}\,LW$$

Because $f_{\Box\Box B}[\mu]$ contains only one GLR rule whose conclusion contains the formula occurrence $(\Box\Box B)^\circ$ in its antecedent, we have $width(cut_0) = 1$.

Remark 2.16 *Let μ be the left premise derivation of cut_0 from Definition 2.13. Valentini [71, pg 473] defines the width as the cardinality of $GLR^{(2)}$, where $GLR^{(2)}$ in our notation is the set of all instances ρ of GLR such that:*

(a) ρ is a 2-ary GLR rule in μ, and

(b) B is the diagonal formula of every 1-ary GLR rule in μ below ρ, and

(c) B is not introduced by weakening below ρ.

Applying Valentini's original definition to the following derivation in GLS we compute the width of cut_0 as 0 (due to condition (c)):

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\{\pi\}_1^r}{\Box\Box X, \Box X, \Box X, X, \Box\Box C, \Box C, \Box C \Rightarrow C}}{\Box\Box X, \Box X, \Box\Box C \Rightarrow \Box C}\,GLR}{\Box\Box X, \Box X, \Box\Box C, \Box\Box C \Rightarrow \Box C}\,LW(\Box\Box C)}{\Box\Box X, \Box X, \Box\Box C \Rightarrow \Box C}\,LC(\Box\Box C)}{\Box\Box X \Rightarrow \Box\Box C}\,GLR \qquad \Box\Box C, U \Rightarrow W}{\Box\Box X, U \Rightarrow W}\,cut_0
$$

Using the definition in this paper we have $width(cut_0) = 1$. Our definition considers the interplay of the weakening and contraction rules, and is required to obtain the cut-elimination result for GLS. In GLS_V however, there are no contraction rules so Valentini's original definition suffices.

Thus Moen is certainly justified in asking whether Valentini's arguments can be lifted to multiset-based sequents. However, we will see that Moen's claims about failure of cut-elimination in the new setting are incorrect.

2.3.2 Invertibility of the logical rules for *GLS*

An inference rule in the sequent calculus is called *invertible* if the premise sequents are derivable whenever the conclusion sequent is derivable. We say that a transformation is *height-preserving* if the height of the transformed derivation is $\leq$ the height of the original derivation. In the following, we write $\overline{A_1, \ldots, A_n}$ to mean an occurrence of a formula from $A_1, \ldots, A_n$, when we do not wish to specify which formula it is. For example, in the sequent $\overline{A, B}, X \Rightarrow Y$, the formula occurrence $\overline{A, B}$ could be either A or B. If this occurrence appears as an initial sequent $\overline{A, B} \Rightarrow B$, for example, then it is possible to deduce that the occurrence $\overline{A, B}$ refers to the formula occurrence B.

The following result is a *generalised* version of the invertibility result for the logical rules in *GLS*, in the sense that we select some number of occurrences of a formula whose main connective is non-modal, and show how to 'decompose' those occurrences into the constituent subformulae. In the statement of Lemma 2.17, if we set $m = 0$ we obtain an invertibility result in the 'flavour' of von Plato's [77] proof for the calculus $G0c$ for classical logic. Our statement differs slightly because we use the 'projective' form of the rules for $L\wedge$ and $R\vee$ so there is a *single* principal formula in the premise sequent of these rules — rather than corresponding non-projective form found in von Plato, shown below:

$$\frac{A, B, X \Rightarrow Y}{A \wedge B, X \Rightarrow Y} \qquad\qquad \frac{X \Rightarrow Y, A, B}{X \Rightarrow Y, A \vee B}$$

Lemma 2.17 (general invertibility for logical rules) *The statements that follow concern derivations in GLS. For all $m \geq 0$,*

(i) If $(\neg A)^{m+1}, X \Rightarrow Y$ is derivable, then $X \Rightarrow Y, A^{m+1}$ is derivable.

(ii) If $X \Rightarrow Y, (\neg A)^{m+1}$ is derivable, then $A^{m+1}, X \Rightarrow Y$ is derivable.

(iii) If $(A \wedge B)^{m+1}, X \Rightarrow Y$ is derivable, then $A^l, B^{m-l}, \overline{A, B, A \wedge B}, X \Rightarrow Y$ is derivable for some l, $0 \leq l \leq m$. Moreover, the transformations are height-preserving.

(iv) If $X \Rightarrow Y, (A \wedge B)^{m+1}$ is derivable, then $X \Rightarrow Y, A^{m+1}$ and $X \Rightarrow Y, B^{m+1}$ are derivable. Moreover, the transformations are height-preserving.

(v) If $(A \vee B)^{m+1}, X \Rightarrow Y$ is derivable, then $A^{m+1}, X \Rightarrow Y$ and $B^{m+1}, X \Rightarrow Y$ are derivable. Moreover, the transformations are height-preserving.

(vi) If $X \Rightarrow Y, (A \vee B)^{m+1}$ is derivable, then $X \Rightarrow Y, A^l, B^{m-l}, \overline{A, B, A \vee B}$ is derivable for some l, $0 \leq l \leq m$. Moreover, the transformations are height-preserving.

(vii) If $(A \supset B)^{m+1}, X \Rightarrow Y$ is derivable, then $X \Rightarrow Y, A^{m+1}$ and $B^{m+1}, X \Rightarrow Y$ are derivable.

(viii) If $X \Rightarrow Y, (A \supset B)^{m+1}$ is derivable, then $A^{m+1}, X \Rightarrow Y, B^{m+1}$ is derivable.

Proof. Let us illustrate the proof for (iii) and (vii). The other cases are similar.

Proof of (iii). Suppose that τ is a derivation of $(A \wedge B)^{m+1}, X \Rightarrow Y$. Proof by induction on the height of τ. We need to obtain a derivation of

$$A^l, B^{m-l}, \overline{A, B, A \wedge B}, X \Rightarrow Y$$

for some l such that $0 \leq l \leq m$.

First suppose that τ is the initial sequent $A \wedge B \Rightarrow A \wedge B$. Then there is nothing to do since this is already in the form $\overline{A, B, A \wedge B} \Rightarrow A \wedge B$, where $\overline{A, B, A \wedge B}$ is an occurrence of $A \wedge B$.

Next, consider when $A \wedge B$ is not principal in the lowest rule ρ in τ (we show when ρ is unary, the binary case is similar). Then τ is of the form:

$$\dfrac{\begin{array}{c}\vdots\\ (A \wedge B)^{m+1}, X' \Rightarrow Y'\end{array}}{(A \wedge B)^{m+1}, X \Rightarrow Y}\ \rho$$

Notice that it must be the case that $\rho \neq GLR$, since $A \wedge B$ cannot occur in the conclusion sequent of a GLR rule as every formula in that sequent is necessarily boxed. Also, we do not exclude the possibility that the sequent $(A \wedge B)^{m+1}, X' \Rightarrow Y'$ is an initial sequent. Denote the height of this derivation by $h+1$, so the height of the premise derivation of ρ is h. By the induction hypothesis we obtain a derivation of $A^l, B^{m-l}, \overline{A, B, A \wedge B}, X' \Rightarrow Y'$ of height h, for some l, $0 \leq l \leq m$. Applying the rule ρ to the this sequent we obtain a derivation of $A^l, B^{m-l}, \overline{A, B, A \wedge B}, X \Rightarrow Y$ of height $h+1$ as required.

Finally, suppose that $A \wedge B$ is principal in the lowest rule ρ in τ. If $\rho = L\wedge(A)$ (the case when $\rho = L\wedge(B)$ is similar) then τ has the form

$$\dfrac{\begin{array}{c}\vdots\\ A, (A \wedge B)^{m}, X \Rightarrow Y\end{array}}{(A \wedge B)^{m+1}, X \Rightarrow Y}\ L\wedge$$

Denote the height of this derivation by $h+1$. If $m = 0$, then the sequent $A, (A \wedge B)^m, X \Rightarrow Y$ is simply $A, X \Rightarrow Y$ and this is the required derivation so there is nothing more to do. Else, if $m > 0$, by the induction hypothesis we obtain a derivation of $A, A^l, B^{m-l-1}, \overline{A, B, A \wedge B}, X \Rightarrow Y$ of height h, for some l, $0 \leq l \leq m-1$. This is a derivation of the required form of height h, so there is nothing more to do. The remaining possibility to consider is when $\rho = LC(A \wedge B)$. Then τ has the following form

$$\dfrac{\dfrac{\{\pi\}_1^r}{(A \wedge B)^{m+2}, X \Rightarrow Y}}{(A \wedge B)^{m+1}, X \Rightarrow Y}\ LC(A \wedge B)$$

Denote the height of this derivation by $h+1$. By the induction hypothesis we obtain a derivation of $A^l, B^{m+1-l}, \overline{A, B, A \wedge B}, X \Rightarrow Y$ of height h, where $0 \leq l \leq m+1$. If $l = 0$ then we can apply the rule $LC(B)$ to obtain the sequent $B^m, \overline{A, B, A \wedge B}, X \Rightarrow Y$. Otherwise apply the rule $LC(A)$ to obtain the sequent $A^{l-1}, B^{m+1-l}, \overline{A, B, A \wedge B}, X \Rightarrow Y$. In each case, the derivation is of the required form and has height $h+1$ so we are done.

Proof of (vii). Suppose that τ is a derivation of $(A \supset B)^{m+1}, X \Rightarrow Y$. Proof by induction on the height of τ. We will show how to obtain a derivation of $B^{m+1}, X \Rightarrow Y$. The transformations to $X \Rightarrow Y, A^{m+1}$ are analogous.

If τ is the initial sequent $A \supset B \Rightarrow A \supset B$ then the following derivation suffices:

$$\dfrac{\dfrac{B \Rightarrow B}{A, B \Rightarrow B}\, LW(A)}{B \Rightarrow A \supset B}\, R{\supset}$$

Incidentally, notice that this transformation is not height-preserving.

Next, consider when $A \supset B$ is not principal in the last rule ρ in τ (we show when ρ is unary, the binary case is similar). Then τ is of the form:

$$\dfrac{\dfrac{\{\pi\}_1^r}{(A \supset B)^{m+1}, X' \Rightarrow Y'}}{(A \supset B)^{m+1}, X \Rightarrow Y}\, \rho$$

Notice that it must be the case that $\rho \neq GLR$. By the induction hypothesis we obtain a derivation of $B^{m+1}, X' \Rightarrow Y'$. Applying the rule ρ to the sequent $B^{m+1}, X' \Rightarrow Y'$ we obtain a derivation of $B^{m+1}, X \Rightarrow Y$ as required.

Finally, suppose that $A \supset B$ is principal in the final rule ρ in τ. If is the case that $\rho = L{\supset}(A \supset B)$ then τ has the form

$$\dfrac{\dfrac{\{\pi\}_1^r}{(A \wedge B)^m, X \Rightarrow Y, A} \qquad \dfrac{\{\sigma\}_1^s}{B, (A \wedge B)^m, X \Rightarrow Y}}{(A \supset B)^{m+1}, X \Rightarrow Y}\, L{\supset}$$

If $m = 0$, then the sequent $B, (A \supset B)^m, X \Rightarrow Y$ is simply $B, X \Rightarrow Y$ so there is nothing more to do. Else, if $m > 0$, by the induction hypothesis applied to the right premise of $L{\supset}$ we obtain a derivation of $B^{m+1}, X \Rightarrow Y$. Once again, this is the required derivation so there is nothing more to do. The remaining possibility to consider is when $\rho = LC(A \supset B)$. Then τ has the form

$$\dfrac{\dfrac{\{\pi\}_1^r}{(A \supset B)^{m+2}, X \Rightarrow Y}}{(A \supset B)^{m+1}, X \Rightarrow Y}\, LC(A \supset B)$$

By the induction hypothesis we obtain a derivation of $B^{m+2}, X \Rightarrow Y$. Now apply the rule $LC(B)$ to obtain a derivation of $B^{m+1}, X \Rightarrow Y$ as required. Q.E.D.

Note that the above results are not height-preserving in general. However, they are height-preserving for (iii)—(vi). This fact will crucial for obtaining the cut-elimination result. If we use the non-projective form of the rules for $L\wedge$ and $R\vee$ then the corresponding transformations will no longer be height-preserving.

Embedded inside the induction in the above proof is the notion of tracing the formula $A \bullet B$ (for $\bullet \in \{\wedge, \vee, \supset\}$) or $\neg A$ upwards from the end-sequent. The presence of the GLR rule does not cause a problem when tracing this formula as it is impossible to encounter, along this path, a GLR rule instance *before* the introduction rule for the principal connective in $A \bullet B$ or $\neg A$. This is because every formula in the conclusion sequent of a GLR rule is necessarily boxed.

Note that proving the result for $m \geq 0$ rather than just $m = 0$ actually simplifies matters. For example, in the proof of item (vii) when the last rule ρ in the derivation is $LC(A \supset B)$, we applied the induction hypothesis directly. In contrast, von Plato has to explicitly trace the formula $A \supset B$ upwards from the endsequent in order to obtain the result.

2.4 Cut-elimination for GLS

The main task for cut-elimination is to show that if $\Box X, X, \Box B \Rightarrow B$ is cut-free derivable in GL, then there is a cut-free derivation of $\Box X, X \Rightarrow B$. This is the content of Lemma 2.20. The cut-elimination theorem follows immediately from this lemma.

Before proceeding with the technical details let us provide an outline of the proof of Lemma 2.20. Let τ be a cut-free derivation of $\Box X, X, \Box B \Rightarrow B$. Then we define the width $n(\tau)$ as the number of occurrences of the following schema, where no GLR rule occurrences appear between GLR_1 and the endseqent.

$$\dfrac{\dfrac{\dfrac{\Box G, G, (\Box B)^n, B^n, \Box C \Rightarrow C}{\Box G, (\Box B)^n \Rightarrow \Box C}\, GLR_1}{\vdots}}{\Box X, X, \Box B \Rightarrow B}$$

If $n(\tau) = 0$ this indicates that the $\Box B$ formula occurrence in the endsequent of τ has either been introduced by $LW(\Box B)$ or can be traced to the initial sequent $\Box B \Rightarrow \Box B$. In the former case, the weakening rule is deleted. In the latter case, the required result can be obtained by substituting the derivation $\tau/^{GLR}\Box X \Rightarrow \Box B$ in place of the initial sequent.

If $n(\tau) = k + 1$, each occurrence of the above schema in τ is transformed so that the GLR_1 rule occurrence is deleted. Observe that the conclusion of the GLR_1 rule can *almost* be obtained by the derivation

$$\dfrac{\Box C \Rightarrow \Box C}{\Box G, \Box B, \Box C \Rightarrow \Box C}\ LW(\Box G, \Box B)$$

There is now an unwanted $\Box C$ formula occurrence in the antecedent that has to be removed by the appropriate transformations. After $k + 1$ such transformations we obtain the base case.

We are now ready to formalise this argument. We begin with the following decomposition lemma.

Lemma 2.18 *Let τ be a cut-free derivation of the form $\{\pi\}_1^r/^{\rho} X, \Box B \Rightarrow Y$ and suppose that $\partial^\circ(\Box B, f_{\Box B}[\tau]) > 0$. If*

(i) $\rho = GLR$ then $f_{\Box B}[\tau] = \{\pi\}_1^r/^{GLR} X, (\Box B)^\circ \Rightarrow Y$.

(ii) $\rho \neq GLR$ then we can write the annotated derivation $f_{\Box B}[\tau]$ in the form $d[\mathsf{stub}] \hookleftarrow d'$ such that

$$\partial^\circ(\Box B, d[\mathsf{stub}] \hookleftarrow d') = \partial^\circ(\Box B, d[\mathsf{stub}]) + \partial^\circ(\Box B, d').$$

Furthermore, denote the endsequent of d' as $U \Rightarrow W$. Then for any multiset M, and any derivation d'' with endsequent $U, M \Rightarrow W$, we have that $d[\mathsf{stub}] \oplus M$ and d'' are compatible.

Proof. If $\rho = GLR$ then it follows immediately from Definition 2.8 that $f_{\Box B}[\tau] = \{\pi\}_1^r/^{GLR} X, (\Box B)^\circ \Rightarrow Y$.

Now suppose that $\rho \neq GLR$.

Because $\partial^\circ(\Box B, f_{\Box B}[\tau]) > 0$, $f_{\Box B}[\tau]$ must have the following form, where

(1) $n \geq 1$, and

(2) $\Box G$ contains no annotated formulae, and

(3) GLR_1 is a 1-ary GLR rule in $f_{\Box B}[\tau]$, and

(4) η may contain branches:

$$\begin{array}{c} \{\pi'\}_1^s \\ \hline \Box G, G, (\Box B)^n, B^n, \Box A \Rightarrow A \\ \hline \Box G, (\Box B)^{\circ n} \Rightarrow \Box A \\ \eta \\ X, (\Box B)^* \Rightarrow Y \end{array}\ GLR_1$$

We remind the reader that since η is allowed to contain branches, the last rule in η may be either unary or binary. We can identify the annotated derivation $f_{\Box B}[\tau]$ with $d[\mathsf{stub}] \hookleftarrow d'$ where $d[\mathsf{stub}]$ (below left) is an annotated sstub-derivation and d' (below right) is an annotated derivation:

$$\begin{array}{c} \mathsf{stub} \\ \eta \\ X, (\Box B)^* \Rightarrow Y \end{array} \qquad\qquad \dfrac{\dfrac{\{\pi'\}_1^s}{\Box G, G, (\Box B)^n, B^n, \Box A \Rightarrow A}}{\Box G, (\Box B)^{\circ n} \Rightarrow \Box A}\ GLR$$

From Lemma 2.11(b) we have

$$\partial^\circ(\Box B, f_{\Box B}[\tau]) = \partial^\circ(\Box B, d[\mathsf{stub}] \hookleftarrow d') = \partial^\circ(\Box B, d[\mathsf{stub}]) + \partial^\circ(\Box B, d').$$

Write the endsequent of d' as $U \Rightarrow W$. Since GLR_1 is a 1-ary GLR rule in $f_{\Box B}[\tau]$ the path (through η) between the leaf stub in $d[\mathsf{stub}]$ and the endsequent $X, (\Box B)^* \Rightarrow Y$ of $d[\mathsf{stub}]$ contains no GLR rule instances. From Definition 2.5 and the compatibility of $d[\mathsf{stub}]$ and d', for any multiset M and any derivation d'' with endsequent $U, M \Rightarrow W$, it follows that $d[\mathsf{stub}] \oplus M$ and d'' are compatible.

Q.E.D.

Definition 2.19 (rank of a cut) *For a topmost cut cut_0 the rank $rk(cut_0)$ is the triple (d, n, h) where d is the degree of the cut-formula, n is the width of cut_0, and h is the cut-height of cut_0.*

Lemma 2.20 *Let τ be the following derivation where cut_0 is a topmost cut:*

$$\dfrac{\dfrac{\dfrac{\{\pi\}_1^r}{\Box X, X, \Box B \Rightarrow B}}{\Box X \Rightarrow \Box B}\ GLR \qquad \dfrac{\{\sigma\}_1^s}{\Box B, U \Rightarrow W}}{\Box X, U \Rightarrow W}\ cut_0$$

and suppose ($\star$): for any derivation δ, every topmost cut in δ with rank $<$ $rk(cut_0)$ is eliminable.

Then there is a transformed cut-free derivation τ' of $X, \Box X \Rightarrow B$.

Proof. Let μ denote the subderivation $\{\pi\}_1^r / \Box X, X, \Box B \Rightarrow B$ of τ.

Case width$(\mathbf{cut_0}) = \mathbf{0}$: Hence $\partial^\circ(\Box B, f_{\Box B}[\mu]) = 0$. Then the annotated derivation $f_{\Box B}(\mu)$ must have final parametric ancestors of the form $(\Box B)^\circ \Rightarrow \Box B$ or $X' \Rightarrow Y' /^{LW(\Box B)} X', (\Box B)^\circ \Rightarrow Y'$ only.

Let $\Box B^{(*|\circ)}$ stand for an annotated occurrence of $\Box B$ where the annotation is not known. Consider the substitution $(f_{\Box B}[\mu])\{\Box B^{(*|\circ)} := \Box X\}$ obtained by replacing every occurrence

1. of $(\Box B)^*$ with $\Box X$, and
2. of $(\Box B)^\circ \Rightarrow \Box B$ with a derivation of $\Box X \Rightarrow \Box B$ (the left premise derivation of cut_0), and
3. of $\dfrac{X' \Rightarrow Y'}{X', (\Box B)^\circ \Rightarrow Y'}\, LW(\Box B)$ with $\dfrac{X' \Rightarrow Y'}{X', \Box X \Rightarrow Y'}\, LW^*(\Box X)$

As the endsequent of $f_{\Box B}[\mu]$ was $\Box X, X, (\Box B)^* \Rightarrow B$ we have that

$$(f_{\Box B}[\mu])\{\Box B^{(*|\circ)} := \Box X\}$$

is a cut-free derivation of $\Box X, X, \Box X \Rightarrow B$. Applying repeated left contraction gives a cut-free derivation of $\Box X, X \Rightarrow B$.

Case $\mathbf{width(cut_0) > 0}$: Hence $\partial^\circ(\Box B, f_{\Box B}[\mu]) > 0$. First suppose that the last rule in μ is GLR. Then μ must be of the form:

$$\dfrac{\dfrac{\{\pi'\}_1^s}{\Box\Box X', \Box X', \Box X', X', \Box\Box A, \Box A, \Box A \Rightarrow A}}{\Box\Box X', \Box X', \Box\Box A \Rightarrow \Box A}\, GLR$$

where $X = \Box X'$ and $B \equiv \Box A$.

Then the following is a derivation of $\Box X, X \Rightarrow B$, with $deg(cut_1) = deg(cut_0)$ and $width(cut_1) = 0 < width(cut_0)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box A \Rightarrow \Box A}{\Box A, A, \Box\Box A \Rightarrow \Box A}\, LW^*(A, \Box\Box A)}{\Box A \Rightarrow \Box\Box A}\, GLR \qquad \dfrac{\{\pi'\}_1^s}{\Box\Box X', \Box X', \Box X', X', \Box\Box A, \Box A, \Box A \Rightarrow A}}{\Box A, \Box\Box X', \Box X', \Box X', X', \Box A, \Box A \Rightarrow A}\, cut_1}{\Box\Box X', \Box X', \Box X', X', \Box A \Rightarrow A}\, LC^*(\Box A)}{\Box\Box X', \Box X' \Rightarrow \Box A}\, GLR$$

The required derivation is obtained by using $(\star)$ to eliminate cut_1.

If the last rule in μ is not GLR, we can write $f_{\Box B}[\mu]$ as $d[\mathsf{stub}] \hookleftarrow d'$, where $d[\mathsf{stub}]$ and d' (below left and right respectively) are constructed as in the proof of Lemma 2.18, so $n \geq 1$, $\Box G$ does not contain annotated formulae, and the path in $d[\mathsf{stub}]$ through η from the top-sequent stub to the end-sequent $\Box X, X, (\Box B)^* \Rightarrow B$ contains no occurrence of the GLR rule:

$$\begin{array}{c} \mathsf{stub} \\ \eta \\ \Box X, X, (\Box B)^* \Rightarrow B \end{array} \qquad\qquad \dfrac{\dfrac{\{\pi'\}_1^t}{\Box G, G, (\Box B)^n, B^n, \Box A \Rightarrow A}}{\Box G, (\Box B)^{\circ n} \Rightarrow \Box A}\, GLR$$

By Lemma 2.18, $\partial^\circ(\Box B, d[\mathsf{stub}] \hookleftarrow d') = \partial^\circ(\Box B, d[\mathsf{stub}]) + \partial^\circ(\Box B, d')$.

Let d'' be the annotated derivation

$$\dfrac{\Box A \Rightarrow \Box A}{A, \Box A, \Box G, (\Box B)^{\circ n} \Rightarrow \Box A}\ LW^*(A, \Box G, (\Box B)^n)$$

Then $d[\mathsf{stub}] \oplus \langle A, \Box A\rangle$ and d'' are compatible (Lemma 2.18). Note that $\partial^\circ(\Box B, d') = 1$ and $\partial^\circ(\Box B, d'') = 0$. Let Λ_{11} be the derivation:

$$\dfrac{\dfrac{\dfrac{\lfloor d[\mathsf{stub}] \oplus \langle A, \Box A\rangle \hookleftarrow d''\rfloor}{\Box A, \Box X \Rightarrow \Box B}\ GLR \qquad \dfrac{\{\pi\}_1^r}{\Box X, X, \Box B \Rightarrow B}}{\Box A, \Box X, \Box X, X \Rightarrow B}\ cut_1(\Box B)}{\Box A, \Box X, X \Rightarrow B}\ LC^*$$

Let Λ_{12} be the derivation

$$\dfrac{\dfrac{\dfrac{\lfloor d[\mathsf{stub}] \oplus \langle A, \Box A\rangle \hookleftarrow d''\rfloor}{\Box A, \Box X \Rightarrow \Box B}\ GLR \qquad \dfrac{\{\pi'\}_1^t}{\Box G, G, \Box B, B, \Box A \Rightarrow A}\ LC^*(B^{n-1}, (\Box B)^{n-1})}{\Box A, \Box A, \Box X, \Box G, G, B \Rightarrow A}\ cut_2(\Box B)}{\Box A, \Box X, \Box G, G, B \Rightarrow A}\ LC$$

Consider the derivation Λ_1:

$$\dfrac{\dfrac{\dfrac{\Lambda_{11} \qquad \Lambda_{12}}{\Box A, \Box X, X, \Box A, \Box X, \Box G, G, B \Rightarrow A}\ cut_3(B)}{\Box A, \Box X, X, \Box G, G, \Rightarrow A}\ LC^*}{\Box X, \Box G \Rightarrow \Box A}\ GLR$$

For $i \in \{1, 2\}$, observe that $deg(cut_i) = deg(cut_0)$. Furthermore,

$$\begin{aligned}
width(cut_i) &= \partial^\circ(\Box B, f_{\Box B}(\lfloor d[\mathsf{stub}] \oplus \langle A, \Box A\rangle \hookleftarrow d''\rfloor)) && \text{Def. of width}\\
&= \partial^\circ(\Box B, d[\mathsf{stub}] \oplus \langle A, \Box A\rangle \hookleftarrow d'') && \text{By inspection}\\
&= \partial^\circ(\Box B, d \oplus \langle A, \Box A\rangle[\mathsf{stub}]) + \partial^\circ(\Box B, d'') && \text{Lemma 2.11(b)}\\
&< \partial^\circ(\Box B, d \oplus \langle A, \Box A\rangle[\mathsf{stub}]) + \partial^\circ(\Box B, d') &&\\
&= \partial^\circ(\Box B, d[\mathsf{stub}]) + \partial^\circ(\Box B, d') && \text{Lemma 2.11(a)}\\
&= \partial^\circ(\Box B, d[\mathsf{stub}] \hookleftarrow d') && \text{Lemma 2.11(b)}\\
&= width(cut_0)
\end{aligned}$$

Because $deg(cut_i) = deg(cut_0)$ and the premises of both cut_1 and cut_2 are cut-free, by appealing twice to $(\star)$ we can in turn eliminate cut_1 and cut_2. In the resulting derivation, since $deg(cut_3) < deg(cut_0)$ we can eliminate cut_3 by $(\star)$. We thus obtain a cut-free derivation Λ_2 of $\Box X, \Box G \Rightarrow \Box A$.

Let Λ_3 be the annotated derivation

$$f_{\Box B}\left[\dfrac{\Lambda_2}{\Box X, \Box G, \Box B \Rightarrow \Box A}\ LW^*((\Box B)^n)\right]$$

Clearly $\partial^\circ(\Box B, \Lambda_3) = 0$. Furthermore, by Lemma 2.18, $d[\mathsf{stub}] \oplus \Box X$ and Λ_3 are compatible. Recall that $\lfloor\cdot\rfloor$ is the forgetful map. The endsequent of $\lfloor (d[\mathsf{stub}] \oplus X) \hookleftarrow \Lambda_3\rfloor$ is thus $\Box X, \Box X, X, \Box B \Rightarrow B$. Consider the derivation Λ_4:

$$\cfrac{\cfrac{\cfrac{\lfloor (d[\mathsf{stub}] \oplus \Box X) \hookleftarrow \Lambda_3 \rfloor}{\Box B, \Box X, X \Rightarrow B}\, LC^*(\Box X)}{\Box X \Rightarrow \Box B}\, GLR \qquad \cfrac{\{\pi\}_1^r}{\Box X, X, \Box B \Rightarrow B}}{\cfrac{X, \Box X, \Box X \Rightarrow B}{X, \Box X \Rightarrow B}\, LC^*(\Box X)}\, cut_4(\Box B)$$

By a similar calculation to the above we obtain $width(cut_4) < width(cut_0)$. Because $deg(cut_4) = deg(cut_0)$ and the premises of cut_4 are cut-free, appealing to $(\star)$ we can eliminate cut_4. We thus obtain a cut-free derivation of $X, \Box X \Rightarrow B$ as required. Q.E.D.

Without loss of generality it suffices to consider a derivation concluded by a cut-rule with cut-free premise derivations.

Theorem 2.21 (Cut-elimination) *Let τ be a derivation concluded by an instance cut_0 of the cut-rule with cut-free premise derivations. Then there is a transformed cut-free derivation τ' with identical endsequent.*

Proof. Induction on the rank (d, n, h) of cut_0 under the standard lexicographic ordering. We say that the cut-formula is *left principal* if an occurrence of the cut-formula in the succedent of the left premise is a principal formula. The term *right principal* is defined analogously. This follows standard practice.

1 Cut with an initial sequent as premise. This is the base case. The transformations are standard (see [54, 70]).

2 Cut with neither premise an initial sequent.

(a) Cut-formula is left and right principal.

First suppose that the cut-formula is boxed. There are several possibilities:

(i) the cut-formula is left and right principal by the GLR rule. The derivation must then be in SNF:

$$\cfrac{\cfrac{\cfrac{\{\pi\}_1^r}{\Box X, X, \Box B \Rightarrow B}}{\Box X \Rightarrow \Box B}\, GLR \qquad \cfrac{\cfrac{\{\sigma\}_1^s}{\Box B, \Box U, B, U, \Box C \Rightarrow C}}{\Box B, \Box U \Rightarrow \Box C}\, GLR}{\Box X, \Box U \Rightarrow \Box C}\, cut_0$$

The induction hypothesis implies that for any derivation δ, any topmost cut in δ with rank $< rank(cut_0)$ is eliminable. This is precisely condition $(\star)$ in Lemma 2.20. Hence we can obtain a cut-free derivation of $\Box X, X \Rightarrow B$. Consider the derivation

$$\cfrac{\Box X, X \Rightarrow B \qquad \cfrac{\cfrac{\cfrac{\{\pi\}_1^r}{\Box X, X, \Box B \Rightarrow B}}{\Box X \Rightarrow \Box B}\, GLR \qquad \cfrac{\{\sigma\}_1^s}{\Box B, \Box U, B, U, \Box C \Rightarrow C}}{\Box X, \Box U, B, U, \Box C \Rightarrow C}\, cut_1}{\cfrac{\Box X, X, \Box X, \Box U, U, \Box C \Rightarrow C}{\Box X, \Box U \Rightarrow \Box C}\, GLR}\, cut_2$$

Observe that $rk(cut_1) = (d, n, h-1)$. By the induction hypothesis we can eliminate cut_1. In the resulting derivation, since $deg(cut_2) < d$, the result follows from another application of the induction hypothesis.

(ii) the cut-formula $\Box B$ is left principal by the GLR rule and right principal by $LC(\Box B)$.

Then τ is as below where both premises of cut_0 are cut-free and $m \geq 0$:

$$\dfrac{\dfrac{\dfrac{\{\pi\}_1^r}{\Box X, X, \Box B \Rightarrow B}}{\Box X \Rightarrow \Box B}\,GLR \qquad \dfrac{\dfrac{\{\sigma\}_1^s}{(\Box B)^{m+2}, U \Rightarrow W}\,\rho}{\Box B, U \Rightarrow W}\,LC^{m+1}(\Box B)}{\Box X, U \Rightarrow W}\,cut_0$$

In general ρ need not be the GLR rule. However if $\rho \neq GLR$ then either (1) $\rho = LW(\Box B)$ and we delete ρ and the $LC(\Box B)$ rule that follows, or (2) $\Box B$ is not principal by ρ.

In the former case the result is immediate. In the latter case the result is obtained by applying ρ *after* cut_0 (as opposed to before the $LC^{m+1}(\Box B)$ rules as it currently stands) and invoking the induction hypothesis. Observe that this is possible even if ρ is a binary rule.

If $\rho = GLR$ it follows that $U \equiv \Box V$ and $W \equiv \Box C$ for some multiset V and formula C, and $s = 1$ and $\sigma_1 \equiv \{\sigma'\}_1^{s'}/(\Box B)^{m+2}, B^{m+2}, \Box V, V, \Box C \Rightarrow C$. Thus τ must be of the form

$$\dfrac{\dfrac{\dfrac{\{\pi\}_1^r}{\Box X, X, \Box B \Rightarrow B}}{\Box X \Rightarrow \Box B}\,GLR \qquad \dfrac{\dfrac{\dfrac{\{\sigma'\}_1^{s'}}{(\Box B)^{m+2}, B^{m+2}, \Box V, V, \Box C \Rightarrow C}}{(\Box B)^{m+2}, \Box V \Rightarrow \Box C}\,\rho = GLR}{\Box B, \Box V \Rightarrow \Box C}\,LC^{m+1}(\Box B)}{\Box X, \Box V \Rightarrow \Box C}\,cut_0$$

A derivation of $\Box X, X \Rightarrow B$ is obtained as in (i) using Lemma 2.20. Consider the derivation:

$$\dfrac{\dfrac{\Box X, X \Rightarrow B \qquad \dfrac{\dfrac{\Box X \Rightarrow \Box B \qquad \dfrac{\dfrac{\{\sigma'\}_1^{s'}}{(\Box B)^{m+2}, B^{m+2}, \Box V, V, \Box C \Rightarrow C}}{\Box B, B^{m+2}, \Box V, V, \Box C \Rightarrow C}\,LC^{m+1}(\Box B)}{\Box X, B^{m+2}, \Box V, V, \Box C \Rightarrow C}\,cut_1}{\Box X, B, \Box V, V, \Box C \Rightarrow C}\,LC^{m+1}(B)}{\dfrac{\Box X, X, \Box X, \Box V, V, \Box C \Rightarrow C}{\Box X, X, \Box V, V, \Box C \Rightarrow C}\,LC^*}\,cut_2}{\Box X, \Box V \Rightarrow \Box C}\,GLR$$

Now cut_1 has identical degree and width compared to cut_0, and smaller cut-height. Hence, we can eliminate cut_1 using the induction hypothesis. In the resulting derivation $deg(cut_2) < d$ so the result follows from the induction hypothesis.

(iii) the cut-formula $\Box B$ is left principal by $RC(\Box B)$ and right principal by the GLR rule.

Then τ has the following form where both premises of cut_0 are cut-free:

$$\dfrac{\dfrac{\dfrac{\{\pi\}_1^r}{X \Rightarrow Y, \Box B, \Box B}}{X \Rightarrow Y, \Box B} RC_1 \quad \dfrac{\sigma_1}{\Box B, \Box U \Rightarrow \Box C} GLR}{X, \Box U \Rightarrow Y, \Box C} cut_0$$

Because the conclusion of (any) GLR rule has exactly one formula in the succedent, it follows that at least one of the $\Box B$ formula occurrences in the succedent of the premise of RC_1 can be traced upwards in $\{\pi\}_1^r$ to $RW(\Box B)$ rule application(s). In particular, when tracing upwards, it is impossible to encounter a GLR rule application *before* encountering a $RW(\Box B)$ rule application. Deleting these $RW(\Box B)$ rule applications and the RC_1 contraction rule certainly preserves the derivation structure because all the binary rules excluding the cut-rule are additive. This new derivation contains a single instance of cut with identical degree of cut-formula and reduced cut-height compared to cut_0. Furthermore, observe that it must be the case that the width is $\leq n$. The result follows from the induction hypothesis.

If the calculus uses multiplicative binary rules instead, the result still holds, although the transformations are slightly more complicated.

In each instance, the proof can be formalised using an annotation function similar in structure to $f_{\Box B}$. See Section 2.7.

(iv) the cut-formula $\Box B$ is left and right principal by $RC(\Box B)$ and $LC(\Box B)$ respectively. A combination of the strategies in (ii) and (iii) suffice.

(v) the cut-formula $\Box B$ is either left or right principal by $RW(\Box B)$ or $LW(\Box B)$ respectively. Trivial.

Next, suppose that the cut-formula is not boxed.

(vi) the cut-formula B is is left and right principal by the respective left and right introduction rules. The transformations are standard (see [54, 70]) — derivation τ is transformed to a derivation τ' containing cuts $\{cut_i\}_{i\geq 1}$ on strictly smaller cut-formulae (i.e. $deg(cut_i) < d$ for $i \geq 1$).

(vi) the cut-formula B is right principal by $LC(B)$. Then τ has the following form, where B is principal by ρ:

$$\dfrac{\dfrac{\{\pi\}_1^r}{X \Rightarrow Y, B}\,\rho(B) \qquad \dfrac{\dfrac{\{\sigma\}_1^s}{B, B, U \Rightarrow W}}{B, U \Rightarrow W}\,LC(B)}{X, U \Rightarrow Y, W}\,cut_0$$

Since we have assumed that B is not boxed, it follows that $\rho \neq GLR$. This is the well-known case of 'contractions above cut' that arises in cut-elimination for Gentzen's [25] sequent calculus LK. Gentzen's solution was to introduce a new multicut rule. There are several approaches for obtaining cut-elimination for classical sequent calculi avoiding Gentzen's multicut rule (for example [77, 10, 6]). We adapt the transformations proposed by von Plato [77] for the classical calculus $G0c$. The transformations there relied on the invertibility of all logical rules in $G0c$. The analogous results for GLS were proved in Lemma 2.17. We illustrate with a few cases.

Suppose $B = C \wedge D$. Consider the following derivation.

$$\dfrac{\dfrac{\{\pi\}_1^r}{X \Rightarrow Y, C \wedge D}\,\rho(C \wedge D) \qquad \dfrac{\dfrac{\{\sigma\}_1^s}{C \wedge D, C \wedge D, U \Rightarrow W}}{C \wedge D, U \Rightarrow W}\,LC(C \wedge D)}{X, U \Rightarrow Y, W}\,cut_0$$

Let h_1 and h_2 (> 1) denote the heights of the left and right premise derivations respectively. Applying Lemma 2.17 to the sequent $C \wedge D, C \wedge D, U \Rightarrow W$ we obtain a derivation of $C^l, D^{1-l}, \overline{C, D, C \wedge D}, U \Rightarrow W$ of height $h_2 - 1$, for some l, $0 \leq l \leq 1$; from the sequent $X \Rightarrow Y, C \wedge D$ we obtain derivations of $X \Rightarrow Y, C$ and $X \Rightarrow Y, D$, each of height h_1.

If $\overline{C, D, C \wedge D}$ is the formula occurrence $C \wedge D$ and $l = 0$ we proceed as follows (the case when $l = 1$ is similar):

$$\dfrac{\dfrac{X \Rightarrow Y, D \qquad \dfrac{X \Rightarrow Y, C \wedge D \qquad D, C \wedge D, U \Rightarrow W}{D, X, U \Rightarrow Y, W}\,cut_0}{X, X, U \Rightarrow Y, Y, W}\,cut_1}{X, U \Rightarrow Y, W}\,LC^*$$

Notice that cut_0 has height $h_1 + h_2 - 1 < h_1 + h_2$, and cut_1 has reduced degree compared to cut_0. If $\overline{C, D, C \wedge D}$ is the formula occurrence D (the case when it is C is similar) and $l = 0$ (the case when $l = 1$ is similar) then we proceed as follows, where each instance of the cut-rule in the following has reduced degree compared to cut_0:

$$\dfrac{\dfrac{X \Rightarrow Y, D \qquad \dfrac{X \Rightarrow Y, D \qquad D^2, U \Rightarrow W}{D, X, U \Rightarrow Y, W}}{X, X, U \Rightarrow Y, Y, W}}{X, U \Rightarrow Y, W}\,LC^*$$

Next, suppose $B = C \supset D$. Consider the following derivation:

$$\dfrac{\dfrac{\{\pi\}_1^r}{X \Rightarrow Y, C \supset D}\rho(C \supset D) \qquad \dfrac{\dfrac{\{\sigma\}_1^s}{C \supset D, C \supset D, U \Rightarrow W}}{C \supset D, U \Rightarrow W} LC(C \supset D)}{X, U \Rightarrow Y, W} cut_0$$

Applying Lemma 2.17 to the sequent $C \supset D, C \supset D, U \Rightarrow W$ we obtain derivations of $U \Rightarrow W, C^2$ and $D^2, U \Rightarrow W$. From $RC(C)$ and $LC(D)$ respectively, we obtain $U \Rightarrow W, C$ and $D, U \Rightarrow W$. Applying Lemma 2.17 to the sequent $X \Rightarrow Y, C \supset D$ we obtain a derivation of $C, X \Rightarrow Y, D$. Then the following derivation suffices, where each instance of the cut-rule has reduced degree compared to cut_0.

$$\dfrac{\dfrac{\dfrac{U \Rightarrow W, C \qquad C, X \Rightarrow Y, D}{X, U \Rightarrow Y, W, D} cut \qquad D, U \Rightarrow W}{X, U, U, \Rightarrow Y, W, W} cut}{X, U \Rightarrow Y, W} LC^*, RC^*$$

The remaining cases can be resolved in a similar fashion.

(b) Cut-formula is left principal only.

(c) Cut-formula neither left nor right principal.

We analyse the last inference rule in the *right (left)* premise derivation of cut_0. The standard transformations suffice here (for example [54, 70]). In particular, observe that for any instance cut_1 of the cut-rule appearing in a transformed derivation, it must be the case that $width(cut_1) \leq n$. Q.E.D.

2.5 A comparison with Valentini's original proof

We have already noted in Remark 2.16 that the move from sequents built from sets to sequents built from multisets necessitates a sharpening of the definition of width, in particular, to account for the interplay between weakening and contraction rules. To achieve this we used the notion of an annotated derivation, to trace upwards from the end-sequent.

An aspect of Valentini's proof that is unclear is whether the width is non-increasing in all introduced cuts. When eliminating cuts from a derivation δ containing *multiple* instances of the cut-rule (as occurs following an application of Valentini's move to a topmost cut where the cut-formula is left and right principal by the GLR rule), we always choose to eliminate a topmost cut (with conclusion sequent $\mathcal{S}$, say). Suppose that we then obtain a cutfree derivation δ'

of $\mathcal{S}$. Inserting the derivation δ' at $\mathcal{S}$ in δ we obtain a new derivation δ''. Now, to ensure that it is possible to (ultimately) eliminate those cuts below $\mathcal{S}$ in δ'', it is essential to know that the width of those cuts has not increased. In other words, we want to ensure that the width of a lower (non-topmost) cut does not increase under the cut-elimination transformations to topmost cuts.

In general, it is possible for the width of a lower cut — ie a non-topmost cut — to increase under the cut-elimination transformations. For example, consider a transformation that reduces some topmost cut instance cut_b (for "before") to the derivation below containing the cut instance cut_a (for "after") where $\{\pi\}_1^r$ and $\{\sigma\}_1^s$ need not be cut-free:

$$
\begin{prooftree}
\AxiomC{$\{\pi\}_1^r$}
\AxiomC{$\{\sigma\}_1^s$}
\RightLabel{cut_a}
\BinaryInfC{$G \Rightarrow H$}
\end{prooftree}
$$

The cut-elimination transformations which ultimately turn cut_a into a topmost cut may produce a derivation where $width(cut_a) > width(cut_b)$.

In the proof of Lemma 2.20, cut_4 is the only lower cut that relies on width for elimination. Observe that $width(cut_4)$ does not increase despite the reductions above it. This is because the cut_4 in that proof is 'shielded' by the GLR instance concluding Λ_1.

To see this, observe that derivation Λ_4 in Lemma 2.20 can be written as follows, where σ, η are cutfree:

$$
\begin{prooftree}
\AxiomC{σ}
\RightLabel{GLR_2}
\UnaryInfC{$\Box X, \Box G \Rightarrow \Box A$}
\RightLabel{$LW^*((\Box B)^n)$}
\UnaryInfC{$\Box X, \Box G, (\Box B)^n \Rightarrow \Box A$}
\UnaryInfC{$\lfloor \eta \rfloor \oplus \Box X$}
\RightLabel{$LC^*(\Box X)$}
\UnaryInfC{$\Box B, \Box X, X \Rightarrow B$}
\RightLabel{GLR_1}
\UnaryInfC{$\Box X \Rightarrow \Box B$}
\AxiomC{$\{\pi\}_1^r$}
\UnaryInfC{$\Box X, X, \Box B \Rightarrow B$}
\RightLabel{$cut_4(\Box B)$}
\BinaryInfC{$X, \Box X, \Box X \Rightarrow B$}
\RightLabel{$LC^*(\Box X)$}
\UnaryInfC{$X, \Box X \Rightarrow B$}
\end{prooftree}
$$

Crucially, because GLR_2 is a 2-ary GLR rule in the left premise derivation of cut_0, the width of cut_4 is independent of the structure of σ. In other words, if we substituted σ with any other cutfree derivation σ' with identical end-sequent, the width of cut_4 would remain unchanged. This shielding provided by the GLR_2 rule is crucial for the success of the proof.

We conclude by noting that it has long been recognised that the contraction rule poses special problems for cut-elimination. Hence, it is of independent interest to find syntactic proofs of cut-elimination for the calculus built from multisets, even when cut-admissibility is known for the calculus without cut. This is one reason for the numerous syntactic proofs of cut-elimination for GL that have been proposed, in many different proofs systems. The present work uses the traditional sequent calculus, and the intention is that the results can be extended to new logics and new calculi. Indeed in Chapter 3 we will see a generalisation of this proof applied to the logic Go.

2.6 Moen's Val-II(core) is not Valentini's reduction

We have carefully examined Moen's slides titled "The proposed algorithms for eliminating cuts in the provability calculus GLS do not terminate" [51].

Moen sets out to reduce a cut in SNF using the transformation he titles Val-II(core). Moen claims that Val-II(core) is the "...core of Valentini's reduction" [51]. Yet Val-II(core) does not appear in [71]. However it appears in [64, page 322] with the comment "this reduction is not sufficient".

Thus Moen is incorrect in claiming that he has demonstrated that Valentini's algorithm does not terminate — Moen is using the wrong algorithm. In fact, for his concrete derivation ϵ, the width of the cut-formula is 1 so the reduction is immediate. Applying the base case transformations, and then the classical transformations, we obtained a cut-free derivation of the end-sequent of ϵ.

2.7 Incorporating multiplicative binary rules

Excluding the cut-rule, the sequent calculus GLS (Table 2.1) contains only additive binary rules. However, in Valentini's original sequent calculus GLS_V, the rule $L\supset_m$ introducing the connective $\supset$ in the antecedent is multiplicative:

$$\frac{X \Rightarrow Y, A \qquad B, U \Rightarrow W}{A \supset B, X, U \Rightarrow Y, W}\, L\supset_m$$

Let GLS_m be the sequent calculus obtained from GLS by substituting the additive rule $L\supset$ with the multiplicative rule $L\supset_m$ rule (the subscript is for "multiplicative"). It is easy to pass between these rules using the appropriate weakening and contraction rules. Thus GLS_m is sound and complete for GL. In fact, the proof of cut-elimination for GLS can easily be adapted to GLS_m.

Given a derivation in GLS_m it is clear that we can obtain a cutfree derivation as follows: first convert each instances of $L\supset_m$ to $L\supset$, then use Theorem 2.21 to obtain a cutfree derivation in GLS. Finally, obtain a cutfree derivation in GLS_m by replacing each instance of $L\supset$ in the cutfree derivation in GLS with $L\supset_m$.

Alternatively, we could extend the annotation function f_C (Definition 2.8) to handle the multiplicative binary rule by inserting the following case into Table 2.2 (recall that Table 2.2 describes the helper function $\Phi_{\Box B}$; f_C invokes $\Phi_{\Box B}$ in Definition 2.8.3): Suppose the annotated derivation δ has the form

$$\dfrac{\dfrac{\{\pi\}_1^r}{G', (\Box B)^k \Rightarrow H'} \qquad \dfrac{\{\pi'\}_1^s}{G'', (\Box B)^l \Rightarrow H''}}{G, (\Box B)^{*n} \Rightarrow H}\ L\supset_m$$

If $k \geq n$, then let $\Phi_{\Box B}[\delta]$ be the derivation

$$\dfrac{\Phi_{\Box B}\left[\dfrac{\{\pi\}_1^r}{G', (\Box B)^{*n}, (\Box B)^{(k-n)} \Rightarrow H'}\right] \qquad \dfrac{\{\pi'\}_1^s}{G'', (\Box B)^l \Rightarrow H''}}{G, (\Box B)^{*n} \Rightarrow H}\ L\supset_m$$

If $k < n$, then let $\Phi_{\Box B}[\delta]$ be the derivation

$$\dfrac{\Phi_{\Box B}\left[\dfrac{\{\pi\}_1^r}{G', (\Box B)^{*k} \Rightarrow H'}\right] \qquad \Phi_{\Box B}\left[\dfrac{\{\pi'\}_1^s}{G'', (\Box B)^{*(n-k)}, (\Box B)^{l-n+k} \Rightarrow H''}\right]}{G, (\Box B)^{*n} \Rightarrow H}\ L\supset_m$$

It is now straightforward to adapt the cut-elimination proof for the GLS calculus to the new setting.

2.8 A decision procedure for Ip using GLS

G. Sambin suggested[1] that it might be possible to use the decision and countermodel construction procedure [64] for GL to obtain the corresponding procedure for propositional intuitionistic logic Ip [16]. We demonstrate that this is indeed the case. Of course, such procedures for Ip are well-known. The novelty here is the use of the Gödel translation [27, 16] and the decision procedure for GL. Our main contribution is showing that the countermodel obtained using the auxiliary calculus GLS' introduced below has the persistence property (Lemma 2.23). The proof of the intuitionistic countermodel (Theorem 2.25) follows from this result.

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2.8.1 Terminology and basic results

Before we proceed, let us introduce some standard terminology.

Propositional intuitionistic logic Ip is defined using the propositional language $\mathcal{L}$ [16]. We obtain the modal language $\mathcal{ML}$ by augmenting the propositional language $\mathcal{L}$ with the modal operator $\Box$. Let **For**$\mathcal{L}$ (resp. **For**$\mathcal{ML}$) and **Var**$\mathcal{L}$ (**Var**$\mathcal{ML}$) denote the set of well-formed formulae and propositional variables of the language $\mathcal{L}$ ($\mathcal{ML}$).

A *frame* is a pair (W, R) where W is a non-empty set of states and R is a binary relation on W. If $u, v \in W$ such that Ruv then we say that v is *above* u (in F). A frame (W, R) has the property $\mathcal{P}$ (think reflexivity, transitivity, antisymmetry etc.) if the binary relation R has the property $\mathcal{P}$. If a frame $F = (W, R)$ contains some sequence of (not necessarily distinct) points $w_1, w_2, \dots w_n$ from W such that $Rw_1w_2, Rw_2w_3, \dots, Rw_{n-1}w_n$ then we say that F contains an *R-chain*. If F contains an R-chain for arbitrarily large n, we say that F contains an *∞-R-chain*. A *proper R-chain* is an R-chain where the points are distinct. When R is transitive, a *cluster* $\mathcal{C}$ is a maximal subset of W such that for all distinct states $w, w' \in \mathcal{C}$ we have Rww' and $Rw'w$. A cluster is *proper* if it consists of two or more states.

Let F be a frame. A *model* based on F is the pair (F, V) where V is a valuation function assigning a subset $V(p) \subseteq W$ to each propositional variable p. Define the satisfaction relation $M, w \models D$ (read as 'D is *satisfiable* in M at state w') by induction on the structure of the formula $D \in$ **For**$\mathcal{ML}$ as follows:

$$
\begin{aligned}
&M, w \models p \text{ iff } w \in V(p)\\
&M, w \models \neg A \text{ iff not } M, w \models A\\
&M, w \models A \vee B \text{ iff } M, w \models A \text{ or } M, w \models B\\
&M, w \models A \wedge B \text{ iff } M, w \models A \text{ and } M, w \models B\\
&M, w \models A \supset B \text{ iff } M, w \models A \text{ implies } M, w \models B\\
&M, w \models \Box A \text{ iff for all } v \in W, \text{ if } Rwv \text{ then } M, v \models A
\end{aligned}
$$

The negation of $M, w \models D$ is written $M, w \not\models D$. We say that D is *falsifiable* on a model M if there is some state w such that $M, w \not\models D$.

Let F be a frame. A formula A is *valid* at a state w in F (written $F, w \models A$) if it is the case that $M, w \models A$ for every model M based on F. A formula is valid on a frame (written $F \models A$) if it is valid at each state on the frame. Also, a formula A is valid on a class $\mathcal{F}$ of frames (written $\mathcal{F} \models A$) if that formula is valid on each frame in the class. Finally, a logic L is *sound* for $\mathcal{F}$ if $A \in L$ implies $\mathcal{F} \models A$, and a logic L is *complete* for $\mathcal{F}$ if $\mathcal{F} \models A$ implies $A \in L$.

It is well-known [64] that GL is sound and complete for the class of frames that are transitive and contain no ∞-R-chains. Such frames are necessarily irreflexive. A model whose underlying frame is transitive and contains no ∞-R-chains will be called a *GL-model*.

Suppose that L_1 and L_2 are logics in the languages $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. A function $f : \mathbf{For}\mathcal{L}_1 \mapsto \mathbf{For}\mathcal{L}_2$ is called an *embedding* of L_1 into L_2 if

$$\forall A \in \mathbf{For}\mathcal{L}_1 . A \in L_1 \text{ iff } f(A) \in L_2$$

Consider the following function T taking $\mathbf{For}\mathcal{L}$ into $\mathbf{For}\mathcal{ML}$: for all $p \in \mathbf{Var}\mathcal{L}$ and all $A, B \in \mathbf{For}\mathcal{L}$:

$$\begin{aligned} T(p) &= \Box p \\ T(\neg A) &= \Box \neg T(A) \\ T(A \vee B) &= T(A) \vee T(B) \\ T(A \wedge B) &= T(A) \wedge T(B) \\ T(A \supset B) &= \Box(T(A) \supset T(B)) \end{aligned}$$

This function is called the *Gödel translation* [27, 16]. It is well-known that T is an embedding of Ip into $S4$ and Ip into Grz (see [16]). Notice that for every occurrence of the propositional variable q in $T(A)$ for $A \in \mathbf{For}\mathcal{L}$, it must be the case that q appears in $T(A)$ as the subformula $\Box q$. In other words, all propositional variables in $T(A)$ are boxed.

For $A \in \mathbf{For}\mathcal{ML}$, let $T^{\boxtimes}(A)$ be the formula obtained by simultaneous replacement of all occurrences of $\Box$ in A with $\boxtimes$, where $\boxtimes B$ is abbreviation for $B \wedge \Box B$. The map $T^{\boxtimes}$ is known to be an embedding of Grz into GL (see [16]). It follows that $T_{GL} = T^{\boxtimes}T$ is an embedding of Ip into GL. Notice that for every occurrence of the propositional variable q in $T_{GL}(A)$ for $A \in \mathbf{For}\mathcal{L}$, it must be the case that q appears in $T_{GL}(A)$ as the subformula $q \wedge \Box q$.

2.8.2 Decision and countermodel procedure for GL

Sambin and Valentini [64, pg 326-7] have presented the following decision and countermodel procedure for GL. We begin by defining the auxiliary calculus GLS' — adapted from the version appearing in [64] in order to incorporate sequents built from multisets — obtained from GLS (Table 2.1) as follows:

(i) In GLS', initial sequents are of the form $P, \Box W \Rightarrow Q$ where P and Q are multisets of propositional variables, and W is a multiset of formulae. If $P \cap Q \neq \emptyset$ then we call this sequent an *axiom*.

(ii) The GLR rule is replaced with the following rule, where P and Q are finite sets of propositional variables and $Y = \{A_1, \ldots, A_m\}$:

$$\frac{X, \Box X, \Box A_1 \Rightarrow A_1 \qquad \ldots \qquad X, \Box X, \Box A_m \Rightarrow A_m}{P, \Box X \Rightarrow \Box Y, Q}\ RR$$

Its meaning is that if *one* of the premises is derivable, then the conclusion in derivable. Observe that RR is not a rule in the usual sense (hence the dashed line). The rule RR is admissible in GLS because the conclusion of RR is derivable in GLS whenever one of the premises of RR is derivable in GLS. Also, RR is invertible in GLS in the sense that if the conclusion sequent is derivable in GLS, then at least one of the premises is derivable in GLS.

(iii) Replace the rules $L\wedge$ and $R\vee$ in GLS with the following rules:

$$\frac{A, B, X \Rightarrow Y}{A \wedge B, X \Rightarrow Y}\ L\wedge' \qquad\qquad \frac{X \Rightarrow Y, A, B}{X \Rightarrow Y, A \vee B}\ R\vee'$$

(iv) Delete the weakening rules LW and RW, and delete the contraction rules LC and RC.

(v) Delete the cut-rule.

The idea is to use the calculus GLS' for backward proof search in GL. By backward proof search we mean repeated backward application of the rules of GLS' (ie. matching the conclusion sequent to obtain the premise sequents) until an axiom is obtained or no further backward application is possible. The resulting object is called a *searchtree*. A *search* is obtained by choosing a particular branch of the searchtree at each application of RR. We use the term *proper search* to refer to a search whose every initial sequent is an axiom. A sequent is said to be derivable in GLS' if there is a proper search for that sequent. See Sambin and Valentini [64] for the proof that there can be no repetition of a sequent along any branch of a searchtree. Since GLS' can produce only finitely many sequents for a given input, it follows that the GLS' calculus terminates under backward proof search for any input.

The calculi GLS and GLS' derive the same set of sequents

The first thing to do is show that derivations in GLS and proper searches in GLS' derive the same set of sequents. That is, a sequent $\mathcal{S}$ is derivable in GLS iff there is a proper search of some searchtree in GLS' with endsequent $\mathcal{S}$. (If) From a proper search in GLS' it is straightforward to obtain a derivation of the identical endsequent in GLS.

To prove the (Only-if) it suffices to show that the steps (i)–(v) for obtaining GLS' from GLS do not reduce the set of derivable sequents. Clearly every initial sequent of GLS is derivable in GLS'. Furthermore, it is clear that GLR is a special case of RR so replacing the former rule with the latter does not reduce the set of derivable sequents. Similar comments apply to the rules $L\wedge'$ and $R\vee'$. Notice that weakening has been absorbed into the initial sequents and the RR rule. Although only propositional variables (as opposed to arbitrary formulae) can be introduced into a sequent via weakening in GLS', observe that this does not reduce the set of derivable sequents. Neither does deletion of the cut-rule reduce the set of derivable sequents because of the cut-elimination result for GLS.

It remains to show that the deletion of the contraction rules does not reduce the set of derivable sequents. This result is facilitated by (i) the dispensing of the initial sequents $A \Rightarrow A$ in GLS — in contrast, an axiom in GLS' has a common *propositional variable* in the antecedent and succedent, (ii) absorbing weakening into the initial sequents and the RR rule, and (iii) replacing the rules $L\wedge$ and $R\vee$ with $L\wedge'$ and $R\vee'$. Due to these changes, by a standard induction on the height of the derivation, we can show height-preserving invertibility of the logical rules and hence admissibility of the contraction rules as required. We omit the details.

Decision and countermodel procedure for GL

For a given input $X \Rightarrow Y$, if the searchtree contains a proper search, then we can directly obtain a derivation of $X \Rightarrow Y$ in GLS. On the other hand, suppose that the searchtree contains no proper search. It follows that $X \Rightarrow Y$ is not derivable in GLS. The task then is to construct a GL-model M such that $\bigwedge X \supset \bigvee Y$ is falsifiable on M. Such a model is called a *countermodel* for $\bigwedge X \supset \bigvee Y$ (to simplify the notation we will continue to write this formula as $X \Rightarrow Y$). In this manner a decision and countermodel procedure for GL is obtained.

Sambin and Valentini [64] present the following method for constructing a countermodel. First observe that an initial sequent in the searchtree that is not an axiom must have the form $L, \Box W \Rightarrow M$ where L and M are sets of propositional variables. Given a searchtree containing no proper search, for each initial sequent $L, \Box W \Rightarrow M$ in the searchtree that is not an axiom (so $L \cap M = \emptyset$), obtain a valuation falsifying that sequent at a state x such that Rxy for no y by setting $x \in V(p)$ iff $p \in L$. Then descend the searchtree, extending the countermodel where necessary to falsify each successive sequent. The proof is by induction on the height of the searchtree.

The existing model suffices when logical rules are encountered. However, it becomes necessary to add new states to the model when the RR rule is encountered. Let us illustrate this step. Consider an instance of RR with premise sequents $\Box X, X, \Box A_i \Rightarrow A_i$ $(1 \leq i \leq m)$ and conclusion sequent $P, \Box X \Rightarrow \Box Y, Q$ where $Y = \{A_1, \ldots, A_m\}$. By the induction hypothesis we have that $x_1, \ldots x_m$ are the roots of m finite irreflexive transitive antisymmetric frames $F_1, \ldots, F_m$ with valuations $V_1, \ldots, V_m$ such that $F_i, x_i \not\models \Box X, X, \Box A_i \Rightarrow A_i$ for $1 \leq i \leq m$. Let us construct a GL-model falsifying the sequent $P, \Box X \Rightarrow \Box Y, Q$. Consider the transitive closure F of the frame obtained by setting a new node y below each of the frames with root x_i so Ryx_i. A frame has the antisymmetry property if the binary relation R satisfies $Rxy \wedge Ryx \rightarrow x = y$ for all states x, y. By construction, we observe that F is antisymmetric. To see this, intuitively, all the arrows point in the same general 'direction', upwards from y, so the antecedent of the antisymmetry condition is never true for any instantiation of states, and hence antisymmetry holds trivially.

Let V be the valuation that agrees with V_i on all states of the subtree with root x_i and such that $P \Rightarrow Q$ is falsified at y (ie. set $y \in V(p)$ for those propositional variables $p \in P \cap Q^{\perp}$, where $Q^{\perp}$ is the complement of Q). Denote this model (F, V). Notice that y forces $\Box X$ as each x_i forces $X, \Box X$. However $(F, V), y \not\models \Box A_i$ for each i since $(F, V), x_i \not\models A_i$. We conclude that $(F, V), y \not\models P, \Box X \Rightarrow \Box Y, Q$. By construction, F is finite, irreflexive, transitive and antisymmetric and hence contains no ∞-R-chains. Thus (F, V) is a GL-model falsifying $P, \Box X \Rightarrow \Box Y, Q$. We will use the name *$GLS'$-countermodel* to refer to this model.

Motivating the termination of backward proof search

The termination of backward proof search in GLS' can be explained as follows. From a syntactic viewpoint, the reason is that the diagonal formula in the conclusion sequent of RR reappears in the antecedent of the premise sequent. This ensures that if we encounter the same diagonal formula again while searching backwards on a branch, then an axiom immediately results forcing termination (no loops). Since $\Box X, X, \Box B \Rightarrow B$ is derivable iff $\Box X, X \Rightarrow B$ is derivable (the forward direction is the key result for Valentini's cut-elimination — see [71] and [32]), the diagonal formula that appears in the antecedent of the premise of the RR rule can be viewed as an artifice to ensure a finite frame (equivalently, to ensure termination without loop-check).

An explanation can also be given via the GLS'-countermodel construction. Consider the following instance of the RR rule:

$$\frac{\Box A \Rightarrow A}{\Rightarrow \Box A}\, RR$$

The GLS'-countermodel procedure constructs a model falsifying $\Box A$ at a frame rooted at y, by falsifying $\Box A \Rightarrow A$ at some state x above y. That is, by falsifying A at x *and* forcing A at every v above x. Of course, to falsify $\Box A$ at y it is enough to falsify A at x, but the former construction ensures that we do not try to repeatedly falsify A leading to a loop.

2.8.3 Lifting the method to intuitionistic logic

Propositional intuitionistic logic Ip (see [16]) is defined in the usual way in the language $\mathcal{L}$. Let us begin by presenting the semantics for Ip.

An *intuitionistic frame* is defined to be a reflexive, transitive and antisymmetric frame. An *intuitionistic model* is defined as the pair (F, V) where $F = (W, R)$ is an intuitionistic frame and V is a valuation function assigning a subset $V(p) \subseteq W$ for each propositional variable p such that V also satisfies the following *persistence* property: for all states $w, v \in W$ and propositional variables p, $w \in V(p)$ and Rwv implies $v \in V(p)$.

Let $M = (F, V)$ be an intuitionistic model based on the intuitionistic frame (W, R) and w a state in W. Define the satisfaction relation $M, w \models_i D$ by induction on the structure of the formula $D \in \mathbf{For}\mathcal{L}$ as follows:

$M, w \models_i p$ iff $w \in V(p)$

$M, w \models_i \neg A$ iff for all $v \in W$, if Rwv then not $M, v \models_i A$

$M, w \models_i A \vee B$ iff $M, w \models_i A$ or $M, w \models_i B$

$M, w \models_i A \wedge B$ iff $M, w \models_i A$ and $M, w \models_i B$

$M, w \models_i A \supset B$ iff for all $v \in W$, if Rwv then $M, v \models_i A$ implies $M, v \models_i B$

The negation of $M, w \models_i D$ is written $M, w \not\models_i D$. We say that the formula D is *falsifiable* on an intuitionistic model M if there is some state w such that $M, w \not\models_i D$. The notions of validity, soundness and completeness are defined as in Section 2.8.1 with the word "frame" substituted with "intuitionistic frame". It is well-known that Ip is sound and complete for the class of intuitionistic frames [16].

Let us look at how to construct a decision/countermodel procedure for Ip. Given a formula $A \in \mathbf{For}\mathcal{L}$, compute the modal formula $T_{GL}(A)$. Via backward proof search using the GLS' calculus obtain a searchtree for $\Rightarrow T_{GL}(A)$. If the searchtree contains a proper search, then it follows that $T_{GL}(A) \in GL$. Since T_{GL} is an embedding of Ip into GL, it follows that $A \in Ip$. On the other hand, if the searchtreee for $\Rightarrow T_{GL}(A)$ does not contain a proper search, using the procedure in Section 2.8.2 we can construct a GLS'-countermodel M for $T_{GL}(A)$. Of course, M is not an intuitionistic model because the underlying frame is irreflexive. Moreover an intuitionistic model has the persistence property while no such restriction applies to a GL-model. Nevertheless, we will show how to construct an intuitionistic model M' from M such that formula A is falsifiable on M'. As a result we will have obtained a countermodel for A as required.

We have already observed that every occurrence of a propositional variable q in $T_{GL}(A)$, for $A \in \mathbf{For}\mathcal{L}$, must appear in the context $q \wedge \Box q$. This observation is crucial for the following result.

Lemma 2.22 *Let δ be a searchtree with endsequent $\Rightarrow T_{GL}(A)$ for $A \in \mathbf{For}\mathcal{L}$. Then, in every instance ρ of the rule RR (shown below) in δ:*

$$\frac{X, \Box X, \Box A_1 \Rightarrow A_1 \quad \ldots \quad X, \Box X, \Box A_m \Rightarrow A_m}{P, \Box X \Rightarrow \Box Y, Q} \; RR$$

for every $p \in P$, it is the case that $p \in X$.

Proof. Let p be an arbitrary propositional variable in P.

First suppose that this formula occurrence moves to the succedent in some sequent below the conclusion sequent $P, \Box X \Rightarrow \Box Y, Q$ of ρ and prior to encountering another RR rule instance or the endsequent. Then the occurrence p appears in the succedent as the subformula of some formula $D(p)$. Note that the occurrence p must appear in $D(p)$, (i) under the scope of an odd number of negation signs, and (ii) not in the scope of a $\Box$. Since the propositional variable p must appear in the endsequent $\Rightarrow T_{GL}(A)$ in the context $p \wedge \Box p$, due to (i) it must be the case that p appears in $D(p)$ as the subformula $p \wedge \Box p$ and in the scope of an odd number of negation signs (to be precise, read implications $M \supset N$ in $D(p)$ as $\neg M \vee N$). Then by (ii) it follows that $p \wedge \Box p$ is not in the scope of a $\Box$ in $D(p)$. Notice that every formula in the conclusion sequent of ρ is either a propositional variable or a boxed formula. Hence an occurrence of $\Box p$ in $D(p)$ in the scope of an odd number of negation signs and not in the scope of a $\Box$ implies that this occurrence must have been a member of $\Box X$ in the conclusion sequent of ρ. Thus $\Box p \in \Box X$ and hence $p \in X$.

Next suppose that the formula occurrence p remains in the antecedent prior to encountering another RR rule instance, (or the endsequent in case ρ is a bottommost RR rule). In fact, if ρ is a bottommost RR rule, then p cannot remain in the antecedent because the endsequent is $\Rightarrow T_{GL}(A)$ (there is no formula in the antecedent). If ρ is not a bottommost RR rule, the searchtree has the following form, where $Y = \{A_1, \ldots, A_m\}$ and $Y' = \{B_1, \ldots, B_n\}$:

$$\dfrac{\Box X, X, \Box A_1 \Rightarrow A_1 \quad \ldots \quad \Box X, X, \Box A_m \Rightarrow A_m}{P, \Box X \vdash \Box Y, Q}\ RR \equiv \rho$$

$$\vdots$$

$$\text{no } RR \text{ rule instance}$$

$$\vdots$$

$$\dfrac{\Box X', X', \Box B_1 \Rightarrow B_1 \quad \ldots \quad \ldots \quad \Box X', X', \Box B_n \Rightarrow B_n}{P', \Box X' \vdash \Box Y', Q'}$$

Since a formula cannot acquire boxes between RR rules instances, we have $\Box X' \subseteq \Box X$. Now if $p \in X'$, then $\Box p \in \Box X'$ and hence $p \in X$. Else, if $p \notin X'$, then the formula occurrence p must have been the principal formula of some logical rule. It follows that p appears in the subformula $p \wedge \Box p$ in some formula $C(p)$ in X'. Moreover, the subformula $p \wedge \Box p$ must be in the scope of an even number of negation signs (as before, read implications $M \supset N$ in $C(p)$ as the formula $\neg M \vee N$) and cannot be in the scope of a $\Box$. It follows that $\Box p \in \Box X$ and hence we have that $p \in X$.

This exhausts all the possibilities, so we conclude that $p \in X$. Q.E.D.

Lemma 2.23 *Suppose that $A \in$ **For**$\mathcal{L}$ such that $A \notin Ip$. Then the GLS'-countermodel $M = (F, V)$ obtained according to the procedure in Section 2.8.2 has the persistence property.*

Proof. Revisiting the construction in Section 2.8.2, in order to show the persistence property, it suffices to consider the situation when a RR rule ρ (with conclusion sequent $P, X \Rightarrow Y, Q$) is encountered in the searchtree. Recall that this is the only situation where new states are added to the model. The construction stipulates that we take the transitive closure of the frame obtained by placing a new state y under the states x_i (ie. Ryx_i), where F_i denotes the frame with root x_i obtained from the induction hypothesis ($1 \leq i \leq m$). The model is obtained by setting $y \in V(p)$ for $p \in P \cap Q^{\perp}$. To prove persistence, for all states x above y we must show that $x \in V(p)$. By the construction and Lemma 2.22, every state x_i forces $\Box p, p$. Since every state above y is either x_i ($1 \leq i \leq m$) or some state above x_i, the result follows. Q.E.D.

The *reflexive closure* $\mathcal{R}^r$ of $\mathcal{R}$ is defined as

$$R^r xy \text{ iff } x = y \text{ or } Rxy.$$

Given a modal frame $F = (W, R)$ and a model $M = (F, V)$ on it, the frame $F^r = (W, R^r)$ and the model $M^r = (F^r, V)$ are called the *reflexivizations* of F and M respectively.

Theorem 2.24 *[Reflexivization] For every model M, every state x in M and every $\mathcal{ML}$-formula A,*

$$M, x \models T^{\boxtimes}(A) \textit{ iff } M^r, x \models A.$$

Proof. See [16]. Q.E.D.

Theorem 2.25 *Suppose that $A \in$ **For**$\mathcal{L}$ such that $A \notin Ip$. Then the reflexivization M^r of the GLS'-countermodel M for $T_{GL}(A)$ is a finite intuitionistic model falsifying A.*

Proof. From Section 2.8.2 we know that the model M is a finite, irreflexive, transitive, antisymmetric model such that $M, y \not\models T_{GL}(A)$ for some state y. Moreover from Lemma 2.23 we know that M has the persistence property. Since $T_{GL} = T^{\boxtimes}T$, by Theorem 2.24 it follows that $M^r, y \not\models T(A)$, where M^r is a finite, reflexive, transitive, antisymmetric model with the persistence property.

Clearly M^r is an intuitionistic model. It suffices to show that $M^r, y \not\models_i A$. To show this we prove the stronger result that $M^r, y \models_i A$ iff $M^r, y \models T(A)$. Proof by induction on the structure of A.

Suppose that A is the propositional variable p. Then $M^r, y \models_i p$ iff $y \in V(p)$. By the persistence property, every state above y is also in $V(p)$. The latter occurs iff $M^r, y \models \Box p$.

If $A = \neg B$, then $M^r, y \models_i \neg B$ iff for all z, if Ryz then $M^r, z \not\models_i B$. By the induction hypothesis this occurs iff for all z, if Ryz then $M^r, z \not\models T(B)$. This in turn occurs iff $M^r, y \models \Box\neg T(B)$.

If $A = B \supset C$, then $M^r, y \models_i B \supset C$ iff for all z, if Ryz then $M^r, z \models_i B$ implies $M^r, z \models_i C$. By the induction hypothesis, this occurs iff for all z, if Ryz then $M^r, z \models T(B)$ implies $M^r, z \models T(C)$. This in turn occurs iff $M^r, y \models \Box(T(B) \supset T(C))$.

The cases when $A = C \vee D$ and $A = C \wedge D$ are straightforward. Q.E.D.

Bounding the depth of the countermodel.

The above procedure suggests a simple bound on the height of the countermodel for $A \in \mathbf{For}\mathcal{L}$ such that $A \notin Ip$. It is the maximum modal depth — in other words, the maximum nesting of $\Box$ — in A.

Finite model property.

Theorem 2.25 implies the finite model property [16] for Ip.

2.8.4 Related work

Gentzen's [25] single-formula succedent calculus LJ for intuitionistic logic Ip contains the following rule $L\supset$ for introducing the connective $\supset$ into the antecedent:

$$\frac{A \supset B, X \Rightarrow A \qquad B, X \Rightarrow C}{A \supset B, X \Rightarrow C}\; L\supset$$

Notice that the principal formula $A \supset B$ is repeated in the left premise. This ensures admissibility of the contraction rule. However, this repetition also means that a loop check is required to obtain termination for the calculus.

In the calculus LJT [22], the above $L\supset$ rule is split into the following four rules, the motivation being a fine-analysis of the structure of A:

$$\frac{B, p, X \Rightarrow A}{p \supset B, p, X \Rightarrow A}\; L\supset_1 \qquad \frac{C \supset (D \supset B), X \Rightarrow A}{(C \wedge D) \supset B, X \Rightarrow A}\; L\supset_2$$

$$\frac{C \supset B, D \supset B, X \Rightarrow A}{(C \vee D) \supset B, X \Rightarrow A}\; L\supset_3 \qquad \frac{D \supset B, X \Rightarrow C \supset D \qquad B, X \Rightarrow A}{(C \supset D) \supset B, X \Rightarrow A}\; L\supset_4$$

The rule of contraction is admissible in LJT (see [23] for a direct proof; only $L\supset_4$ is non-invertible and hence requires special care). Furthermore, it is possible to define a measure μ under which the conclusion sequents have strictly greater weight that the premises. Thus termination is guaranteed without the use of a loopcheck. Intuitively, the left premise of $L\supset_4$, (i) contains enough information to ensure admissibility of contraction, and (ii) is simpler than the conclusion sequent under the measure μ.

Note that we are not suggesting that the method of embedding Ip into GL (Section 2.8.3) is better for proof search than the calculus LJT.

2.9 Adapting the proof of cut-elimination for some other logics

Let K be the basic modal logic and let $K\mathcal{A}_1 \ldots \mathcal{A}_n$ denote the axiomatic extension of K obtained by the addition of axioms $\mathcal{A}_1, \ldots, \mathcal{A}_n$ to K. Consider the following axioms:

$$\begin{array}{ll} 4: & \Box p \supset \Box\Box p \quad \text{(transitivity)} \\ T: & \Box p \supset p \quad \text{(reflexivity)} \\ \mathcal{G}: & \Box(\Box p \supset p) \supset \Box p \quad \text{(Löb's axiom)} \\ \mathcal{G}rz: & \Box(\Box(p \supset \Box p) \supset p) \supset p \\ \mathcal{G}o: & \Box(\Box(p \supset \Box p) \supset p) \supset \Box p \end{array}$$

As we have already noted, the Gödel–Löb provability logic GL can be axiomatised as $K\mathcal{G}$. The logics Grz and Go are axiomatised as $K\mathcal{G}rz$ and $K\mathcal{G}o$ respectively. The following results are well-known:

$$\begin{array}{ccc} 4 \in GL & \mathcal{G}o \in GL & \\ 4 \in Go & & \\ 4 \in Grz & T \in Grz & \mathcal{G}o \in Grz \end{array}$$

An alternative axiomatisation for Grz is $KT\mathcal{G}o$ [31].

The logic GL is sound and complete for the class of frames that are transitive and contain no ∞-$\mathcal{R}$-chains [64]. It is known that Go is sound and complete for the class $\mathfrak{F}$ of frames that are transitive, contain no proper clusters, and contain no proper ∞-$\mathcal{R}$-chains [30]. Indeed, it is easy to see how to construct a countermodel to show that $T \notin Go$. Finally, Grz is sound and complete for the subclass of frames in $\mathfrak{F}$ satisfying reflexivity [30].

The similarity of the axioms $\mathcal{G}rz$ and $\mathcal{G}o$ with Löb's axiom $\mathcal{G}$ — which leads to a similarity in the corresponding sequent rules — suggests that the proof of cut-elimination for GL may be adapted to the logics Grz and Go. As shown in [2], a sequent calculus $GrzS$ for Grz can be obtained by replacing the GLR rule in GLS with the following modal rules:

$$\frac{B, X \Rightarrow Y}{\Box B, X \Rightarrow Y}\ GRZa \qquad \frac{\Box X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B}\ GRZb$$

Notice the similarity of the $GRZb$ rule to the GLR rule — in particular, in both rules the $\Box B$ formula passes from the succedent of the conclusion sequent to the antecedent of the premise sequent. A semantic proof of cut-elimination for this calculus is given in [2]. Borga and Gentilini [9] present a syntactic proof of cut-elimination for this calculus where the sequents are built from sets. This proof bears much similarity to the proof for GLS. The extension to sequents built from multisets is straightforward.

A sequent calculus GoS for Go is easily obtained from GLS by replacing the GLR rule with the following:

$$\frac{\Box X, X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B}\ GoR$$

There is no existing cut-elimination procedure for Go. We present a solution in Chapter 3. Some of the ideas in the proof for GL are employed, although the transformations for Go seem to require a deeper analysis of the derivation structures than the proofs for GL and Grz.

The logic GL_{lin} is obtained by the addition of the following axiom to GL:

$$\Box(p \wedge \Box p \supset \Box q) \vee \Box(q \wedge \Box q \supset \Box p)$$

Valentini [72] has extended the ideas in the proof of cut-elimination for GL to obtain cut-elimination for this logic for a calculus where the sequents are built from sets.

The logics $S4.3.1$ and $S4Dbr$ are axiomatised respectively, as $KT4.3Dum$ and $KT4Dbr$ [30], where the axioms

$$\begin{aligned} Dum: &\quad \Box(\Box(p \supset \Box p) \supset p) \supset (\Diamond\Box p \supset \Box p) \\ Dbr: &\quad \Box(\Box(p \supset \Box p) \supset p) \supset (\Box\Diamond\Box p \supset \Box p) \end{aligned}$$

have a similar form to the Go axiom. In the case of $S4.3.1$, Shimura [67] has presented a proof of cut-elimination that requires a cutfree calculus for the logic $S5$ as an oracle. However this is a rather strong requirement as all the existing cutfree systems for $S5$ are obtained via modification of the traditional sequent calculus [66, 20, 36, 56]. Thus it would be interesting to see if we can exploit the syntactic similarity of the axioms to the Go axiom — which leads to a similarity in the corresponding sequent rules — in order to obtain cut-elimination for traditional sequent calculi for these logics.

Form of annotated derivation δ	$\Phi_{\Box B}[\delta]$
$(\Box B)^{*} \Rightarrow \Box B$	$(\Box B)^{\circ} \Rightarrow \Box B$
$\dfrac{\dfrac{\{\pi\}_1^r}{G, (\Box B)^{n-1} \Rightarrow H}}{G, (\Box B)^{*n} \Rightarrow H}\, LW(\Box B)$	$\dfrac{\Phi_{\Box B}\left[\frac{\{\pi\}_1^r}{G,(\Box B)^{*n-1} \Rightarrow H}\right]}{G, (\Box B)^{\circ}, (\Box B)^{*n-1} \Rightarrow H}\, LW(\Box B)$
$\dfrac{\dfrac{\{\pi\}_1^r}{G, (\Box B)^{n+1} \Rightarrow H}}{G, (\Box B)^{*n} \Rightarrow H}\, LC(\Box B)$	$\dfrac{\Phi_{\Box B}\left[\frac{\{\pi\}_1^r}{G,(\Box B)^{*n+1} \Rightarrow H}\right]}{G, (\Box B)^{*n} \Rightarrow H}\, LC(\Box B)$
$\dfrac{\dfrac{\{\pi\}_1^r}{G', (\Box B)^{n} \Rightarrow H'}}{G, (\Box B)^{*n} \Rightarrow H}\, \rho \neq GLR$	$\dfrac{\Phi_{\Box B}\left[\frac{\{\pi\}_1^r}{G',(\Box B)^{*n} \Rightarrow H'}\right]}{G, (\Box B)^{*n} \Rightarrow H}\, \rho$
$\dfrac{\dfrac{\{\pi\}_1^r}{\Box G, G, (\Box B)^{n}, B^n, \Box A \Rightarrow A}}{\Box G, (\Box B)^{*n} \Rightarrow \Box A}\, \text{GLR}$	$\dfrac{\frac{\{\pi\}_1^r}{\Box G, G, (\Box B)^{n}, B^n, \Box A \Rightarrow A}}{\Box G, (\Box B)^{\circ n} \Rightarrow \Box A}\, \text{GLR}$
$\dfrac{\dfrac{\{\pi\}_1^r}{G', (\Box B)^{n} \Rightarrow H'} \qquad \dfrac{\{\pi'\}_1^s}{G'', (\Box B)^{n} \Rightarrow H''}}{G, (\Box B)^{*n} \Rightarrow H}\, \rho \neq cut$	
	$\dfrac{\Phi_{\Box B}\left[\frac{\{\pi\}_1^r}{G',(\Box B)^{*n} \Rightarrow H'}\right] \qquad \Phi_{\Box B}\left[\frac{\{\pi'\}_1^s}{G'',(\Box B)^{*n} \Rightarrow H''}\right]}{G, (\Box B)^{*n} \Rightarrow H}\, \rho$
antecedent of $ES(\delta)$ does not contain a $(\Box B)^{*}$ formula occurrence	δ

Table 2.2: Definition of $\Phi_{\Box B}$. Multisets G and $\Box G$ contain no occurrences of annotated formulae. An annotated derivation δ in the left column is mapped under $\Phi_{\Box B}$ to the annotated derivation in the right column. Due to space restrictions, in the case where ρ is a binary rule, δ and $\Phi_{\Box B}[\delta]$ appear on separate rows.

Chapter 3

Syntactic Cut-elimination for *Go*

We present a syntactic proof of cut-elimination for the logic *Go*. The logic has a syntactically similar axiomatisation to Grzegorczyk's logic *Grz* and provability logic *GL*. In fact, *Go* can be viewed as the non-reflexive counterpart of *Grz*, and *GL* can be viewed as the irreflexive counterpart of *Grz*. Although proofs of cut-elimination for *GL* and *Grz* have appeared in the literature, to our knowledge, this is the first proof of cut-elimination for *Go*. The proof seems to require a deeper analysis of the derivation structures than the proofs for *GL* and *Grz*, and new transformations are developed here.

3.1 Introduction

The logic *Go* is the smallest normal modal logic containing K and the schemata $\Box p \supset \Box\Box p$ and $\Box(\Box(p \supset \Box p) \supset p) \supset \Box p)$. The logic is sound and complete with respect to the class of transitive frames with no proper clusters and no proper ∞-$\mathcal{R}$-chains [30] (see Section 2.8.1 for a definition of these terms), and it is a proper subsystem of both Gödel-Löb logic *GL* (also known as provability logic) and Grzegorzyk's logic *Grz*. A survey of some results on *Go* can be found in Litak [44], where the logic is called the weak Grzegorczyk logic *wGrz*.

A sequent calculus *GoS* for *Go* (see Table 3.1) can be obtained by the addition of the following modal rule *GoR* to a suitable calculus for classical propositional logic:

$$\frac{\Box X, X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B}\, GoR$$

Observe that *GoS* contains the cut-rule. Showing that it is always possible via constructive transformation to eliminate the cuts in a given derivation to obtain a cutfree derivation of the same endsequent is called *syntactic cut-elimination*. It is one of the most important results in the proof theory of a logic and the existence of such a transformation is a highly desirable property for a sequent calculus. The first such proof was given by Gentzen [25] who recognised the importance of a constructive procedure in his celebrated *Hauptsatz* or 'main theorem' where syntactic cut-elimination is presented for the classical and intuitionistic sequent calculi LK and LJ respectively. A proof has been presented for Grz [9], and while there has been some controversy [51] regarding Valentini's [71] proof for GL, the issues are now resolved [32]. Here we show syntactic cut-elimination for GoS. To our knowledge, this is the first proof of syntactic cut-elimination for Go. We observe that cut-elimination for Go is not just a simple variation of the proofs for GL and Grz. Indeed, although Valentini's [71] transformations for GL remain an inspiration for our transformations, the proof presented here appears to generalise the methods used for GL and Grz. In particular, new transformations are introduced, and the proof uses a quaternary induction measure (three induction variables suffice for GL and Grz).

In the *Hauptsatz*, Gentzen relied on a primary induction on the degree of the cut-formula and secondary induction on cut-height. Suppose that cut_1 and cut_2 denote two occurrences of the cut-rule in some derivation. Write $cut_1 < cut_2$ to mean that cut_1 is less than cut_2 under the above measure. If we attempt a proof for Go following the proof of the *Hauptsatz* we quickly find that the only case deserving special attention is the case when the cut-formula is principal in both premises by GoR. Consider the following derivation where we assume without loss of generality that both premises of cut_0 are cutfree:

$$\dfrac{\dfrac{\Box X, X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B}\,GoR \qquad \dfrac{\Box B, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}{\Box B, \Box U \Rightarrow \Box C}\,GoR}{\Box X, \Box U \Rightarrow \Box C}\,cut_0$$

It is not obvious how to proceed from here. However, making use of the cut-rule observe that it is easy to construct a derivation of $\Box X, X \Rightarrow B$ from the left premise of cut_0. Indeed, the following suffices:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box X \Rightarrow \Box B}{\Box X, \Box(B \supset \Box B), B \Rightarrow \Box B}\,LW^*}{\Box X, \Box((B \supset \Box B) \supset \Box(B \supset \Box B)) \Rightarrow B \supset \Box B}\,R\supset}{\Box X \Rightarrow \Box(B \supset \Box B)}\,GoR \qquad \Box X, X, \Box(B \supset \Box B) \Rightarrow B}{\Box X, \Box X, X \Rightarrow B}\,cut}{\Box X, X \Rightarrow B}\,LC^*$$

If we could obtain a *cutfree* derivation of $\Box X, X \Rightarrow B$ then we may proceed

$$\dfrac{\dfrac{\dfrac{\Box X, X \Rightarrow B \qquad \dfrac{\Box X \Rightarrow \Box B \qquad \Box B, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}{\Box X, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}\,cut_1}{\Box X, \Box X, X, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}\,cut_2}{\Box X, X, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}\,LC^*}{\Box X, \Box U \Rightarrow \Box C}\,GoR$$

where, $cut_1 < cut_0$ and $cut_2 < cut_0$. The result would then follow directly from the induction hypothesis. This situation parallels the approach to cut-elimination for the calculi GLS for GL [71, 32, 8] and $GrzS$ for Grz [9]. In GLS for example, it is sufficient to obtain a derivation of $\Box X, X \Rightarrow B$ from a derivation of $\Box X, X, \Box B \Rightarrow B$. In $GrzS$, a derivation of $\Box X \Rightarrow B$ from a derivation for $\Box X, \Box(B \supset \Box B) \Rightarrow B$ suffices. Thus the obvious approach for GoS would be to draw on the syntactic proofs of cut-elimination for GLS and $GrzS$. We discuss the difficulties in adapting those proofs to GoS in Section 3.4.

Finally, we remind the reader that it is straightforward to show that the cut-rule is redundant by proving that the calculus without the cut-rule is sound and complete for the frame semantics of Go (see [1]). However the drawback of such a semantic (as opposed to syntactic) proof is that we have no effective method of constructing the cutfree derivation.

3.2 Basic definitions and notation

Formulae are constructed in the usual way from propositional variables using the logical connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication) and the modal operator $\Box$. Propositional variables are written using $p, q, \ldots$ and formulae are denoted by $A, B, C, \ldots$. Multisets of formulae are denoted by $X, Y, \ldots$. We write $\boxtimes A$ to denote the multiset $\langle \Box A, A \rangle$. Let X be the multiset $\langle A_1, \ldots, A_n \rangle$. Then we write $\Box X$ and $\boxtimes X$ to mean the following multisets respectively:

$$\langle \Box A_1, \ldots, \Box A_n \rangle \qquad\qquad \langle \Box A_1, \ldots, \Box A_n, A_1, \ldots, A_n \rangle$$

The notation A^m denotes the multiset $\langle A_1, \ldots, A_m \rangle$. A *sequent* is a tuple (X, Y) of multisets X and Y and is written $X \Rightarrow Y$. The symbols $\cup$ and $\subseteq$ are used to denote multiset union and the multiset inclusion relation respectively. The multiset X and Y are called respectively the *antecedent* and *succedent* of the sequent. The sequent calculus GoS is defined in Table 3.1.

Initial sequents: $A \Rightarrow A$ for each formula A

Logical rules:

$$\frac{X \Rightarrow Y, A}{X, \neg A \Rightarrow Y}\, L\neg \qquad \frac{A, X \Rightarrow Y}{X \Rightarrow Y, \neg A}\, R\neg$$

$$\frac{A_i, X \Rightarrow Y}{A_1 \wedge A_2, X \Rightarrow Y}\, L\wedge \qquad \frac{X \Rightarrow Y, A_1 \qquad X \Rightarrow Y, A_2}{X \Rightarrow Y, A_1 \wedge A_2}\, R\wedge$$

$$\frac{A_1, X \Rightarrow Y \qquad A_2, X \Rightarrow Y}{A_1 \vee A_2, X \Rightarrow Y}\, L\vee \qquad \frac{X \Rightarrow Y, A_i}{X \Rightarrow Y, A_1 \vee A_2}\, R\vee$$

$$\frac{X \Rightarrow Y, A \qquad B, X \Rightarrow Y}{A \supset B, X \Rightarrow Y}\, L\supset \qquad \frac{A, X \Rightarrow Y, B}{X \Rightarrow Y, A \supset B}\, R\supset$$

Modal rule:

$$\frac{\Box X, X, \Box(A \supset \Box A) \Rightarrow A}{\Box X \Rightarrow \Box A}\, GoR$$

Structural rules:

$$\frac{X \Rightarrow Y}{A, X \Rightarrow Y}\, LW \qquad \frac{X \Rightarrow Y}{X \Rightarrow Y, A}\, RW$$

$$\frac{A, A, X \Rightarrow Y}{A, X \Rightarrow Y}\, LC \qquad \frac{X \Rightarrow Y, A, A}{X \Rightarrow Y, A}\, RC$$

Cut-rule:

$$\frac{X \Rightarrow Y, A \qquad A, U \Rightarrow W}{X, U \Rightarrow Y, W}\, cut$$

Table 3.1: The sequent calculus GoS. Note: $i \in \{1, 2\}$ in the rules $L\wedge$ and $R\vee$.

A *derivation* (in GoS) is defined recursively with reference to Table 3.1 in the usual manner as follows:

(i) for any formula A, the initial sequent $A \Rightarrow A$ is a derivation, and

(ii) an application of a logical, modal, structural or cut-rule to derivations concluding its premise(s) is a derivation.

For the logical and structural rules in GoS, the multisets X and Y are called the *context*. As usual [70], in the conclusion of each of these rules, the formula not in the context is called the *principal formula*. For the GoR rule in Table 3.1, the $\Box A$ in the succedent of the conclusion sequent is the principal formula. Furthermore, the formula A in that rule is called the *diagonal formula*. A formula occurring in some sequent in a derivation is called *principal* if it is the principal formula of the rule deriving that sequent. We sometimes write $\rho(A)$ to indicate that the rule ρ makes A principal.

In the cut-rule in Table 3.1, the formula A is the *cut-formula*. A derivation is said to be *cutfree* if it contains no instances of the cut-rule. Viewing a derivation as a tree, we call the root of the tree the *endsequent* of the derivation. We use the phrase 'upwards' informally to mean the direction from the endsequent to the initial sequents. 'Downwards' is the direction towards the endsequent. The phrases 'above' and 'below' are used with respect to these directions. If there is a derivation with endsequent $X \Rightarrow Y$ we say that $X \Rightarrow Y$ is *derivable* in *GoS*.

Let $\bigwedge X$ ($\bigvee Y$) denote the conjunction (disjunction) of all formula occurrences in X (Y). It is straightforward to show that a sequent $X \Rightarrow Y$ is derivable in *GoS* iff the formula $\bigwedge X \supset \bigvee Y$ is a theorem of the logic *Go*. In other words, *GoS* is sound and complete with respect to *Go* and thus *GoS* is a sequent calculus for *Go*. We observe that the cut-elimination result shows that the calculus minus the cut-rule is sound and complete for *Go*.

Finally we define the height, cut-height, and degree of a formula in the standard manner.

Definition 3.1 (height, cut-height, degree) *The* height $h(\tau)$ *of a derivation τ is the greatest number of successive applications of rules in it plus one. The* cut-height s *of an instance of the cut-rule with premise derivations τ_1 and τ_2 is $h(\tau_1) + h(\tau_2)$. The* degree $|A|$ *of a formula A is defined as the number of symbol occurrences in A from* $\{\Box, \neg, \wedge, \vee, \supset\}$

3.2.1 Preliminary results

Lemma 3.2 (height-preserving invertibility of $L\supset$) *Suppose that τ is a cutfree derivation of $A \supset B, X \Rightarrow Y$. There there is an effective height-preserving transformation to cutfree derivations of $X \Rightarrow Y, A$ and $B, X \Rightarrow Y$.*

Proof. Because *GoS* contains contraction rules, we actually need to prove the stronger statement: if τ is a cutfree derivation of $(A \supset B)^{m+1}, X \Rightarrow Y$ then there are cutfree derivations of $A^{m+1}, X \Rightarrow Y$ and $X \Rightarrow B^{m+1}, Y$. The argument is a standard induction on the height of τ so we omit the details. Q.E.D.

Lemma 3.3 *Let τ be a cutfree derivation of $X, \Box(B \supset \Box B)^{m+1} \Rightarrow Y$. Then there is an effective transformation to a cutfree derivation τ' of $X, (\Box B)^{m+1} \Rightarrow Y$.*

Proof. Induction on the height of τ. Consider the last rule in τ.

For example, consider the case when the last rule in τ is the *GoR* rule:

$$\frac{\boxtimes X, \Box(B \supset \Box B)^{m+1}, (B \supset \Box B)^{m+1}, \Box(C \supset \Box C) \Rightarrow C}{\Box X, \Box(B \supset \Box B)^{m+1} \Rightarrow \Box C}$$

By height-preserving invertibility of $L\supset$ (Lemma 3.3) we have a derivation of

$$\boxtimes X, \Box(B \supset \Box B)^{m+1}, (\Box B)^{m+1}, \Box(C \supset \Box C) \Rightarrow C$$

Applying the induction hypothesis we obtain

$$\boxtimes X, (\Box B)^{m}, (\Box B)^{m}, \Box(C \supset \Box C) \Rightarrow C$$

The result follows from repeated application of the *LC* rule.

As another example, consider when the last rule in τ is *LC*:

$$\frac{X, \Box(B \supset \Box B)^{m+2} \Rightarrow Y}{X, \Box(B \supset \Box B)^{m+1} \Rightarrow Y}$$

From the induction hypothesis we can obtain a derivation of $X, (\Box B)^{m+2} \Rightarrow Y$. The result follows from an application of the *LC* rule.

The other cases are similar. Q.E.D.

We will use the diagonal formula as the label for a *GoR* rule, writing "C is a *GoR* rule in τ" to refer to an occurrence of a *GoR* rule in τ with diagonal formula C. Although it is certainly possible for a derivation to contain multiple *GoR* rule occurrences with the identical diagonal formula, we will ensure that the context identifies the intended occurrence. This labelling will greatly simplify the notation.

Let C_1 and C_2 denote two different occurrences of the *GoR* rule in τ. We say that C_1 *is above* C_2 if, tracing upwards, there is a path upwards in τ from C_2 to C_1.

We say that *boxes persist upwards* in τ if, for all occurrences C_1 and C_2 of *GoR* in τ (with conclusion sequents $\Box X_1 \Rightarrow \Box A_1$ and $\Box X_2 \Rightarrow \Box A_2$ say), C_1 is above C_2 implies that $\Box X_2 \cup \langle \Box(A_2 \supset \Box A_2) \rangle \subseteq \Box X_1$.

The rule immediately above the endsequent in a non-trivial derivation (ie the last rule in the derivation) is called the *final* rule. A derivation τ ending as $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B /^{GoR} \Box X \Rightarrow \Box B$ (so the final rule is *GoR* with diagonal formula B) is called *implication-forced* if every occurrence of *GoR* in τ apart from the final rule is preceeded by $L\supset$ $(B \supset \Box B)$. In other words, every *GoR* rule with the exception of the final rule occurs in the context

$$\dfrac{\dfrac{\Box(B\supset\Box B),\Box X,X,\Box(A\supset A)\Rightarrow A,B \qquad \Box B,\Box(B\supset\Box B),\Box X,X,\Box(A\supset A)\Rightarrow A}{\Box(B\supset\Box B),B\supset\Box B,\Box X,X,\Box(A\supset A)\Rightarrow A}\,L{\supset}}{\Box(B\supset\Box B),\Box X\Rightarrow\Box A}\,GoR$$

Definition 3.4 (normal derivation) *A cutfree derivation τ with final rule GoR is called* normal *if boxes persist upwards and τ is implication-forced.*

The following lemma shows that any cutfree derivation where the final rule is GoR can be transformed into a normal derivation with the same endsequent.

Lemma 3.5 (normal derivation lemma) *Let τ be a cutfree derivation ending as $\boxtimes X,\Box(B\supset\Box B)\Rightarrow B/^{GoR}\Box X\Rightarrow\Box B$. Then there is an effective transformation to a normal derivation τ' ending as $\boxtimes X,\Box(B\supset\Box B)\Rightarrow B/^{GoR}\Box X\Rightarrow\Box B$.*

Proof. We will transform τ to ensure that boxes persist upwards, and then transform the resulting derivation to ensure that it is implication-forced.

To illustrate the transformation required to ensure that boxes persist upwards, suppose that τ has the following form:

$$\begin{array}{c}
p_i\Rightarrow p_i\\
\vdots\\
\dfrac{\boxtimes Y_r,\Box(A_r\supset\Box A_r)\Rightarrow A_r}{\Box Y_r\Rightarrow\Box A_r}\,GoR\\
\vdots\\
\dfrac{\boxtimes Y_{r-1},\Box(A_{r-1}\supset\Box A_{r-1})\Rightarrow A_{r-1}}{\Box Y_{r-1}\Rightarrow\Box A_{r-1}}\,GoR\\
\vdots\\
\dfrac{\boxtimes Y_1,\Box(A_1\supset\Box A_1)\Rightarrow A_1}{\Box Y_1\Rightarrow\Box A_1}\,GoR
\end{array}$$

The idea is to transform the proof to the following using appropriate weakening and contraction rules — LW^* indicates some number of applications of the LW rule (read the following proof diagram downwards from the initial sequent):

$$\dfrac{p_i \Rightarrow p_i}{\boxtimes Y_1, \boxtimes(A_1 \supset \Box A_1), \ldots, \boxtimes Y_{r-1}, \boxtimes(A_{r-1} \supset \Box A_{r-1}), \boxtimes Y_r, p_i \Rightarrow p_i}$$

$$\vdots$$

$$\dfrac{\dfrac{\boxtimes Y_1, \boxtimes(A_1 \supset \Box A_1), \ldots, \boxtimes Y_{r-1}, \boxtimes(A_{r-1} \supset \Box A_{r-1}), \boxtimes Y_r, \Box(A_r \supset \Box A_r) \Rightarrow A_r}{\Box Y_1, \Box(A_1 \supset \Box A_1), \ldots, \Box Y_{r-1}, \Box(A_{r-1} \supset \Box A_{r-1}), \Box Y_r \Rightarrow \Box A_r}\,GoR}{\boxtimes Y_1, \boxtimes(A_1 \supset \Box A_1), \ldots, \boxtimes Y_{r-1}, \Box(A_{r-1} \supset \Box A_{r-1}), \Box Y_r \Rightarrow \Box A_r}\,LW^*$$

$$\vdots$$

$$\dfrac{\boxtimes Y_1, \boxtimes(A_1 \supset \Box A_1), \ldots, \boxtimes Y_{r-1}, \Box(A_{r-1} \supset \Box A_{r-1}) \Rightarrow A_{r-1}}{\Box Y_1, \Box(A_1 \supset \Box A_1), \ldots, \Box Y_{r-1} \Rightarrow \Box A_{r-1}}\,GoR$$

$$\vdots$$

$$\dfrac{\boxtimes Y_1, \Box(A_1 \supset \Box A_1) \Rightarrow A_1}{\Box Y_1 \Rightarrow \Box A_1}\,GoR$$

We omit the details as the proof is straightforward, if tedious.

In this manner, from τ we can obtain a derivation τ' ending as $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B /^{GoR} \Box X \Rightarrow \Box B$ such that boxes persist upwards in τ'. Thus every occurrence of GoR aside from the final rule has the form

$$\dfrac{\Box(B \supset \Box B), B \supset \Box B, \Box X, X, A \supset \Box A \Rightarrow A}{\Box(B \supset \Box B), \Box X \Rightarrow \Box A}$$

By invertibility of $L\supset$ (Lemma 3.2) we can obtain cutfree derivations δ_1 and δ_2 of, respectively,

$$\Box(B \supset \Box B), \Box X, X, A \supset \Box A \Rightarrow A, B$$

and

$$\Box(B \supset \Box B), \Box B, \Box X, X, A \supset \Box A \Rightarrow A$$

Then replace the subderivation ending $\Box(B \supset \Box B), \Box X \Rightarrow \Box A$ in τ with the following derivation:

$$\dfrac{\dfrac{\begin{matrix}\delta_1 \\ \Box(B \supset \Box B), \Box X, X, A \supset \Box A \Rightarrow A, B\end{matrix} \qquad \begin{matrix}\delta_2 \\ \Box(B \supset \Box B), \Box B, \Box X, X, A \supset \Box A \Rightarrow A\end{matrix}}{\Box(B \supset \Box B), B \supset \Box B, \Box X, X, A \supset \Box A \Rightarrow A}\,L\supset}{\Box(B \supset \Box B), \Box X \Rightarrow \Box A}\,GoR$$

Apply this argument to all non-final GoR rules in τ' to obtain a cutfree derivation τ'' that is implication-forced. Observe that boxes persist upwards in τ'' because boxes persist upwards in τ'. It follows that τ'' is a normal derivation ending as $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B /^{GoR} \Box X \Rightarrow \Box B$. Q.E.D.

Let C be an arbitrary non-final GoR rule occurrence in the normal derivation τ. Because τ is implication-forced, C must appear as follows, with a $L\supset$ rule immediately above the GoR rule:

$$\dfrac{\dfrac{\Box(B\supset\Box B),\Box X, X, C\supset\Box C \Rightarrow C, B \qquad \Box(B\supset\Box B),\Box B,\Box X, X, C\supset\Box C\Rightarrow C}{\Box(B\supset\Box B), B\supset\Box B,\Box X, X, C\supset\Box C\Rightarrow C}\,L\supset}{\Box(B\supset\Box B),\Box X\Rightarrow\Box C}\,GoR$$

In the above context where $L\supset$ is immediately above an occurrence C of the GoR rule (remember that we use the diagonal formula C as a label for the GoR rule occurrence), we will write $\mathcal{S}_L(C)$ and $\mathcal{S}_R(C)$ respectively, to denote the left and right premise of that $L\supset$ rule occurrence. Now suppose that C_1 is an occurrence of the GoR rule above C. If C_1 occurs above the sequent $\mathcal{S}_L(C)$ then we say that C_1 is *left-above* C. Similarly, if C_1 occurs above the sequent $\mathcal{S}_R(C)$ then we say that C_1 is *right-above* C. If there is no GoR rule on the path between C_1 and C then we say that C_1 is *immediately* left-above (resp. right-above) C.

Definition 3.6 (topmost sequent) *Let τ be a normal derivation ending as $\boxtimes X, \Box(B\supset\Box B)\Rightarrow B/^{GoR}\Box X\Rightarrow\Box B$. A sequent $\mathcal{S}$ in derivation τ is called* topmost *if each $\Box B$ and $\Box(B\supset\Box B)$ formula occurring in the antecedent of $\mathcal{S}$ is introduced in every branch above $\mathcal{S}$ via the initial sequents $\Box B\Rightarrow\Box B$ and $\Box(B\supset\Box B)\Rightarrow\Box(B\supset\Box B)$ or weakening rules, and prior to encountering a GoR rule.*

Intuitively, tracing upwards along each branch of the derivation from the topmost sequent, we will encounter the initial sequents or weakening rules for $\Box B$ and $\Box(B\supset\Box B)$ *before* encountering a GoR rule. For example, consider the following proof diagram.

$$\dfrac{\dfrac{\dfrac{\vdots}{\Box(B\supset\Box B),\Box(C\supset\Box C)\Rightarrow C^{\dagger}}\,GoR}{\Box(B\supset\Box B),\Box(C\supset\Box C)\Rightarrow B, C} \qquad \dfrac{\dfrac{\vdots}{\Box(C\supset\Box C)\Rightarrow C}\,GoR}{\Box(B\supset\Box B),\Box B,\Box(C\supset\Box C)\Rightarrow C^{\ddagger}}\,LW}{\dfrac{\Box(B\supset\Box B), B\supset\Box B,\Box(C\supset\Box C)\Rightarrow C^{*}}{\Box(B\supset\Box B)\Rightarrow\Box C}\,GoR}\,L\supset$$
$$\vdots$$

The sequent marked with † is *not* a topmost sequent because there is a GoR rule immediately above it — thus violating the condition that, viewed upwards, the $\Box(B\supset\Box B)$ formula in the antecedent is introduced prior to encountering a GoR rule. However the sequent marked with ‡ *is* a topmost sequent because both the $\Box(B\supset\Box B)$ and $\Box B$ formulae in the antecedent are introduced via weakening and there is no GoR rule in-between ‡ and the weakening rules. Finally, the sequent marked with $*$ is *not* a topmost sequent because, tracing upwards, there is a branch above it (the left premise of $L\supset$) where the $\Box(B\supset\Box B)$ formula is *not* introduced via an initial sequent or weakening prior to encountering a *GoR* rule.

Lemma 3.7 (topmost sequent lemma) *Let τ be a normal derivation ending as $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B/^{GoR}\Box X \Rightarrow \Box B$ and suppose that $\Gamma \Rightarrow \Delta$ is a topmost sequent in τ. Then there is a cutfree derivation of $\Gamma^* \Rightarrow \Delta$ where Γ^* is the multiset obtained from Γ by deleting all occurrences of $\Box B$ and $\Box(B \supset \Box B)$.*

Proof. Since $\Gamma \Rightarrow \Delta$ is a topmost sequent, each $\Box B$ formula in Γ must have been introduced by a $LW(\Box B)$ weakening rule above $\Gamma \Rightarrow \Delta$, or it can be traced to a $\Box B \Rightarrow \Box B$ initial sequent. Similarly, each $\Box(B \supset \Box B)$ formula in Γ must have been introduced by a $LW(\Box(B \supset \Box B)$ weakening rule above $\Gamma \Rightarrow \Delta$, or it can be traced to a $\Box(B \supset \Box B) \Rightarrow \Box(B \supset \Box B)$ initial sequent.

A cutfree derivation of $\Gamma^* \Rightarrow \Delta$ can be obtained as follows.

Substitute any $LW(\Box B)$ or $LW(\Box(B \supset \Box B)$ weakening rules above the sequent $\Gamma \Rightarrow \Delta$ with $LW^*(\Box X)$, and substitute the derivation τ of $\Box X \Rightarrow \Box B$ for occurrences of the initial sequent $\Box B \Rightarrow \Box B$. Finally substitute the following derivation in place of the initial sequent $\Box(B \supset \Box B) \Rightarrow \Box(B \supset \Box B)$.

$$\dfrac{\dfrac{\dfrac{\Box X \Rightarrow \Box B}{B, \boxtimes X, \Box((B \supset \Box B) \supset \Box((B \supset \Box B))) \Rightarrow \Box B}\,LW^*}{\boxtimes X, \Box((B \supset \Box B) \supset \Box((B \supset \Box B))) \Rightarrow B \supset \Box B}\,R{\supset}}{\Box X \Rightarrow \Box(B \supset \Box B)}\,GoR$$

By inspection, the obvious derivation that can be obtained from these transformations is a cutfree derivation of $\Gamma^* \Rightarrow \Delta$. Q.E.D.

3.3 Cut-elimination for *Go*

In this section, we consider exclusively a normal derivation τ ending as follows:

$$\dfrac{\boxtimes X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B}\,GoR$$

Recall that the (GoR) rule ending τ is called the *final rule*.

Before we proceed, we remind the reader once more that an occurrence in τ of a *GoR* rule is referred to by the diagonal formula of that rule — even if there are multiple occurrences of *GoR* with identical diagonal formula in τ the context will make it clear which occurrence is meant. For example, the final rule in τ is a *GoR* rule with diagonal formula B so we refer to this occurrence as the final rule B.

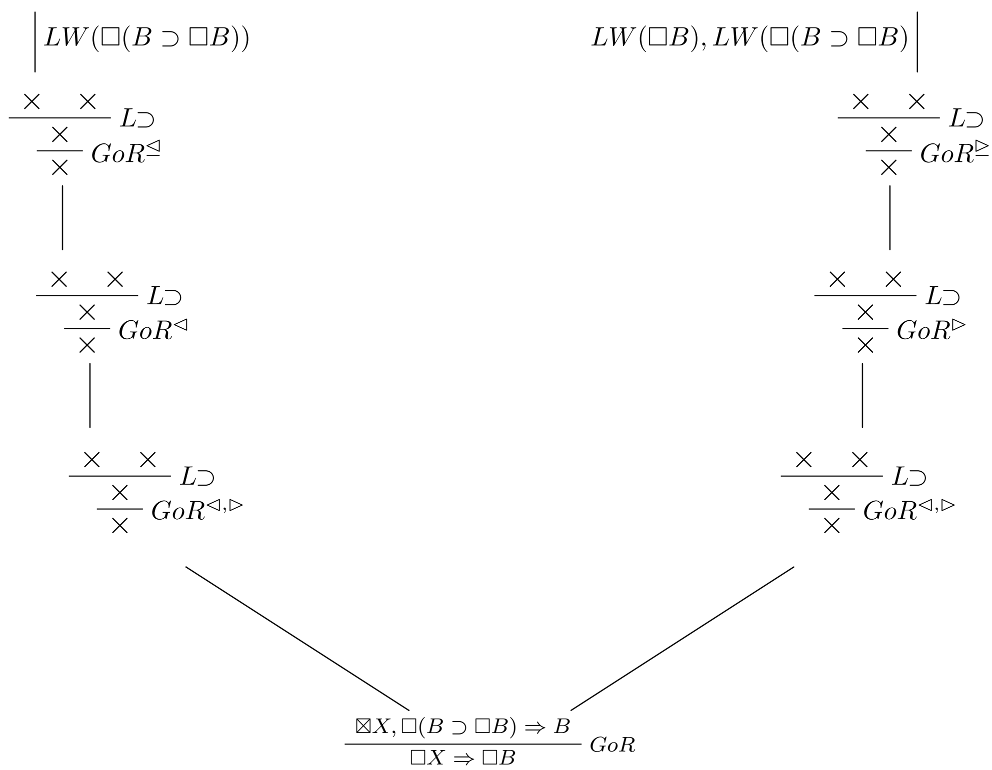


Figure 3.1: A schematic representation of a fragment of a normal derivation ending as $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B /^{GoR} \Box X \Rightarrow \Box B$. The solid lines represent portions of the derivation that are GoR-free. The symbol $\lhd$ denotes a leftflush rule and $\rhd$ denotes a rightflush rule (each wrt B). The symbols $\unlhd$ and $\unrhd$ respectively denote a MLL rule and a MRR rule (once again wrt to B).

Definition 3.8 (leftflush, rightflush rules) *Let C be some GoR rule in a normal derivation τ. The set of GoR rules that are* leftflush (rightflush) *rule wrt C is precisely the set defined by the following recursive definition:*

(i) any GoR rule immediately above the final rule B is leftflush and rightflush wrt B

(ii) any GoR rule that is left-above (right-above) a GoR rule C is leftflush (rightflush) wrt C

(iii) any GoR rule that is left-above (right-above) a rule that is itself leftflush (rightflush) wrt C is said to to be leftflush (rightflush) wrt C

Intuitively, D is leftflush wrt C if D is encountered by repeatedly tracing through GoR rules left-above C. The intuition for rightflush is analogous. See Figure 3.1

for an illustration of these terms.

Notice that it is never the case that C is leftflush (rightflush) wrt itself. Also the final rule B is not leftflush (rightflush) wrt to any rule, although from (i) every rule immediately above the final rule is both leftflush and rightflush wrt B.

The following observation is crucial for the success of the proof.

Suppose that the *GoR* rule C in a normal derivation τ is leftflush wrt the final rule B. Then the rule C has a conclusion sequent of the form $\Box Y, \Box(B \supset \Box B)^{k+1} \Rightarrow \Box C$, where $\Box Y$ contains no $\Box B$ or $\Box(B \supset \Box B)$ formulae that are parametric ancestors of the diagonal formula in the final rule. In particular, this means that

$$\mathcal{S}_L(C) = \boxtimes Y, \Box(B \supset \Box B)^{k+1} \Rightarrow B^{k+1}, C$$
$$\mathcal{S}_R(C) = \boxtimes Y, \Box(B \supset \Box B)^{k+1}, \Box B^{k+1} \Rightarrow C$$

Definition 3.9 (depth) *The* depth *of a GoR rule ρ in a derivation τ is the number of GoR rules between the premise of ρ and the endsequent of τ.*

For example, for a derivation concluding with a *GoR* rule ρ, the depth of ρ in that derivation is 1.

Definition 3.10 (left-, right-topmost) *Suppose that τ is a normal derivation. A GoR rule C in τ is called* left-topmost (right-topmost) *if the sequent $\mathcal{S}_L(C)$ ($\mathcal{S}_R(C)$) is a topmost sequent.*

Definition 3.11 (MLL rule wrt C) *Let C be some occurrence of the GoR rule in a normal derivation τ and suppose that D is a leftflush rule (wrt C) and left-topmost rule. Then D is called an MLL* rule wrt *C if there is no leftflush (wrt C) and left-topmost rule below D.*

The term MLL stands for 'minimal leftflush left-topmost'. Although a normal derivation may contain distinct MLL rules (wrt C) A and B, it must be the case that A and B lie on different branches above C. Intuitively, an MLL rule is the leftflush left-topmost rule closest to C. Similarly we define

Definition 3.12 (MRR rule wrt C) *Let C be some occurrence of the GoR rule in a normal derivation τ and suppose that D is a rightflush rule (wrt C) and right-topmost rule. Then D is said to be an MLL* rule wrt *C if there is no rightflush (wrt C) and right-topmost rule below C.*

The term MRR stands for 'minimal rightflush right-topmost'. See Figure 3.1 for an illustration of the terms MLL and MRR.

Definition 3.13 (leftwidth, rightwidth) *The* leftwidth $lw(\tau)$ *is defined as the sum of the depths of each MLL rule (wrt the final rule) in a normal derivation τ. Similarly, define the* rightwidth $rw(\tau)$ *as the sum of the depths of each RLL rule (wrt the final rule) in τ.*

Definition 3.14 (width of cut) *The* width *of an instance of cut is defined when the left premise of cut is principal by the GoR rule, as the leftwidth $lw(\delta)$ of the left premise derivation δ.*

To simplify the notation, in the following we will omit writing the full antecedent of each sequent, dropping context terms such as $\Box X$, and ignore formula multiplicities. For example, instead of $\Box X, X, \Box(B \supset \Box B) \Rightarrow B$ we write $\Box(B \supset \Box B) \Rightarrow B$. Similarly, the rule

$$\frac{\boxtimes X, \boxtimes Y, \boxtimes(B \supset \Box B)^{m+1}, \Box(C \supset \Box C) \Rightarrow C}{\Box X, \Box Y, \Box(B \supset \Box B)^{m+1} \Rightarrow \Box C}$$

becomes

$$\frac{\boxtimes(B \supset \Box B), \Box(C \supset \Box C) \Rightarrow C}{\Box(B \supset \Box B) \Rightarrow \Box C}$$

It is straightforward to extend the proof to the general case.

Let τ be a normal derivation ending as $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B/^{GoR}\Box X \Rightarrow \Box B$.

Part I. obtain $\boxtimes(C \supset \Box C) \Rightarrow B$ for an arbitrary MLL rule C wrt B

First suppose that $lw(\tau) = 0$. Then it must be the case that there are no MLL rules wrt the final rule. This implies that the $\Box(B \supset \Box B)$ formula is introduced via weakening or initial sequents immediately above $\Box(B \supset \Box B) \Rightarrow B$ in every branch. Thus, $\Box(B \supset \Box B) \Rightarrow B$ is already a topmost sequent. We can immediately obtain a cutfree derivation of $\boxtimes X, X \Rightarrow B$ from Lemma 3.7.

Now suppose that $lw(\tau) > 0$. We can schematically represent τ as follows — an *arbitrary* MLL rule C wrt B is highlighted in bold:

$$\dfrac{\dfrac{\overset{\text{(topmost sequent)}}{\mathcal{S}_2 = \Box(B \supset \Box B), \Box(C \supset \Box C) \Rightarrow B, C} \qquad \overset{\vdots}{\mathcal{S}_3 = \Box(B \supset \Box B), \Box B, \Box(C \supset \Box C) \Rightarrow C}}{\dfrac{B \supset \Box B, \Box(B \supset \Box B), \Box(C \supset \Box C) \Rightarrow C}{\mathcal{S}_1 = \Box(B \supset \Box B) \Rightarrow \Box C}\,\textbf{GoR}} \atop {\vdots \atop \Box(B \supset \Box B) \Rightarrow B}}{\Rightarrow \Box B}$$

First, replace the subderivation of sequent $\mathcal{S}_1$ in τ with the derivation

$$\frac{\Box C \Rightarrow \Box C}{\Box(B \supset \Box B), \Box C \Rightarrow \Box C} LW$$

thus deleting an MLL rule from τ. We can then obtain a derivation τ' of $\Box C \Rightarrow \Box B$. Although the GoR rule ρ below $\Box(B \supset \Box B), \Box C \Rightarrow \Box C$ in τ' may now be an MLL rule (observe that ρ could not have been an MLL rule in τ because then C would not have been an MLL), by inspection it is clear that the depth of ρ is strictly less than the depth of the MLL rule C in τ. Thus $lw(\tau') < lw(\tau)$. From $\Box(B \supset \Box B) \Rightarrow B$ we can obtain a derivation of $\Box B \Rightarrow B$ (Lemma 3.3), and so we have

$$\frac{\Box C \Rightarrow \Box B \qquad \Box B \Rightarrow B}{\Box C \Rightarrow B} cut$$

where the cut has width $< lw(\tau)$.

From $\Box(B \supset \Box B), \Box(C \supset \Box C) \Rightarrow B, C$ (this is $\mathcal{S}_2$) by Lemma 3.7 we can obtain directly a derivation of

$$\Box(C \supset \Box C) \Rightarrow B, C$$

Using $\Box C \Rightarrow B$ and the above sequent, from $L\supset$ we get

$$\boxtimes(C \supset \Box C) \Rightarrow B \tag{3.1}$$

Part II. transform τ so $\mathcal{S}_R(D)$ is a topmost sequent

To show this we will prove a stronger statement $(*)$:

> If D is either the MLL rule C or a rightflush rule wrt to the MLL rule C in τ, then we can transform the subderivation above D in τ so that $\mathcal{S}_R(D)$ is a topmost sequent.

Let δ denote the subderivation in τ deriving the conclusion sequent of D. The proof is by induction on the rightwidth $rw(\delta)$ of δ.

Case I. Suppose that D is the MLL rule C.

If $rw(\delta) = 0$, then it must be the case that there are no MRR rules wrt C. This implies that every $\Box(B \supset \Box B)$ formula in the antecedent of $\mathcal{S}_R(C)$ ($=\mathcal{S}_3$) is introduced via weakening or initial sequents above $\mathcal{S}_R(C)$ in every branch. Thus, $\mathcal{S}_R(C)$ is already a topmost sequent.

If $rw(\delta) > 0$, there must be a GoR rule F immediately right-above the MLL rule C (highlighted in bold for clarity):

$$\cfrac{\cfrac{\times \qquad \mathcal{S}_4 = \Box B, B, \Box(B \supset \Box B), \boxtimes(C \supset \Box C), \Box(F \supset \Box F) \Rightarrow F}{\Box B, B, \boxtimes(B \supset \Box B), \boxtimes(C \supset \Box C), \Box(F \supset \Box F) \Rightarrow F}\;L\supset}{\Box B, \Box(B \supset \Box B), \Box(C \supset \Box C) \Rightarrow \Box F}\;GoR$$

$$\vdots$$

$$\cfrac{\cfrac{\times \quad \Box(B \supset \Box B), \Box B, \Box(C \supset \Box C) \Rightarrow C}{\Box(B \supset \Box B), B \supset \Box B, \Box(C \supset \Box C) \Rightarrow C}\;L\supset}{\Box(B \supset \Box B) \Rightarrow \Box C}\;\mathbf{GoR}$$

Let δ' be the subderivation deriving $\Box B, \Box(B \supset \Box B), \Box(C \supset \Box C) \Rightarrow \Box F$. Since $rw(\delta') < rw(\delta)$, by the induction hypothesis it follows that $\mathcal{S}_R(F)$ (=$\mathcal{S}_4$) is a topmost sequent. Hence, from Lemma 3.7 we have a derivation of $B, \boxtimes(C \supset \Box C), \Box(F \supset \Box F) \Rightarrow F$. Now, making use of the derivation of (3.1) we obtained before, we have

$$\cfrac{\cfrac{\cfrac{\boxtimes(C \supset \Box C) \Rightarrow B \qquad B, \boxtimes(C \supset \Box C), \Box(F \supset \Box F) \Rightarrow F}{\boxtimes(C \supset \Box C), \Box(F \supset \Box F) \Rightarrow F}\;cut}{\Box(C \supset \Box C) \Rightarrow \Box F}\;GoR}{\Box(B \supset \Box B), \Box B, \Box(C \supset \Box C) \Rightarrow \Box F}\;LW$$

$$\vdots$$

$$\cfrac{\cfrac{\times \quad \Box(B \supset \Box B), \Box B, \Box(C \supset \Box C) \Rightarrow C}{\Box(B \supset \Box B), B \supset \Box B, \Box(C \supset \Box C) \Rightarrow C}\;L\supset}{\Box(B \supset \Box B) \Rightarrow \Box C}\;\mathbf{GoR}$$

Because of the left weakening rule we introduced above, the rightwidth of C in the above derivation is $< rw(\delta)$, and so by the induction hypothesis it follows that $\mathcal{S}_R(C)$ is a topmost sequent.

Case II. Suppose that D is a rightflush rule wrt C. If $rw(\delta) = 0$ then there are no MRR rules wrt D, so $\mathcal{S}_R(D)$ must be topmost. Else, if $rw(\delta) > 0$, then there must be a *GoR* rule G immediately right-above D:

$$\cfrac{\cfrac{\times \qquad \Box B, B, \Box(B \supset \Box B), \boxtimes(C \supset \Box C), \boxtimes(D \supset \Box D), \Box(G \supset \Box G) \Rightarrow G}{\Box B, B, \boxtimes(B \supset \Box B), \boxtimes(C \supset \Box C), \boxtimes(D \supset \Box D), \Box(G \supset \Box G) \Rightarrow G}\;L\supset}{\Box B, \Box(B \supset \Box B), \Box(C \supset \Box C), \Box(D \supset \Box D) \Rightarrow \Box G}\;GoR$$

$$\vdots$$

$$\cfrac{\cfrac{\times \quad \Box(B \supset \Box B), \Box B, \boxtimes(C \supset \Box C), \Box(D \supset \Box D) \Rightarrow D}{\boxtimes(B \supset \Box B), \boxtimes(C \supset \Box C), \Box(D \supset \Box D) \Rightarrow D}\;L\supset}{\Box(B \supset \Box B), \Box(C \supset \Box C) \Rightarrow \Box D}\;GoR$$

From the induction hypothesis and Lemma 3.7 we can obtain a derivation of

$$B, \boxtimes(C \supset \Box C), \boxtimes(D \supset \Box D), \Box(G \supset \Box G) \Rightarrow G$$

Once again, making use of the derivation of (3.1) we obtained before, we have

$$\dfrac{\dfrac{\dfrac{\boxtimes(C \supset \Box C) \Rightarrow B \qquad B, \boxtimes(C \supset \Box C), \boxtimes(D \supset \Box D), \Box(G \supset \Box G) \Rightarrow G}{\boxtimes(C \supset \Box C), \boxtimes(D \supset \Box D), \Box(G \supset \Box G) \Rightarrow G}\,cut}{\Box(C \supset \Box C), \Box(D \supset \Box D) \Rightarrow \Box G}\,GoR}{\Box(B \supset \Box B), \Box B, \Box(C \supset \Box C), \Box(D \supset \Box D) \Rightarrow \Box G}\,LW$$

$$\vdots$$

$$\dfrac{\dfrac{\times \quad \Box(B \supset \Box B), \Box B, \boxtimes(C \supset \Box C), \Box(D \supset \Box D) \Rightarrow D}{\boxtimes(B \supset \Box B), \boxtimes(C \supset \Box C), \Box(D \supset \Box D) \Rightarrow D}\,L\supset}{\Box(B \supset \Box B), \Box(C \supset \Box C) \Rightarrow \Box D}\,GoR$$

Because of the left weakening rule we introduced above, the rightwidth of D in the above derivation is $< rw(\delta)$, and so by the induction hypothesis it follows that $\mathcal{S}_R(D)$ is a topmost sequent.

We have proved all the cases for the inductive step and so $(*)$ is proved.

Part III. obtain a derivation of $\Rightarrow \Box B$ *with reduced leftwidth*

In Part II we showed how to obtain a derivation where $\mathcal{S}_R(C)$ is a topmost sequent. Then, by Lemma 3.7 we have a derivation of $\Box(C \supset \Box C) \Rightarrow C$. Finally,

$$\dfrac{\dfrac{\Box(C \supset \Box C) \Rightarrow C}{\Rightarrow \Box C}\,GoR}{\Box(B \supset \Box B) \Rightarrow \Box C}\,LW$$

Replace the subderivation of $\mathcal{S}_1$ in τ (see Part I) with the above derivation to ultimately obtain a derivation τ'' of $\Rightarrow \Box B$ where $lw(\tau'') < lw(\tau)$. Then the following cut has width $< lw(\tau)$:

$$\dfrac{\Rightarrow \Box B \qquad \Box B \Rightarrow B}{\Rightarrow B}\,cut$$

We have proved the following result.

Lemma 3.15 *Let τ be a normal derivation ending as $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B/^{GoR}\Box X \Rightarrow \Box B$. Then there is an effective transformation to a derivation τ' of $\Box X, X \Rightarrow B$, where each cut-rule in τ' has degree $< |\Box B|$, or degree $|\Box B|$ and width $< lw(\tau)$.*

Theorem 3.16 *Syntactic cut-elimination holds for GoS.*

Proof. Without loss of generality, let τ be a derivation containing a single instance *cut* of the cut-rule as the final rule. We need to show that there is a cutfree derivation of the identical sequent.

Primary induction on the degree of the cut-formula, secondary induction on the width of the left premise derivation of cut, and ternary induction on the cut-height. (Observe that the proof of Lemma 3.15 uses an induction on rightwidth, so this proof implicitly uses a quaternary induction measure). In the following, for instances cut_1 and cut_2 of the cut-rule, we write $cut_1 < cut_2$ to mean that cut_1 is less than cut_2 under the above measure.

When the cut-formula is not a boxed-formula, the standard transformations suffice (we explain how to deal with the contraction rules below). If the cut-formula is a boxed-formula, first transform the left premise derivation and then the right-premise derivation in the usual manner to obtain the situation where the cut-formula is principal by the GoR rule in both premises. This is the case discussed in the Introduction. Note that although the standard transformations introduce new cuts, by inspection it is easily seen that the width of these introduced cuts is $< n$. From Lemma 3.5 we can write the left premise derivation as a normal derivation ending as (say) $\boxtimes X, \Box(B \supset \Box B) \Rightarrow B /^{GoR} \Box X \Rightarrow \Box B$. Using Lemma 3.15 we can obtain a cutfree derivation of $\Box X, X \Rightarrow B$. Proceed as directed in the Introduction.

Since we use sequents built from multisets, we also need to specify how to deal with the contraction rules. In fact, there are two possible approaches for dealing with 'contractions above cut'. If we are prepared to use the multicut rule $(m, n > 0)$

$$\dfrac{X \Rightarrow Y, A^m \qquad A^n, U \Rightarrow W}{X, U \Rightarrow Y, W}\, mcut$$

then we can obtain a cutfree derivation by taking a detour via the calculus $GoS + mcut$. This is the approach Gentzen [25] takes in his proof of the *Hauptsatz*. If we wish to avoid making a detour via a new calculus, instead of using the multicut rule we can adapt the transformations described in [77] for classical logic. The only new case to deal with is a derivation of the following form:

$$\dfrac{\dfrac{\Box X, X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B}\, GoR \qquad \dfrac{\dfrac{\Box B, \Box B, B, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}{\Box B, \Box B, \Box U \Rightarrow \Box C}\, GoR}{\Box B, \Box U \Rightarrow \Box C}\, LC}{\Box X, \Box U \Rightarrow \Box C}\, cut_0$$

Then the following transformation suffices, where a derivation of $\Box X, X \Rightarrow B$ can be obtained from Lemma 3.15.

$$\dfrac{\dfrac{\Box X, X \Rightarrow B \qquad \dfrac{\dfrac{\Box X \Rightarrow \Box B \qquad \dfrac{\Box B, \Box B, B, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}{\Box B, B, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}\, LC}{B, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}\, cut_1}{\Box X, B, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}\, LC}{\Box X, X, \Box U, U, \Box(C \supset \Box C) \Rightarrow C}\, cut_2}{\Box X, \Box U \Rightarrow \Box C}\, GoR$$

because $cut_1 < cut_0$ (reduced cut-height) and $cut_2 < cut_0$ (reduced degree of the cut-formula). This is similar to the approach for avoiding multicut in cut-elimination for GLS [32]. Q.E.D.

3.4 Conclusion

We fill a gap in the literature by presenting a syntactic proof of cut-elimination for GoS. We conclude by comparing this proof to the existing proofs of cut-elimination for Grz and GL.

The calculus GLS for GL can be obtained by substituting the GoR rule in Table 3.1 with the GLR rule:

$$\frac{\Box X, X, \Box B \Rightarrow B}{\Box X \Rightarrow \Box B}$$

The calculus $GrzS$ for Grz can be obtained by substituting the GoR rule in Table 3.1 with the following rules:

$$\frac{B, X \Rightarrow Y}{\Box B, X \Rightarrow Y}\, GRZa \qquad\qquad \frac{\Box X, X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B}\, GRZc$$

Informally, the proof for GoS appears to be more intricate than the proof for the calculus GLS [71, 32] because of the necessity of dealing with the formula $\Box(B \supset \Box B)$ as opposed to $\Box B$ in the premise of the respective modal rules.

Compared with the $GrzS$ calculus, although the $GRZc$ rule also contains $\Box(B \supset \Box B)$ in the premise antecedent, the presence of the $GRZa$ rule enables us to directly transform any sequent of the form $C \supset \Box C, X \Rightarrow Y$ into $\Box(C \supset \Box C), X \Rightarrow Y$ which greatly simplifies the proof. In GoS, we have only the GoR rule at our disposal to 'box' the $C \supset \Box C$ formula in a sequent of the form $C \supset \Box C, X \Rightarrow Y$, and must abide by the restrictions it places on the multisets X and Y. As a result, the proof for GoS seems to require a more detailed study of the structure of derivations in GoS, and a quaternary induction measure, whereas three induction variables suffice for GLS and $GrzS$.

Part II

Display calculi

Chapter 4

Preliminaries

We begin by defining the basic modal logic K and the basic tense logic Kt, and present a semantics for these logics (Section 4.1). In Section 4.2 we introduce some results from correspondence theory that will be used throughout the chapter. The Display calculus [5] is a generalised sequent framework for capturing a variety of logics. In Section 4.3 we present the display calculus DLM for the basic tense logic Kt.

Most of the material here is standard. Our contributions here are as follows. We have identified an error in the definition of Kracht formula given in Blackburn *et al.* [7, Definition 3.58]. To obtain the desired correspondence between Kracht formulae and Sahlqvist formulae, the definition given in [7] needs to be suitably extended. Blackburn *et al.* also show how to compute a Sahlqvist formula from a Kracht formula. We explain how to extend the given algorithm to handle the additional cases that arise due to the new definition of Kracht formula. We have provided a new definition of "properly displays" for display calculi. In our view, this new definition corresponds more closely with the notion of soundness and completeness of a calculus with respect to a logic. Moreover, our new definition does not rely on the translation τ between display structures and tense formula. As a result, the definition works equally well with calculi for the modal language. We prove the equivalence of the two definitions for display calculi for the tense language.

4.1 Introducing modal and tense logics

A formal language consists of strings of symbols. Some of these strings are taken to be meaningful (*formulae*) and the remainder are taken to be meaningless.

From the set of meaningful strings of the language, some strings are chosen to be 'good' (*theorems*). The set of good strings is called a *logic*.

There are two standard ways of specifying the theorems of a logic.

One way is to use the language to describe an object or some class of objects (the set of natural numbers, or the class of directed graphs, for example). The logic is defined as the set of formulae in the language that are true of the object. Since this method relies on the meaning or semantics of the object, we say that the logic has been defined *semantically*.

The other way is to choose a set of formulae as theorems (call this set the *axioms*), and provide a set of inference rules which specify how to produce new theorems from the existing ones. The logic is defined as the set of formulae that can be produced by repeatedly applying the inference rules to already-obtained theorems. Unlike before, the language is not used to directly describe the object, and theorems are specified solely based on the syntax. For this reason we say that the logic has been defined *syntactically*. There are many different systems that can be used to syntactically define a logic — the Hilbert calculus [75, 16], natural deduction and sequent calculi systems [25] and the display calculus [5], to name just a few examples. The reason for this diversity is that each system has advantages and drawbacks from a theoretical and computational perspective, and also in terms of applicability to a given logic.

Let us begin by defining a formal language called the propositional language $\mathcal{L}$. This language is defined using a countably infinite set of propositional variables p_i, the propositional constants $\bot$ and $\top$, the propositional connectives $\neg$ ("not"), $\vee$ ("or"), $\wedge$ ("and") and $\supset$ ("implication"), and the punctuation marks "(" and ")". The set $\mathbf{For}\mathcal{L}$ of formulae of $\mathcal{L}$ is given by the grammar

$$A ::= p_i \mid \bot \mid \top \mid \neg A \mid (A \vee B) \mid (A \wedge B) \mid (A \supset B)$$

where p_i ranges over the set $\mathbf{Var}\mathcal{L}$ of propositional variables. We will use $p, q, r, \ldots$ (possibly with subscripts) to denote propositional variables, and $A, B, \ldots$ to denote formulae (the context will determine the language in question).

For example, the string "$p \supset (p \wedge q)$" is in $\mathbf{For}\mathcal{L}$ because it is derivable using the above grammar, whereas the string "$p_0 \bot \vee$" is not derivable by the above grammar so it is not a formula of $\mathcal{L}$. Notice that it is straightforward to decide whether or not a given string of $\mathcal{L}$ belongs to $\mathbf{For}\mathcal{L}$.

Let us now define a logic called *Classical propositional logic Cp*. We will define Cp syntactically using a *Hilbert Calculus*. A *Hilbert calculus* consists of a set of formulae (axioms) and inference rules of the following form, for formulae $A_1, \ldots, A_n, B$:

$$\frac{A_1 \ldots \quad A_i \quad \ldots A_n}{B}$$

The formulae $A_1, \ldots, A_n$ are called the *premises* of the rule, and the formula B is called the *conclusion*. This rule states that if all the premises of the rule are theorems of the logic, then the conclusion is also a theorem of the logic. The logic defined by a Hilbert calculus is precisely the closure of the axioms under the inference rules. Note that we will occasionally describe the inference rules using words when it is convenient to do so. Finally, we write $A \approx B$ as an abbreviation for the formula $(A \supset B) \wedge (B \supset A)$.

Here is a Hilbert calculus for Cp (see [16]):

Axioms:

(A1) $p \supset (q \supset r)$

(A2) $(p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$

(A3) $p \wedge q \supset p$

(A4) $p \wedge q \supset q$

(A5) $p \supset (q \supset (p \wedge q))$

(A6) $p \supset p \vee q$

(A7) $q \supset p \vee q$

(A8) $(p \supset r) \supset ((q \supset r) \supset (p \vee q \supset r))$

(A9) $\bot \supset p$

(A10) $p \vee (p \supset \bot)$

(A11) $\neg p \approx p \supset \bot$

(A12) $\top \approx \neg\bot$

Inference rules:

Modus ponens: if $A \in Cp$ and $A \supset B \in Cp$, then $B \in Cp$

Uniform substitution of arbitrary formulae for propositional variables in a formula

Let us introduce some basic terminology and notation. A *subformula* A' of the formula A is a formula that occurs as a substring of A. We say that A' is a *proper subformula* of A if it is a subformula and A' is not identical to A. Also, we write $A \in L$ to denote that the formula A is a theorem of the logic L. If $A \approx B \in L$ then we say that A and B are *logically equivalent* in L. We write "iff" as shorthand for "if and only if" in the usual mathematical sense.

Examples of theorems in Cp include

$$\begin{array}{rr} p \supset \top & (p \supset q) \supset (r \wedge p \supset q \vee s) \\ \neg(p \vee q) \approx (\neg p \wedge \neg q) & (p \supset q) \approx (\neg p \vee q) \\ (\neg\neg p) \approx p & \neg(p \wedge q) \approx (\neg p \vee \neg q) \end{array}$$

Using $(p \supset \bot) \approx \neg p$, it is easy to see that (A10) is logically equivalent to $p \vee \neg p$.

First-order classical logic with equality is obtained from Classical propositional logic in the usual way by extending the language $\mathcal{L}$ with the quantifiers $\forall$ and $\exists$ and the equality relation $=$, and the addition of suitable axioms to the Hilbert calculus for Cp to make the logic work. Specific first-order theories can be obtained by the addition of new function and relation symbols (such as $+$, $\times$ for an arithmetic theory, for example) and axioms that capture the properties of these symbols. Since this formulation of first-order logic is standard, we omit the details (see [4] for example).

Theorems of the form $A \approx B$ in first-order classical logic and classical propositional logic are called *classical equivalences*. We will implicitly make use of the result that if A is a theorem of one of these logics and $B \approx C$ is a classical equivalence, then the formula A' obtained by substituting some occurrences of formula B in A with formula C is logically equivalent to A.

Since (A11) states that $\neg A \approx (A \supset \bot)$ and (A12) states that $\top \approx \neg\bot$, it follows that $\neg$ and $\top$ are redundant in the sense that any formula in Cp containing the symbols $\neg$ and $\top$ is equivalent to some formula not containing these symbols. Similarly, the conjunction connective can be defined in terms of $\{\neg, \vee\}$, and the disjunction connective can written in terms of $\{\neg, \wedge\}$. Also the implication connective can be defined in terms of $\{\neg, \vee\}$. Despite this obvious redundancy, we retain these connectives because it is convenient to be able to use them directly, and also because their presence makes it possible to construct formal proof systems with nice properties.

4.1.1 Hilbert calculi for modal and tense logic

The basic modal language $\mathcal{ML}$ can be obtained by augmenting the language $\mathcal{L}$ with the modal operators $\Diamond$ ('diamond') and $\Box$ ('box'). The formulae of the basic modal language is the set **For**$\mathcal{ML}$ given by the grammar

$$A ::= p_i \,|\, \bot \,|\, \top \,|\, \neg A \,|\, (A \vee B) \,|\, (A \wedge B) \,|\, (A \supset B) \,|\, \Diamond A \,|\, \Box A$$

where p_i ranges over the set of propositional variables. Formulae in **For**$\mathcal{ML}$ are called *modal* formulae. For example, $\Box(A \wedge B) \supset \Box A$ and $\Diamond\Box p \supset p$ are modal formulae. Below, we define syntactically the basic modal logic K_H using the following Hilbert calculus:

Axioms: all the axioms of classical propositional logic Cp plus:

$$\Box(p \supset q) \supset (\Box p \supset \Box q) \qquad (Ax - \Box)$$
$$\Box p \approx \neg\Diamond\neg p \qquad (Dual - \Box)$$

Inference rules:

Modus ponens: if $A \in K_H$ and $A \supset B \in K_H$ then $B \in K_H$

Uniform substitution of arbitrary modal formulae for propositional variables in a formula

Necessitation: if $A \in K_H$, then $\Box A \in K_H$

Clearly $Cp \subset K_H$. Next we introduce the basic temporal language $\mathcal{TL}$, obtained by augmenting $\mathcal{ML}$ with the modal operators $\blacklozenge$ ('black diamond') and $\blacksquare$ ('black box'). The formulae of the basic temporal language is the set **For**$\mathcal{TL}$ given by the grammar

$$A ::= p_i \,|\, \bot \,|\, \top \,|\, \neg A \,|\, (A \vee B) \,|\, (A \wedge B) \,|\, (A \supset B) \,|\, \Diamond A \,|\, \Box A \,|\, \blacklozenge A \,|\, \blacksquare A$$

where p_i ranges over the set of propositional variables. Formulae in **For**$\mathcal{TL}$ are called *tense* formulae. Clearly every modal formula is a tense formula.

Define the basic tense logic Kt_H in the language $\mathcal{TL}$ using the following Hilbert calculus:

Axioms: all the axioms of classical propositional logic Cp plus:

$$\Box(p \supset q) \supset (\Box p \supset \Box q) \qquad (Ax - \Box)$$
$$\blacksquare(p \supset q) \supset (\blacksquare p \supset \blacksquare q) \qquad (Ax - \blacksquare)$$
$$\Box A \approx \neg\Diamond\neg A \qquad (Dual - \Box)$$
$$\blacksquare A \approx \neg\blacklozenge\neg A \qquad (Dual - \blacksquare)$$
$$p \supset \Box\blacklozenge p \qquad (Converse1)$$
$$p \supset \blacksquare\Diamond p \qquad (Converse2)$$

Inference rules:

Modus ponens: if $A \in Kt_H$ and $A \supset B \in Kt_H$ then $B \in Kt_H$

Uniform substitution of arbitrary tense formulae for propositional variables in a formula

Necessitation: if $\alpha \in Kt_H$, then $\Box\alpha \in Kt_H$ and $\blacksquare\alpha \in Kt_H$.

For an example of reasoning in the Hilbert calculus, see Lemma 4.47.

Clearly $K_H \subset Kt_H$. In the following subsection, we will see how to define the logics K_H and Kt_H semantically.

4.1.2 Defining the logics K_H and Kt_H semantically

Classical propositional logic Cp defined in the previous section can also be defined semantically using a so-called classical interpretation of the language $\mathcal{L}$ (see [16]). This is sometimes called the truth-table semantics for Cp. The idea is to assign a valuation of either true or false (but not both simultaneously) to each propositional variable. The constant $\bot$ is always false, the constant $\top$ is always true, and the formulae $\neg A$, $A \vee B$, $A \wedge B$ and $A \supset B$ are inductively defined in the usual manner using truth tables. For example, $A \supset B$ is assigned true if and only if A is false or B is true. It can be shown that classical propositional logic Cp consists precisely of those formulae in $\mathbf{For}\mathcal{L}$ that are true under all valuations.

Since the modal and temporal languages contain the operators $\Diamond$ and $\blacklozenge$ and their duals, a more sophisticated semantics is required in order to incorporate these language elements in accordance with their intended meaning. A standard approach is to use Kripke semantics (also known as frame semantics) — see [16, 7] for an exposition.

The abstract framework for Kripke semantics is based on *frames* and *models*.

Definition 4.1 (frame) *A* frame *for the basic modal language ('modal frame') is a pair $F = (W, R)$ such that*

1. *W is a non-empty set ('states'), and*
2. *R is a binary relation on W.*

A modal frame is an instance of a mathematical object called a *relational structure*. A relational structure is simply a non-empty set W together with some positive number of relations on W. For this reason, Kripke semantics are sometimes called *relational semantics*.

Definition 4.2 (model) *A* model *for the basic modal language is a pair $M = (F, V)$, where F is a frame (W, R) for the basic modal language, and V is a function ('valuation function') assigning to each proposition variable p a subset $V(p)$ of W.*

A model M is said to be based on the frame F if $M = (F, V)$ for some valuation V.

Now let $M = ((W, R), V)$ be a model and $w \in W$. Define the *satisfaction* relation $(M, w) \models D$ (read as 'D is *satisfiable* in M at state w') by induction on the structure of the formula $D \in \mathbf{For}\mathcal{ML}$ as follows:

$M, w \models p$ iff $w \in V(p)$

$M, w \models \bot$ never

$M, w \models \top$ always

$M, w \models \neg A$ iff not $M, w \models A$

$M, w \models A \vee B$ iff $M, w \models A$ or $M, w \models B$

$M, w \models A \wedge B$ iff $M, w \models A$ and $M, w \models B$

$M, w \models A \supset B$ iff $M, w \models A$ implies $M, w \models B$

$M, w \models \Diamond A$ iff there exists $v \in W$ such that Rwv and $M, v \models A$

$M, w \models \Box A$ iff for all $v \in W$, if Rwv then $M, v \models A$

The negation of $M, w \models D$ is written $M, w \not\models D$.

Definition 4.3 (validity) *A formula A is* valid *at a state w in frame F (notation: $F, w \models A$) if A is satisfied at w in every model (F, V) based on F. A formula A is valid on a frame F if it is valid at every state in F (notation: $F \models A$). Also, we say that A is valid on a class $\mathcal{F}$ of frames if $F \in \mathcal{F}$ implies that $F \models A$.*

A set of formulae Γ is valid on a frame F (notation: $F \models \Gamma$) if every formula in Γ is valid on F; and Γ is valid on a class of frames $\mathcal{F}$ (notation: $\mathcal{F} \models \Gamma$) if every formula in Γ is valid on $\mathcal{F}$.

The negation of $F, w \models A$ ($F \models A$) is written $F, w \not\models A$ ($F \not\models A$). If $F, w \not\models A$ it follows that there is a model M based on F such that $M, w \not\models A$. In this case we say that A is *falsifiable* on F at w. Moreover, it follows directly from the definition that if $F \not\models A$ then there is some state w in F such that $F, w \not\models A$. We say that A is satisfiable on F if there is a model M based on F and state w such that $M, w \models A$. Similarly, we say that A is falsifiable on F if there is a model M based on F and state w such that $M, w \not\models A$. Clearly, A is falsifiable on F iff $\neg A$ is satisfiable on F.

It is easy to check for any frame F that $F \models Cp$ — it suffices to verify that each axiom of Cp is valid on F, and also that each of the inference rules preserve validity. Note that when Γ is a finite set of formulae, then validity of Γ on a frame F (resp. class of frames $\mathcal{F}$) is equivalent to validity of $\bigwedge \Gamma$ on F ($\mathcal{F}$), where $\bigwedge \Gamma$ denotes the conjunction of all formulae in Γ.

Observe that the definition of validity at a state in a frame requires quantification over all valuation functions V. For each propositional variable p, since $V(p)$ is a subset of W, quantification over all valuations amounts to quantification over all subsets of W. Quantification over sets of propositional variables as opposed to quantifying over a propositional variable hints at the second-order nature of frame validity, discussed in greater detail in the next section.

Define semantically the modal logic $K_{\mathcal{ML}}$ in the language $\mathcal{ML}$ as those formulae in $\mathbf{For}\mathcal{ML}$ that are valid in all modal frames, that is:

$$K_{\mathcal{ML}} = \{A \in \mathbf{For}\mathcal{ML} \mid F \models A \text{ for all modal frames } F\}$$

The following result is a basic result in modal logic (see [16, 7] for example).

Theorem 4.4 $K_{\mathcal{ML}} = K_H$.

In other words, the syntactically specified K_H and the semantically specified $K_{\mathcal{ML}}$ describe the same logic. From now on we will write this logic as K using the syntactic and semantic definitions interchangeably as convenient. The logic K is often called the basic (or minimal) propositional modal logic. The term 'minimal' is due to the fact that K is the weakest system for reasoning about frames.

We would now like to give a semantic specification for the logic Kt_H. Generally, Kripke frames for the temporal language ('temporal frames') should consist of two relations (R_F and R_P say) on the non-empty set W standing for *future* and *past* respectively, corresponding to the operators $\Diamond$ and $\blacklozenge$. Semantics for formulae in the temporal language could then be obtained by replacing the final two lines of the satisfaction relation definition above with the following:

$M, w \models \Diamond A$ iff there exists $v \in W$ such that $R_F wv$ and $M, v \models A$

$M, w \models \Box A$ iff for all $v \in W$, if $R_F wv$ then $M, v \models A$

$M, w \models \blacklozenge A$ iff there exists $v \in W$ such that $R_P wv$ and $M, v \models A$

$M, w \models \blacksquare A$ iff for all $v \in W$, if $R_P wv$ then $M, v \models A$

Satisfaction and validity for temporal frames can be defined analogously to the modal case. However it is easily seen that the converse axioms in Kt are valid on a temporal frame iff the temporal frame satisfies

$$\forall xy. R_F xy \leftrightarrow R_P yx \tag{4.1}$$

If R is a binary relation, let $\breve{R}$ be the *converse* binary relation of R defined as $\breve{R}xy = \{(x, y) \mid Ryx\}$. Clearly $\breve{\breve{R}}xy \leftrightarrow Rxy$. Then (4.1) implies that $R_F \leftrightarrow \breve{R}_P$ (and $R_P \leftrightarrow \breve{R}_F$). As a consequence, it is enough to keep track of a single binary relation (R_F say) since the other relation can be computed from it. A non-empty set with a single binary relation is simply a modal frame, so it follows that modal frames contain enough information to encode the class of temporal frames on which ($Converse1$) and ($Converse2$) are valid.

Specifically, define a *tense frame* to be a temporal frame on which ($Converse1$) and ($Converse2$) — or equivalently (4.1) — is valid. From our discussion, it follows that every tense frame has the form $(W, R, \breve{R})$. This gives rise to an obvious isomorphism between the class of tense frames and modal frames: given the modal frame (W, R), obtain the tense frame $(W, R, \breve{R})$; given the tense frame $(W, R, \breve{R})$, obtain the modal frame (W, R).

Example 4.5 *Let $W = \{u, v, w\}$ and R be given by $\{(u, v), (v, w), (w, w)\}$. Then the basic modal frame $F = (W, R)$ can be presented graphically as*

$$\begin{tikzcd} u \arrow[r] & v \arrow[r] & w \arrow[loop above] \end{tikzcd}$$

The tense frame corresponding to the above relational structure is obtained by augmenting F with the relation $\breve{R}$ given by $\{(v, u), (w, v), (w, w)\}$ to obtain the $F' = (W, R, \breve{R})$. This frame can be presented by the following picture:

$$\begin{tikzcd} u \arrow[r] & v \arrow[r] \arrow[l, dotted, bend left] & w \arrow[loop above] \arrow[l, dotted, bend left] \arrow[loop right, dotted] \end{tikzcd}$$

where the solid arrows correspond to the 'future' relation and the broken arrows correspond to the 'past' relation.

From now on we will simply use the term 'frame' to mean either a modal frame or a tense frame, leaving it to the context to indicate which type is meant.

Define a tense model as a pair (F, V) where $F = (W, R, \breve{R})$ is a tense frame and V is a valuation function, assigning to each propositional variable a subset $V(p)$ of W. We can define the satisfaction relation $M, w \models_t A$ for a tense model M containing state w and tense formula A by extending $\models$ in the obvious way by the addition of the statements:

$$M, w \models_t \blacklozenge A \text{ iff there exists } v \in W \text{ such that } Rvw \text{ and } M, v \models_t A$$
$$M, w \models_t \blacksquare A \text{ iff for all } v \in W, \text{ if } Rvw \text{ then } M, v \models_t A$$

Notice that we have done away with $\breve{R}$ and written the above solely in terms of R for simplicity. Following standard practice we will use the symbol $\models$ to mean the satisfaction relation on the temporal language as well (dropping the subscript 't'), allowing the context to determine if the formula belongs to $\mathbf{For}\mathcal{ML}$ or $\mathbf{For}\mathcal{TL}$, except when this is likely to cause confusion.

We can now define the logic

$$Kt_{\mathcal{TL}} = \{A \in \mathbf{For}\mathcal{TL} \mid F \models A \text{ for all frames } F\}.$$

The following is a basic result of modal logic ([16, 7]):

Theorem 4.6 $Kt_{\mathcal{TL}} = Kt_H$.

Thus the syntactic and semantically specified Kt_H and the semantically specified $Kt_{\mathcal{TL}}$ describe the same logic. From now on we will write this logic as Kt using the syntactic and semantic definitions interchangeably as convenient. The logic Kt is often called the basic (or minimal) normal propositional tense logic.

Any logic $L \supseteq K$ ($L \supseteq Kt$) closed under modus ponens, substitution and necessitation is called *normal*. The axiomatic extensions of K and Kt, defined below, is one such class of normal logics.

Definition 4.7 (axiomatic extensions of K,Kt) *Let Δ be a (possibly empty) set of modal (tense) formulae (the axioms). Then the axiomatic extension of K (Kt) by Δ denoted $K \oplus \Delta$ ($Kt \oplus \Delta$) is the logic obtained by the addition of Δ to the axioms of K (Kt) and closure under the inference rules modus ponens, substitution and necessitation.*

An axiomatic extension of K (Kt) a modal (tense) logic. Observe that although the term 'modal logic' is also used more generally to describe the field of logic dealing with modalities (operators) such as $\Diamond$ and $\blacklozenge$, the overloading of this term to refer specifically to logics in the basic modal language will cause no confusion in practice.

Notice that the axiomatic extensions of K and Kt defined above have been syntactically specified using the Hilbert calculus. It is an obvious question to wonder if we can semantically specify these logics (ie provide theorems analogous to Theorem 4.4 and 4.6). The Sahlqvist correspondence and completeness theorems show how such a semantic specification can be achieved for a large class of axiomatic extensions. Before proceeding, let us introduce a few more definitions.

Definition 4.8 (weakly sound) *Let $\mathcal{F}$ be a class of frames. A logic L is* weakly sound *with respect to $\mathcal{F}$ if for any formula A,*

$$A \in L \textit{ implies } \mathcal{F} \models A.$$

Definition 4.9 (weakly complete) *A logic L is* weakly complete *with respect to a class $\mathcal{F}$ of frames if for any formula A,*

$$\mathcal{F} \models A \textit{ implies } A \in L$$

There is in fact a more powerful notion of soundness and completeness [7] that is encountered in the literature, sometimes referred to as "strong soundness" and "strong completeness". Nevertheless, the present definitions are sufficient for our purposes. We will often drop the word 'weakly' and use the terms *sound* and *complete* to refer to Definition 4.8 and 4.9 respectively.

If logic L is sound and weakly complete with respect to some class $\mathcal{F}$ of frames, from the definitions it follows that

$$A \in L \text{ iff } \mathcal{F} \models A$$

We remind the reader that the symbol $\models$ above is overloaded, standing for the satisfaction relation on both modal and temporal languages. Hence, if $\mathcal{F}_{all}$ denotes the class of all frames, then since $A \in K$ iff $\mathcal{F}_{all} \models A$ and $A \in Kt$ iff $\mathcal{F}_{all} \models_t A$, we freely say that both K and Kt are sound and weakly complete for $\mathcal{F}_{all}$. Of course, this does not mean that $K = Kt$.

4.2 Some results in correspondence theory

Our presentation of correspondence theory follows Blackburn, de Rijke and Venema [7], with the following exceptions. We have chosen to introduce Kracht's restricted quantifiers [38, 40] at an early stage. Although this may seem to complicate the notation somewhat, it will allow us to directly obtain the first half of Theorem 4.31. We also identify a deficiency in Definition 3.58 in [7]. Fixing this definition makes the algorithm for the first half of Theorem 4.31 incomplete. Here we show how to complete the algorithm, and also fix an independent error that occurs in the proof of the algorithm. Another good reference for modal correspondence theory is van Benthem [73]. In particular, this work contains some interesting results on the preservation of first-order formulae for classes of frames that are modally definable. We do not specifically address tense correspondence theory here — the interested reader is directed to van Benthem [74].

We begin by introducing 'frame languages' that make use of frame validity to describe classes of frames (Section 4.2.1). A formula from a frame language 'corresponds' to a tense formula if they each describe the same class of frames. In

Section 4.2.2 we will see how to compute the first-order correspondents for a large syntactically-defined class of tense formulae. Then in Section 4.2.3 we will look at the reverse direction, and see how to compute modal and tense correspondents for a large class of first-order formulae. Note that we limit this exposition to cover the tools that are required for our work.

4.2.1 Basic definitions

The following definition provides us with a way of describing a class of frames using a tense formula.

Definition 4.10 (defining a class of frames) *We say that a modal or tense formula A* defines *a class $\mathcal{F}$ of frames if for every frame F:*

$$F \in \mathcal{F} \text{ iff } F \models A$$

A set Γ of formulae defines a class $\mathcal{F}$ of frames if for all frames F, $F \in \mathcal{F}$ iff $F \models \Gamma$.

If Γ is a finite set of formulae, then Γ defines $\mathcal{F}$ iff $\bigwedge \Gamma$ defines $\mathcal{F}$. Formulae A and B are called *frame equivalent* if for every frame F, $F \models A$ iff $F \models B$.

Since a frame is simply a relational structure, we can also describe a class of frames using non-modal languages. In this section we will define two such *frame languages*. The expressiveness of the frame language determines what classes of frames are definable. We will see that the class of frames defined by any modal or tense formula can also be defined by a formula from an appropriate second-order language. However, it is known that the classes of frames defined by McKinsey's axiom $\Box\Diamond p \supset \Diamond\Box p$ and Löb's axiom $\Box(\Box p \supset p) \supset \Box p$ *cannot* be defined using a first-order language. Furthermore, the class of frames having a single reflexive point is definable using a first-order language ($\exists x.Rxx$), but it is *not* definable using tense formulae (see [7]).

Definition 4.11 (frame languages) *The* first-order frame language $\mathcal{L}^f$ *is the first-order language equipped with equality $=$ and a binary relation symbol R.*

The monadic second-order frame language $\mathcal{L}^f_2$ *is obtained by augmenting $\mathcal{L}^f$ with a countable set of monadic predicate variables and the monadic predicate quantifiers $\tilde{\forall}$ and $\tilde{\exists}$.*

The first-order quantifiers $\forall, \exists$ range over first-order variables $x, y, \ldots$, and the monadic predicate quantifiers $\tilde{\forall}, \tilde{\exists}$ range over monadic predicate variables $P, Q, \ldots$

(possibly with subscripts). Here the term monadic refers to the fact that a predicate variable in $\mathcal{L}_2^f$ ranges over sets of propositional variables only. Observe that quantifying over monadic predicate variables, amounts to quantification over *sets* of first-order variables. Full second-order logic is even more expressive (exceeding our requirements here), containing other sorts of variables as well.

To help distinguish the frame languages from the modal and temporal languages $\mathcal{ML}$ and $\mathcal{TL}$, implication in the frame languages will be written using $\rightarrow$ rather than $\supset$, and the logical constants for true and false are written $\mathbf{t}$ and $\mathbf{f}$ respectively (for the other logical connectives and constants we use the same symbols as in $\mathcal{ML}$ and $\mathcal{TL}$). We will use $\alpha, \beta, \ldots$ to denote formulae from $\mathcal{L}^f$ and $\mathcal{L}_2^f$. Following the standard terminology, a first-order (or monadic predicate variable) occurring in a formula is called *free* if it is not bound by the quantifiers $\forall, \exists$ $(\widetilde{\forall}, \widetilde{\exists})$.

For a formula α from $\mathcal{L}^f$ or $\mathcal{L}_2^f$ with free first-order variables $x_1, \ldots x_n$ and (in the case of $\mathcal{L}_2^f$) monadic predicate variables $P_1, \ldots, P_m$, we write

$$\alpha[w_1/x_1, \ldots, w_n/x_n, Q_1/P_1, \ldots, Q_m/P_m]$$

to denote the formula obtained by uniformly substituting the first-order variable w_i for each free occurrence of x_i $(1 \leq i \leq n)$ and substituting the predicate Q_i for each free occurrence of the predicate variable P_i $(1 \leq i \leq m)$ in α. For brevity, we some times write the above as $\alpha[\{w_i/x_i\}][\{Q_i/P_i\}]$, or even $\alpha[w_1 \ldots w_n][Q_1 \ldots Q_m]$ when the free variables can be identified from the context.

Following Kracht [38, 40] we introduce the so-called *restricted quantifiers* $(\forall y \rhd x)\alpha(y)$ and $(\exists y \rhd x)\alpha(y)$ which are the following abbreviations:

$$\begin{aligned}
(\exists y \rhd x)\alpha(y) &\text{ abbreviates } \exists y(Rxy \wedge \alpha(y))\\
(\exists y \lhd x)\alpha(y) &\text{ abbreviates } \exists y(Ryx \wedge \alpha(y))\\
(\forall y \rhd x)\alpha(y) &\text{ abbreviates } \forall y(Rxy \rightarrow \alpha(y))\\
(\forall y \lhd x)\alpha(y) &\text{ abbreviates } \forall y(Ryx \rightarrow \alpha(y))
\end{aligned}$$

In the above, the variable x is called the *restrictor* of the restricted quantifier. We refer to $(\exists y \rhd x)\alpha(y)$ and $(\exists y \lhd x)\alpha(y)$ as the *existential restricted quantifiers*, writing $\exists^r y\, \alpha(y)$ to avoid specifying which instance is meant, and also to avoid naming the restrictor. Similarly, $(\forall y \rhd x)\alpha(y)$ and $(\forall y \lhd x)\alpha(y)$ are called the *universal restricted quantifiers*, and we write $\forall^r y\, \alpha(y)$ to denote either one of these instances. We also use the terminology *forward* (resp. *backward*) restricted quantifiers to refer to $(\forall y \rhd x)\alpha(y)$ and $(\exists y \rhd x)\alpha(y)$ $((\forall y \lhd x)\alpha(y)$ and $(\exists y \lhd x)\alpha(y))$.

Since each restricted quantifier *abbreviates* a formula in $\mathcal{L}^f$, we can use the restricted quantifiers without having to extend the languages with new symbols.

Consider the following translation from a modal formula into a formula in $\mathcal{L}^f_2$. This is called the *standard translation* [7].

Definition 4.12 (standard translation) *Let x be a first-order variable. Define the following translation $ST_x(\cdot)$ taking modal formulae to the monadic second-order frame language $\mathcal{L}^f_2$:*

$$
\begin{aligned}
ST_x(p_i) &= P_i x \\
ST_x(\bot) &= (x \neq x) \\
ST_x(\top) &= (x = x) \\
ST_x(\neg A) &= \neg ST_x(A) \\
ST_x(A \vee B) &= ST_x(A) \vee ST_x(B) \\
ST_x(A \wedge B) &= ST_x(A) \wedge ST_x(B) \\
ST_x(A \supset B) &= ST_x(A) \rightarrow ST_x(B) \\
ST_x(\Diamond A) &= (\exists y \rhd x) ST_y(A) \\
ST_x(\Box A) &= (\forall y \rhd x) ST_y(A)
\end{aligned}
$$

where y is a first-order variable that has not been used so far in the translation.

Observe that $ST_x(A)$ contains only x as a free variable. The standard translation for a tense formula is obtained by the addition of the following to the above definition:

$$
\begin{aligned}
ST_x(\blacklozenge A) &= (\exists y \lhd x) ST_y(A) \\
ST_x(\blacksquare A) &= (\forall y \lhd x) ST_y(A)
\end{aligned}
$$

Now we will show how to use the frame languages to describe models and frames. Suppose that α is a formula from $\mathcal{L}^f$. To construct a model for $\mathcal{L}^f$, we need to provide an interpretation for the symbol R. Suppose that $M = ((W, \mathcal{R}), V)$ is a modal model (remember that the underlying frame $(W, \mathcal{R})$ is just a relational structure). By interpreting the symbol R in $\mathcal{L}^f$ as the symbol $\mathcal{R}$ from M, we can write things such as $M \models \alpha[ww_1, \ldots, w_n]$ which means that α is satisfied in the usual sense of first-order logic on the model M when the free variables $(x, x_1, \ldots, x_n)$ are substituted with the states $(w, w_1, \ldots, w_n)$ in W.

When α is a formula from $\mathcal{L}_2^f$, we need to also interpret the monadic predicate variables. We will interpret each monadic predicate variable P_i on M as the set $V(p_i)$. Then we can show that for any model (F, V) and modal or tense formula A, by a straightforward induction on the structure of A [7]:

$$(F, V), w \models A \text{ iff } (F, V) \models ST_x(A)[w/x][\{V(p_i)/P_i\}] \tag{4.2}$$

Under this interpretation, quantifying over all valuations is equivalent to quantifying over the free monadic predicate variables $\overline{P} = \{P_1, \ldots, P_n\}$, and thus

$$F, w \models A \text{ iff } F \models \widetilde{\forall} \overline{P}\, ST_x(A)[w/x]$$

Indeed, from the above, by quantifying over all states of the frame, we see that for every frame F:

$$F \models A \text{ iff } F \models \widetilde{\forall} \overline{P}\, \forall x\, ST_x(A). \tag{4.3}$$

Definition 4.13 (defining a class of frames using $\mathcal{L}^f$ or $\mathcal{L}_2^f$) *A formula α from $\mathcal{L}^f$ or $\mathcal{L}_2^f$* defines *a class $\mathcal{F}$ of frames if for every frame F:*

$$F \in \mathcal{F} \textit{ iff } F \models \alpha$$

We will make use of the following lemma without explicit reference.

Lemma 4.14 *Let $\{\phi_1, \ldots, \phi_n\}$ be a set of (i) modal or tense formulae or (ii) formulae from $\mathcal{L}^f$ or $\mathcal{L}_2^f$, such that ϕ_i defines the class $\mathcal{F}_i$ of frames. Then $\wedge_{i=1}^n \phi_i$ defines $\cap_{i=1}^n \mathcal{F}_i$.*

Proof. For,

$$\begin{aligned} F \in \cap_{i=1}^n \mathcal{F}_i &\Leftrightarrow F \in \mathcal{F}_i \text{ for } 1 \leq i \leq n \\ &\Leftrightarrow F \models \phi_i \text{ for } 1 \leq i \leq n \end{aligned}$$

If the $\{\phi_1, \ldots, \phi_n\}$ are formulae from the frame languages, then the above is equivalent to $F \models \wedge_{i=1}^n \phi_i$ as required. If the $\{\phi_1, \ldots, \phi_n\}$ are tense formulae, then $F \models \phi_i$ iff it is the case that $M, w \models \phi_i$ for every model M based on F and every state w in F. This is equivalent to $M, w \models \wedge_{i=1}^n \phi_i$ for every model M based on F and state w. By the definition of validity on a frame, the latter is equivalent to $F \models \wedge_{i=1}^n \phi_i$ so we are done. Q.E.D.

Definition 4.15 (global frame correspondent) *If a class of frames can be defined by a modal or tense formula A and a formula α from $\mathcal{L}^f$ or $\mathcal{L}_2^f$, then we say that A and α are* global frame correspondents *of each other.*

An equivalent formulation of the above definition is

A is a global correspondent of α iff for every frame F, $F \models A$ iff $F \models \alpha$

From (4.3), it follows that any modal or tense formula A is a global correspondent of the $\mathcal{L}_2^f$ formula $\widetilde{\forall \overline{P}}\, \forall x\, ST_x(A)$.

That every tense formula has a second-order global correspondent is not really surprising. After all, the standard translation essentially rehashes the definition of truth relation and validity. The requirement for monadic predicate quantification is clearly due to the fact that validity is defined (Definition 4.3) using a quantification over all valuations of a given frame. Quantifying over sets of variables (as opposed to just over variables) takes us out of the realm of first-order logic and into monadic second-order logic.

At the beginning of the section we noted that McKinsey's axiom and Löb's axiom have no first-order global correspondent. In other words, these formulae define classes of frames that cannot be described using first-order formulae (although the classes can be described using second-order formulae). What *is* surprising however is why certain tense formulae *do* have a *first-order* global correspondent. Consider some well-known examples. It is a standard result that the formulae $p \supset \blacklozenge p$ and $\Box p \supset \Box\Box p$ respectively define the class of reflexive frames and the class of transitive frames. These frame properties can be defined in turn by the $\mathcal{L}^f$ formulae $\forall x Rxx$ and $\forall xyz\,(Rxy \wedge Ryz \to Rxz)$. Thus $\forall x Rxx$ is a first-order global correspondent of $p \supset \blacklozenge p$, and $\forall xyz\,(Rxy \wedge Ryz \to Rxz)$ is a first-order global correspondent of $\Box p \supset \Box\Box p$.

It would be nice if we had some syntactic conditions to determine when a formula has a first-order correspondent. In Section 4.2.2 we introduce the *Sahlqvist formulae* which are a large class of formulae that are known to have first-order correspondents. In Theorem 4.31 we will see that each Sahlqvist formula corresponds to a formula belonging to a fragment of the first-order language called *Kracht formulae*, and also that each Kracht formula corresponds to a Sahlqvist formula.

Kracht formulae

Consider the following recursive abbreviations:

$$\begin{aligned}
\mathcal{R}^1 xy &:= Rxy \\
\mathcal{R}^{n+1} xy &:= (\exists y_{n+1} \rhd x)\mathcal{R}^n y_{n+1} y \quad (n \geq 1)
\end{aligned}$$

Notice that the $\mathcal{R}^n xy$ contains $n-1$ occurrences of the restricted existential quantifier. Since $\mathcal{R}^n$ *abbreviates* a formula in $\mathcal{L}^f$, usage of this symbol does not extend the $\mathcal{L}^f$ language.

Let us introduce the Kracht formulae. Without loss of generality, in this section we work with formulae in which no variable occurs as both free and bound. Also, no distinct quantifier occurrences will bind the same variable. Such a formula is called a *clean* formula.

Definition 4.16 (restrictedly positive) *An $\mathcal{L}^f$ formula is called* tense restrictedly positive *if it is built-up from atomic formulae of the form $u \neq u$, $u = u$, $u = v$ and $\mathcal{R}^s uv$ using $\wedge, \vee$ and the restricted quantifiers only. An $\mathcal{L}^f$ formula is called* modal restrictedly positive *if it is built-up from atomic formulae using $\wedge$, $\vee$ and the forward restricted quantifiers $(\exists y \rhd x)$ and $(\forall y \rhd x)$ only.*

Notice that a restrictedly positive formula will contain at least one free variable.

Definition 4.17 (inherently universal) *An occurrence of the variable y in the clean formula α is* inherently universal *if either y is free, or else y is bound by a restricted quantifier of the form $(\forall y \rhd x)\beta$ or $(\forall y \lhd x)\beta$ which is not in the scope of an existential quantifier.*

When x is not free in β, since

$$(\exists x\, \alpha(x)) \wedge \beta = \exists x\, (\alpha(x) \wedge \beta)$$
$$(\exists x\, \alpha(x)) \to \beta = \forall x\, (\alpha(x) \to \beta)$$

observe that

$$\begin{aligned}
ST_x(\Box^{m+1} p) &= (\forall y_1 \rhd x)(\forall y_2 \rhd y_1) \ldots (\forall y_{m+1} \rhd y_m) P y_{m+1} \\
&= \forall y_1 (Rxy_1 \to \forall y_2 (Ry_1y_2 \to \ldots \forall y_{m+1}(Ry_m y_{m+1} \to Py_{m+1}) \ldots)) \\
&= \forall y_1 \ldots y_{m+1} (Rxy_1 \to (Ry_1y_2 \to \ldots \to (Ry_m y_{m+1} \to Py_{m+1}) \ldots)) \\
&= \forall y_1 \ldots y_{m+1} (Rxy_1 \wedge Ry_1y_2 \wedge \ldots \wedge Ry_m y_{m+1} \to Py_{m+1}) \\
&= \forall y_{m+1} (\exists y_1 (Rxy_1 \wedge (\exists y_2 Ry_1y_2 \wedge \exists y_3 (\ldots))) \to Py_{m+1}) \\
&= \forall y_{m+1} (\mathcal{R}^{m+1} xy_{m+1} \to Py_{m+1})
\end{aligned}$$

Definition 4.18 (modal and tense Kracht formula) *A modal (resp. tense) restrictedly positive formula $\alpha(x)$ containing a single free variable x is called a* modal (tense) Kracht formula *if α is clean and in atomic formulae of the form $u = v$ and $\mathcal{R}^s uv$, either u or v is inherently universal.*

This definition agrees exactly with the definition given by Kracht [40]. We simply write *Kracht formula* to refer to both modal and tense Kracht formulae — the context will determine which type is meant.

4.2.2 From Sahlqvist formulae to first-order formulae

Sahlqvist formulae are a large class of formulae that are known to have first-order correspondents. Moreover, the first-order correspondent of a Sahlqvist formula is effectively computable and expressible as a Kracht formula. For our purposes, it will be sufficient to focus on a proper subclass of the Sahlqvist formulae called the *very simple Sahlqvist formulae* [7]. Furthermore, although the results apply to many different modal languages, here we will work with the basic temporal language (which properly includes the basic modal language). Sahlqvist's original paper can be found at [62].

Let $A^{\not\supset}$ be the tense formula that is obtained from A by eliminating all occurrences of the $\supset$-connective using the classical equivalence $B \supset C \approx \neg B \vee C$.

Definition 4.19 (positive, negative formula) *A tense formula A is positive in the propositional variable p (negative in p) if every occurrence of the monadic predicate variable p in $A^{\not\supset}$ is in the scope of an even (odd) number of negation signs. A formula is called* positive (negative) *if it is positive (negative) in all propositional variables occurring in it.*

Definition 4.20 (upward, downward monotone) *A tense formula A is* upward monotone *in p if for every valuation V' such that (i) $V(p) \subseteq V'(p)$ and (ii) for all $q \neq p$, $V(q) = V'(q)$,*

$$\textit{if } (F, V), w \models A \textit{ then } (F, V'), w \models A$$

A tense formula A is downward monotone *in p if for every valuation V' such that (i) $V(p) \supseteq V'(p)$ and (ii) for all $q \neq p$, $V(q) = V'(q)$,*

$$\textit{if } (F, V), w \models A \textit{ then } (F, V'), w \models A$$

Informally, A is upward monotone in p if whenever A is satisfied on some model at state w, then A is satisfied at w under any valuation that extends the interpretation (valuation) of p and keeps constant the interpretations on all other propositional variables.

Lemma 4.21 *Let A be a tense formula. Then,* *(i) if A is positive in p, then it is upward monotone in p.*

(ii) if A is negative in p, then it is downward monotone in p.

Proof. Straightforward. Q.E.D.

The notions of positive and negative formula, and upward and downward monotonicity can be defined analogously for monadic predicate variables for formulae in $\mathcal{L}_2^f$. An analogous result to the above Lemma applies to these formulae.

Recall the definition of global frame correspondent (Definition 4.15). The Sahlqvist correspondence theorems actually prove a stronger version of correspondence called *local* frame correspondence. Let us introduce this notion before proceeding to the theorem.

Definition 4.22 (local frame correspondents) *Let A be a modal or tense formula, and suppose that $\alpha(x)$ is a formula in $\mathcal{L}^f$ or $\mathcal{L}_2^f$ containing a single free variable x. We say that A and $\alpha(x)$ are* local frame correspondents *of each other if, for all frames F and states w in F:*

$$F, w \models A \text{ iff } F \models \alpha[w/x]$$

It is easy to see that if A and $\alpha(x)$ are local frame correspondents, then it must be the case that A and $\forall x \alpha$ are global frame correspondents.

We will say "A and α are frame correspondents" or simply that A and α are each other's correspondents to mean that A and α are local frame correspondents. When the formula α is from $\mathcal{L}^f$, we will say that α is a first-order correspondent of A. When we wish to refer to the notion of global frame correspondenence we will take care to explicitly use the term "global".

As an aside we observe that global frame correspondence does not imply local frame correspondence. For it is known that $(\Box\Diamond p \to \Diamond\Box p) \wedge (\Diamond\Diamond q \to \Diamond q)$ and the $\mathcal{L}^f$ formula

$$(\forall x \exists y (Rxy \wedge \forall z (Ryz \to z = y))) \wedge (\forall xyz (Rxy \wedge Ryz \to Rxz))$$

are global correspondents, but it is known that the above modal formula does not have a local frame correspondent in $\mathcal{L}^f$ (see [7, page 169]).

We have already noted (see the discussion following equation (4.2) in Section 4.2.1) that for any formula A and all frames F and states w in F:

$$F, w \models A \text{ iff } F \models \widetilde{\forall} \overline{P}\, ST_x(A)[w/x]$$

so A and the $\mathcal{L}_2^f$ formula $\widetilde{\forall \overline{P}}\, ST_x(A)[w/x]$ are local frame correspondents.

We are ready to define very simple Sahlqvist formulae and present the correspondence results.

Definition 4.23 (very simple Sahlqvist formula) *A* very simple Sahlqvist antecedent *in the basic temporal language is a formula built up from* $\top$, $\bot$ *and propositional letters, using only* $\wedge$, $\Diamond$ *and* $\blacklozenge$. *A* (tense) very simple Sahlqvist formula *is an implication* $A \supset B$ *in which* B *is positive and* A *is a very simple Sahlqvist antecedent in the basic temporal language.*

Theorem 4.24 *Let* $D \approx A \supset B$ *be a very simple Sahlqvist formula in the basic temporal language. Then there is a tense Kracht formula* $\alpha_D(x)$ *that is effectively computable from* D *such that* $\alpha_D(x)$ *is a first-order correspondent of* D.

We closely follow the presentation given in [7], the main deviation being the use of the restricted quantifier notation. This enables us to directly obtain the first half of Theorem 4.31. In [7] this has to be worked out separately.

Proof. The 'effectively computable' statement follows from the fact that it is straightforward to write a program to implement the following algorithm.

The second-order translation of D is the formula $\widetilde{\forall \overline{P}}\,(ST_x(A) \to ST_x(B))$. Let us denote $ST_x(B)$ by POS to remind us that this formula is positive. Then $ST_x(D)$ can be written

$$\widetilde{\forall \overline{P}}(ST_x(A) \to POS) \tag{4.4}$$

Without loss of generality, we may assume that no two quantifiers in the above formula bind the same variable, and no quantifier binds x (pre-processing step).

Step 1. Whenever y is not free in β, we have the classical equivalences

$$(\exists^r y\, \alpha(y)) \wedge \beta = \exists^r y\, (\alpha(y) \wedge \beta)$$

and

$$\begin{aligned}((\exists^r y\, \alpha(y)) \to \beta) &= (\exists y \rhd x)\alpha(y) \to \beta \text{ or } (\exists y \lhd x)\alpha(y) \to \beta\\ &= (\exists y(Rxy \wedge \alpha(y))) \to \beta \text{ or } (\exists y(Ryx \wedge \alpha(y))) \to \beta\\ &= \forall y(Rxy \wedge \alpha(y) \to \beta) \text{ or } \forall y(Ryx \wedge \alpha(y) \to \beta)\\ &= \forall y(Rxy \to (\alpha(y) \to \beta)) \text{ or } \forall y(Ryx \to (\alpha(y) \to \beta))\\ &= \forall^r y\, (\alpha(y) \to \beta)\end{aligned}$$

Using these equivalences (in that order) we can convert all existential restricted quantifiers in the antecedent $ST_x(A)$ to universal restricted quantifiers over the main implication in (4.4).

Step 1 results in a formula of the form

$$\widetilde{\forall}\overline{P}\,\forall^r\overline{x}\,(AT \to POS) \tag{4.5}$$

where AT is a conjunction of (translations of) proposition letters, and $\overline{x}$ is a set $\{x_1, \ldots, x_m\}$ of proposition variables not containing x.

Step 2. Let P_i be a unary predicate occurring in (4.5) and let $P_i x_{i_1}, \ldots P_i x_{i_k}$ be all the occurrences of the predicate P_i in the antecedent AT of (4.5). Define the predicate $\sigma(P_i)$ using the following characteristic function:

$$\sigma(P_i)(\omega) = \begin{cases} \mathbf{t} & \omega \in \{x_{i_1}, \ldots x_{i_k}\}, \\ \mathbf{f} & \text{otherwise} \end{cases} \tag{4.6}$$

Let M be an arbitrary model, and let $w, w_1, \ldots, w_m$ be some arbitrary states in M. Now suppose that $M \models AT[w/x][\{w_i/x_i\}]$ and $M \models \sigma(P_i)(u)[w/x][\{w_i/x_i\}]$ for some state u. Due to the definition of $\sigma(P_i)(u)$ it follows that u (under the substitutions $[w/x][\{w_i/y_i\}]$) must be a variable in $\{x_{i_1}, \ldots x_{i_k}\}$. Since each $P_i x_{i_j}$ term occurs as a conjunct in AT, from $M \models AT[w/x][\{w_i/x_i\}]$ it follows that $M \models P_i u[w/x][\{w_i/x_i\}]$. We have shown that

$$M \models AT[w/x][\{w_i/y_i\}] \text{ implies } M \models \forall u(\sigma(P_i)(u) \to P_i u)[w/x][\{w_i/x_i\}] \tag{4.7}$$

Step 3. Instantiate $\sigma(P_i)$ for each P_i in (4.5). This results in a formula of the form

$$\forall^r\overline{x}\,(AT \to POS)[\{\sigma(P_i)/P_i\}]$$

Since $AT[\{\sigma(P_i)/P_i\}]$ is trivially true by the definition of σ, this is equivalent to

$$\forall^r\overline{x}\,(POS[\{\sigma(P_i)/P_i\}]) \tag{4.8}$$

Notice that the above formula is a Kracht formula. More precisely, the substitution $[\{\sigma(P_i)/P_i\}]$ results in a formula containing terms of the form $\sigma(P_i)(u)$ — this term can be reduced to the first-order term

$$(u = x_{i_1}) \vee \ldots \vee (u = x_{i_k})$$

Moreover POS is the formula $ST_x(B)$, which is constructed using the restricted quantifiers. Finally, every variable occurring in the formula is inherently universal because there are no existential restricted quantifiers to worry about.

To complete the proof, it suffices to show that (4.8) is equivalent to (4.5). Certainly (4.5) implies (4.8) since (4.8) is an instantiation of (4.5). Now for the other direction. Recall that (4.5) is shorthand for the formula

$$\widetilde{\forall}\overline{P}\,\overbrace{(\forall x_1 \bowtie y_1)\ldots(\forall x_m \bowtie y_m)}^{\forall^r \overline{x}}(AT \rightarrow POS)$$

for some $\{y_1, \ldots, y_m\}$ where $\bowtie$ means either $\rhd$ or $\lhd$. With respect to this formula, define

$$\mathcal{S}_i = \begin{cases} Rx_iy_i & \text{if } (\forall x_i \rhd y_i) \text{ occurs in } \forall^r \overline{x} \\ Ry_ix_i & \text{if } (\forall x_i \lhd y_i) \text{ occurs in } \forall^r \overline{x} \end{cases}$$

Now observe that formula (4.5) is equivalent to the statement: for any model M and states $w, w_1, \ldots, w_m$,

$$M \models \left(\bigwedge_{i=1}^{m} \mathcal{S}_i[w_i/x_i]\right) \wedge AT[ww_1, \ldots, w_m] \text{ implies } M \models POS[ww_1 \ldots w_m]$$

Thus, in order to show that (4.8) implies (4.5), we assume for arbitrary model M that

$$M \models \forall^r x\,(POS[\{\sigma(P_i)/P_i\}]) \tag{4.9}$$

and

$$M \models \left(\bigwedge_{i=1}^{m} \mathcal{S}_i[w_i/x_i]\right) \wedge AT[ww_1, \ldots, w_m] \tag{4.10}$$

and show that $M \models POS[ww_1 \ldots w_m]$. Instantiating w_i for each x_i in (4.9) and expanding the $\forall^r \overline{x}$ notation we get

$$M \models \mathcal{S}_1[w_1/x_1] \rightarrow (\mathcal{S}_2[w_2/x_2] \rightarrow (\ldots \rightarrow (\mathcal{S}_m[w_m/x_m] \rightarrow POS)\ldots))[\{\sigma(P_i)/P_i\}]$$

Since $M \models \mathcal{S}_i[w_i/x_i]$ for each i from (4.10), we obtain

$$M \models POS[ww_1 \ldots w_m][\{\sigma(P_i)/P_i\}] \tag{4.11}$$

Furthermore since we have $M \models AT[ww_1, \ldots, w_m]$ (4.10), from (4.7) it follows that

$$M \models \forall u(\sigma(P_i)(u) \rightarrow P_iu)[ww_1 \ldots w_m]$$

What this says is that if $u \in \sigma(P_i)[ww_1 \ldots w_m]$ (viewing the monadic predicate as a set), then $u \in P_i[ww_1 \ldots w_m]$. Thus $\sigma(P_i)$ is a *minimal* valuation in the sense that the set $V(p_i)$ for an arbitrary model satisfying (4.9) and (4.10) extends $\sigma(P_i)$. As $POS[ww_1 \ldots w_m]$ is positive, it is upward monotone is all unary predicates occurring in it, so by (4.11) and Lemma 4.21 we get $M \models POS[ww_1 \ldots w_m]$. Q.E.D.

Example 4.25 *Consider the very simply Sahlqvist formula* $\blacklozenge\Diamond p \supset \Diamond\blacklozenge p$. *The second-order translation of this formula is*

$$\tilde{\forall} P((\exists u \lhd x)(\exists v \rhd u)Pv \to (\exists l \rhd x)(\exists m \lhd l)Pm)$$

Step 1. Pushing the existential restricted quantifiers in the antecedent of the implication outwards, we obtain the equivalent formula

$$\tilde{\forall} P(\forall u \lhd x)(\forall v \rhd u)(Pv \to (\exists l \rhd x)(\exists m \lhd l)Pm) \tag{4.12}$$

Step 2. Set $\sigma(P)(\omega)$ *as* $\mathbf{t}$ *if* $\omega = v$ *and* $\mathbf{f}$ *otherwise.*
Step 3. Instantiating $\sigma(P)$ *for the predicate* P *in (4.12) we obtain*

$$(\forall u \lhd x)(\forall v \rhd u)(v = x \to (\exists l \rhd x)(\exists m \lhd l)v = m)$$

The above formula is equivalent to $(\forall u \lhd x)(\forall v \rhd u)(\exists l \rhd x)Rvl$. *Notice that this formula is a Kracht tense formula. From the proof of Theorem 4.24 we know that it is equivalent to (4.12) and hence it is a local correspondent of* $\blacklozenge\Diamond p \supset \Diamond\blacklozenge p$.

Before proceeding, for the sake of completeness we state the definition and result for full Sahlqvist formulae. In the following, a boxed atom in the basic temporal language is a propositional variable preceeded by a (possibly empty) string constructed from $\Box$ and $\blacksquare$'s. Notice that a boxed atom with an empty string is simply a propositional variable.

Definition 4.26 (modal and tense Sahlqvist formulae) *Define a* Sahlqvist antecedent *in the basic temporal language to be a formula built from* $\top$, $\bot$, *boxed atoms, and negative formulae, using* $\wedge$, $\vee$ *and existential modal operators (*$\Diamond$ *and* $\blacklozenge$*). A Sahlqvist implication in the basic temporal language is an implication* $A \supset B$ *in which* B *is positive and* A *is a Sahlqvist antecedent in the basic temporal language.*

A tense Sahlqvist formula *is a formula that is built from Sahlqvist implications in the basic temporal language by freely applying boxes and conjunctions, and by applying disjunctions only between formulae that do not share any proposition letters.*

A modal Sahlqvist formula *is a tense Sahlqvist formula that does not contain either* $\blacksquare$ *or* $\blacklozenge$.

Although we have been working with tense Sahlqvist formulae, corresponding results apply to modal Sahlqvist formulae in the obvious way. Note that we will often drop the word "modal" and "tense" prefixing the term "Sahlqvist/Kracht formula" when the results apply by the uniform usage of either word. By inspection we see that every very simple Sahlqvist formula is also a Sahlqvist formula.

The main result is the Sahlqvist correspondence theorem.

Theorem 4.27 (Sahlqvist correspondence theorem) *Let D be a tense (resp. modal) Sahlqvist formula. Then there is a tense (modal) Kracht formula $\alpha_D(x)$ that is effectively computable from D such that $\alpha_D(x)$ is a first-order correspondent of D.*

The proof is an extension of the proof of Theorem 4.24. See [7] for details.

Observe that if $\{A_1, \ldots, A_n\}$ is a set of Sahlqvist formulae, then together with Lemma 4.14 we can compute a set $\{\alpha_1, \ldots, \alpha_n\}$ of Kracht formulae, each with the single free variable w, such that $\wedge_i A_i$ corresponds to $\wedge_i \forall w \alpha_i(w)$. Since $\wedge_i \forall w \alpha_i(w)$ is frame-equivalent to $\forall w \wedge_i \alpha_i(w)$, it follows that $\wedge_i A_i$ corresponds to $\forall w \wedge_i \alpha_i(w)$. We can rewrite $\wedge_i \alpha_i(w)$ as a (clean) Kracht formula by appropriate renaming of bound variables in the formula to ensure that no distinct quantifier occurrences bind the same variable.

It should be mentioned that the class of Sahlqvist formulae is not the last word on formulae with first-order correspondents. See [7] for (i) an example of a formula that has a local first-order correspondent, and (ii) an example of a formula that has a global first-order correspondent but no local first-order correspondent, where in each case, the respective formula is not equivalent to a Sahlqvist formula. Despite such results, the class of Sahlqvist formulae is expressive enough to be useful in many of the cases encountered in practice.

4.2.3 From Kracht formulae to Sahlqvist formulae

In the previous section we saw how to compute the local first-order correspondent of a Sahlqvist formula. Kracht [38, 40] has identified a class of first-order formulae ('Kracht formulae') such that each Kracht formula is a correspondent of some Sahlqvist formula. In other words, Kracht's result is a converse to the Sahlqvist correspondence theorem.

In this section we focus on modal Kracht formulae. The generalisation to tense Kracht formulae is straightforward. We will write $\exists^{r'} y$ as an abbreviation to mean $(\exists y \rhd x)$ for some x, and $\forall^{r'} y$ as an abbreviation to mean $(\forall y \rhd x)$ for some x. We write $\mathbf{Q}^{r'} y$ when we do not want to specify whether we mean $\exists^{r'} y$ or $\forall^{r'} y$.

Kracht [38, 40] presented a Calculus of Internal Descriptions that can be used to compute the formula correspondent of a given Kracht formula — his calculus uses general frames [7] rather than Kripke frames. Instead of using that calculus, we will use the algorithm presented in Blackburn *et al.* [7, Theorem 3.59] to accomplish this task. We have identified the following errors with this algorithm — we expand on this at the end of this section.

(i) The *type 1* characterisation that used in the proof cannot be obtained using the algorithm. This is because the algorithm makes use of an equivalence that is invalid. Consequently the *type 2* characterisation is unattainable via the algorithm as well. Here we use revised characterisations type $1'$ and type $2'$ respectively.

(ii) The definition of Kracht formula used there [7, Definition 3.58] is deficient.[1] The revised definition (which coincides with Kracht's original definition) introduces new cases for the algorithm that need to be dealt with.

Here we sketch the proof and show how to resolve the above problems. Aside from the resolution of these issues, note that we follow closely the detailed proof given in [7]. For this reason it may be helpful to read the following in conjunction with that proof.

Recall that the notation $\beta(u_1, \ldots, u_n)$ is used to identify the free variables in β. Now we introduce the notation

$$\beta(\mathbf{Q}_1^{r'} y_1, \ldots, \mathbf{Q}_m^{r'} y_m; u_1, \ldots, u_n)$$

to mean a formula β containing restricted quantifier occurrences $\mathbf{Q}_1^{r'} y_1, \ldots, \mathbf{Q}_m^{r'} y_m$ and free variables $u_1, \ldots, u_n$. To simplify the notation we denote the above formula as $\beta(\mathbf{Q}^{r'}\overline{y}; \overline{u})$ for sequences $\overline{y} = (y_1, \ldots, y_m)$ and $\overline{u} = (u_1, \ldots, u_n)$.

Definition 4.28 (type $1'$ formula) *A* type $1'$ *formula is a Kracht formula of the following form containing the single free variable x_0:*

$$\forall^{r'} x_1 \ldots \forall^{r'} x_n\, \beta(\mathbf{Q}_1^{r'} y_1, \ldots, \mathbf{Q}_m^{r'} y_m; x_0, \ldots, x_n, \ldots, y_1, \ldots, y_m) \tag{4.13}$$

such that $n, m \geq 0$ and each variable is restricted by an earlier variable (that is, the restrictor of any x_i is some x_j with $j < i$ and the restrictor of any y_i is either some x_k or some y_j with $j < i$. Furthermore, β is a disjunction of conjunctions of restricted quantifiers and atomic formulae of the form $u \neq u$, $u = u$, $u = v$ and $\mathcal{R}^s uv$ (ie β is in disjunctive normal form DNF).

[1] Thanks to M. Kracht for clarifying the shortcomings in the definition of modal Kracht* formula (in our terminology) given in [7, Definition 3.58].

In this section, u, z_i denote arbitrary variables in $\{x_0, \ldots, x_n, y_1, \ldots y_m\}$ and x denotes an arbitrary variable in $\{x_0, \ldots, x_n\}$.

The first step is to show how to rewrite a given Kracht formula α as a type $1'$ formula. The idea is to use repeatedly the equivalences

$$(\forall^{r'} x\, \delta) \heartsuit \gamma = \forall^{r'} x\, (\delta \heartsuit \gamma)$$

(where $\heartsuit$ uniformly denotes either $\wedge$ or $\vee$) to pull out each universal restricted quantifier in α, quantifying over an inherently universal variable, to the front. This is possible because no universal restricted quantifier quantifying an inherently universal variable can occur within the scope of an existential restricted quantifier (see Definition 4.17). It is easy to verify that the resulting formula is a type $1'$ formula.

In the following definition, a boxed atom in the propositional variable p is a formula of the form $\Box^t p$ for $t \geq 0$.

Definition 4.29 (type 2′ formula) *A* type $2'$ *formula is a formula in $\mathcal{L}_2^f$ of the following form containing the single free variable x_0:*

$$\tilde{\forall} P_1 \ldots \tilde{\forall} P_n \tilde{\forall} Q_{1,1} \ldots \tilde{\forall} Q_{1,r_1} \ldots \tilde{\forall} Q_{n,r_n} \forall^{r'} x_1 \ \ldots \forall^{r'} x_n \left(\bigwedge_{0 \leq i \leq n} ST_{x_i}(\sigma_i) \rightarrow \gamma \right)$$

such that each σ_i is a conjunction of boxed atoms in the propositional variables p_i and $q_{i,1} \ldots q_{i,r_i}$, and γ is a DNF of formulae $ST_x(B)$ ($x \in \{x_i\}$) where B is a modal formula which is positive in every propositional variable and containing only those propositional variables occurring in $\cup\{\sigma_i\}$.

We have seen how to rewrite a Kracht formula as a type $1'$ formula. The next step is to show that every type $1'$ formula can be effectively rewritten as an equivalent type $2'$ formula.

Suppose that we are given the type 1 formula $\forall^{r'} \overline{x}\, \beta(\mathbf{Q}^{r'} \overline{y})$. Let β' be the formula obtained from β by replacing each subformula

$$\begin{gathered} u \neq u \text{ with } ST_u(\bot) \\ u = u \text{ with } ST_u(\top) \\ u = x_i \text{ with } ST_u(p_i) \\ \mathcal{R}^n u x_i \text{ with } ST_u(\Diamond^n p_i) \end{gathered}$$

and each occurrence (indexed by j) of $\mathcal{R}^{m(i,j)} x_i u$ ($m(i,j) \geq 1$) in β with $ST_u(q_{i,j})$. That is, the index j corresponds to the variable x_i and the index r signifies the j-th occurrence of $\mathcal{R}^m x_i u$ in β.

The claim is that $\forall^{r'}\overline{x}\,\beta(\mathbf{Q}^{r'}\overline{y})$ is equivalent to

$$\widetilde{\forall}\overline{PQ}\forall^{r'}\overline{x}\left(\bigwedge_{0\leq i\leq n} ST_{x_i}\left(p_i \wedge \left[\bigwedge_{(i,j)} \Box^{m(i,j)} q_{i,j}\right]\right) \to \beta'(\mathbf{Q}^{r'}\overline{y})\right) \tag{4.14}$$

Remark 4.30 *Notice that we could simply replace $\mathcal{R}^m x_i u$ with $ST_u(\blacklozenge^m p_i)$ in the above (instead of introducing the $ST_u(q_{i,j})$ terms) and proceed in the obvious way if we are content to obtain a (tense) formula containing occurrences of $\blacklozenge$. However our aim is to compute a modal formula from a modal Kracht formula. Hence we are forced to introduce the $\Box$ operator to handle the $\mathcal{R}^n x_i u$ terms.*

Define the predicates $\sigma(P_i)$ and $\sigma(Q_{i,r})$ using the following characteristic functions:

$$\sigma(P_i)(\omega) = \begin{cases} \mathbf{t} & \omega = x_i \\ \mathbf{f} & \text{otherwise} \end{cases}$$

and

$$\sigma(Q_{i,r})(\omega) = \begin{cases} \mathbf{t} & \mathcal{R}^{m(i,r)} x_i \omega \\ \mathbf{f} & \text{otherwise} \end{cases}$$

($\Leftarrow$) Consider the formula obtained from (4.14) by the instantiation of each predicate P_i and $Q_{i,r}$ with $\sigma(P_i)$ and $\sigma(Q_{i,r})$ respectively:

$$\forall^{r'}\overline{x}\left(\bigwedge_{0\leq i\leq n} ST_{x_i}\left(p_i \wedge \left[\bigwedge_{(i,r)} \Box^{m_{i,r}} q_{i,r}\right]\right) \to \beta'(\mathbf{Q}^{r'}\overline{y})\right)[\{\sigma(P_i)/P_i\}][\{\sigma(Q_{i,r})/Q_{i,r}\}] \tag{4.15}$$

Let $[\mathcal{S}]$ denote the substitution $[\{\sigma(P_i)/P_i\}][\{\sigma(Q_{i,r})/Q_{i,r}\}]$. Note that

$$ST_{x_i}(p_i)[\mathcal{S}] = P_i x_i[\mathcal{S}] = (x_i = x_i) = \mathbf{t}$$

and

$$\begin{aligned} ST_{x_i}(\Box^{m(i,r)} q_{i,r})[\mathcal{S}] &= \forall y(\mathcal{R}^{m(i,r)} x_i y \to Q_{i,r} y)[\mathcal{S}] \\ &= \forall y(\mathcal{R}^{m(i,r)} x_i y \to \mathcal{R}^{m(i,r)} x_i y) \\ &= \mathbf{t} \end{aligned}$$

Thus, under the substitution $[\mathcal{S}]$, the entire formula preceeding the implication connective $\to$ in (4.15) becomes $\mathbf{t}$, so that formula is equivalent to

$$\forall^{r'}\overline{x}\,\beta'(\mathbf{Q}^{r'}\overline{y})[\mathcal{S}] \tag{4.16}$$

Since

$$\begin{aligned}
ST_u(\top)[\mathcal{S}] &= (u = u)\\
ST_u(\bot)[\mathcal{S}] &= (u \neq u)\\
ST_u(p_i)[\mathcal{S}] &= P_i u[\mathcal{S}] = (x_i = u)\\
ST_{z_i}(q_{i,r})[\mathcal{S}] &= Q_{i,r} z_i[\mathcal{S}] = \mathcal{R}^{m(i,r)} x_i z_i\\
ST_u(\Diamond^n p_i)[\mathcal{S}] &= (\exists y_n \rhd u) \ldots (\exists y_2 \rhd y_3)(\exists y_1 \rhd y_2) P_i y_1[\mathcal{S}]\\
&= (\exists y_n \rhd u) \ldots (\exists y_2 \rhd y_3) \exists y_1 (R y_2 y_1 \wedge y_1 = x_i)\\
&= (\exists y_n \rhd u) \ldots (\exists y_2 \rhd y_3) R y_2 x_i\\
&= \mathcal{R}^n u x_i
\end{aligned}$$

by inspection, $\forall^{r'} \overline{x}\, \beta'(\mathbf{Q}^{r'} \overline{y})[\mathcal{S}]$ is equivalent to $\forall^{r'} \overline{x}\, \beta(\mathbf{Q}^{r'} \overline{y})$.

($\Rightarrow$) For any model M and states $\overline{w} = \{w, w_1, \ldots, w_m\}$, it suffices to show that

$$M \models \forall^{r'} \overline{x}\, \beta(\mathbf{Q}^{r'} \overline{y}) \tag{4.17}$$

and

$$M \models \bigwedge_{0 \leq i \leq n} ST_{x_i} \left(p_i \wedge \left[\bigwedge_{(i,r)} \Box^{m(i,r)} q_{i,r} \right] \right) [\overline{w}] \tag{4.18}$$

implies

$$M \models \beta(\mathbf{Q}^{r'} \overline{y})[\overline{w}]$$

Note that the above formula is positive for all predicates P_i and $Q_{i,r}$ and hence upward monotone due to Lemma 4.21. We showed above that $\forall^{r'} \overline{x}\, \beta'(\mathbf{Q}^{r'} \overline{y})[\mathcal{S}]$ is equivalent to $\forall^{r'} \overline{x}\, \beta(\mathbf{Q}^{r'} \overline{y})$, and thus from (4.17) we have

$$M \models \beta'(\mathbf{Q}^{r'} \overline{y})[\overline{w}][\mathcal{S}]$$

Furthermore, for all i and j indices, (4.18) implies

$$M \models P_i x_i[\overline{w}] \qquad\qquad M \models \forall y(\mathcal{R}^{m(i,j)} x_i y \to Q_{i,j} y)[\overline{w}]$$

It follows for every $\sigma(P_i)$ and $\sigma(Q_{i,r})$ that

$$M \models \forall y(\sigma(P_i)(y) \to P_i(y))[\overline{w}] \qquad M \models \forall y(\sigma(Q_{i,r})(y) \to Q_{i,r}(y))[\overline{w}]$$

It follows that every $\sigma(P_i)$ and $\sigma(Q_{i,r})$ predicate is a minimal valuation. The result follows due to upward monotonicity.

The final step is to show that every type $2'$ formula can be rewritten as a Sahlqvist formula. First we delete the restricted quantifiers $\mathbf{Q}^{r'} \overline{y}$ in β'. The idea is to work from the innermost quantifier outwards as follows: delete an existential restricted quantifier appearing in β' as $(\exists u \rhd v)ST_u(B)$ by substituting with $ST_v(\Diamond B)$; when a universal restricted quantifier $(\forall u \rhd v)ST_u(B)$ is encountered, substitute this occurrence with $ST_v(\neg\Diamond\neg B)$. The remaining manipulations are usually straightforward in practice, so we omit the algorithmic details.

Notice that we have taken care to map modal Kracht formulae into modal Sahlqvist formulae (see Remark 4.30). The generalisation to tense Kracht formulae is straightforward. Combining this result with Theorem 4.27 we get

Theorem 4.31 (Kracht's theorem) *Every tense (resp. modal) Sahlqvist formula locally corresponds to a tense (modal) Kracht formula. Also, every tense (modal) Kracht formula is a local first-order correspondent of some tense (modal) Sahlqvist formula which can be effectively obtained from the Kracht formula.*

Note that if $\{\alpha_1, \ldots, \alpha_n\}$ is some set of Kracht formulae, then we can compute a set $\{A_1, \ldots, A_n\}$ of Sahlqvist formulae such that $\alpha_1 \wedge \ldots \alpha_n$ corresponds to $A_1 \wedge \ldots \wedge A_n$.

Example 4.32 *Consider the Kracht formula*

$$(\forall u \rhd x)(\forall v \rhd x)(\exists z \rhd u)(\mathcal{R}^1 vz \wedge \mathcal{R}^2 vz) \tag{4.19}$$

This is already a type 1′ formula. Following the above proof, we will replace $\mathcal{R}^1 vz$ *with* $ST_z(q_1)$ $(= Q_1 z)$ *and* $\mathcal{R}^2 vz$ *with* $ST_z(q_2)$ $(= Q_2 z)$*, so that the above formula is equivalent to the following type 2′ formula:*

$$\tilde{\forall} Q_1 \tilde{\forall} Q_2 (\forall u \rhd x)(\forall v \rhd x)\left(ST_v(\Box q_1 \wedge \Box\Box q_2) \to (\exists z \rhd u)(ST_z(q_1) \wedge ST_z(q_2))\right)$$

The manipulation from here is straightforward. Since $(\exists z \rhd u)ST_z(q_1 \wedge q_2)$ *can be replaced with* $ST_u(\Diamond(q_1 \wedge q_2))$ *we have*

$$\tilde{\forall} Q_1 \tilde{\forall} Q_2 (\forall u \rhd x)(\forall v \rhd x)\left(ST_v(\Box q_1 \wedge \Box\Box q_2) \to ST_u(\Diamond(q_1 \wedge q_2))\right)$$

This is equivalent to

$$\tilde{\forall} Q_1 \tilde{\forall} Q_2 \neg(\exists u \rhd x)(\exists v \rhd x) ST_v(\Box q_1 \wedge \Box\Box q_2) \wedge \neg ST_u(\Diamond(q_1 \wedge q_2))$$

This simplies to

$$\tilde{\forall} Q_1 \tilde{\forall} Q_2 \neg ST_x(\Diamond(\Box q_1 \wedge \Box\Box q_2) \wedge \Diamond\neg\Diamond(q_1 \wedge q_2))$$

This means that (4.19) corresponds to the Sahlqvist formula

$$\Diamond(\Box q_1 \wedge \Box\Box q_2) \wedge \Diamond\neg\Diamond(q_1 \wedge q_2) \supset \bot$$

or simply $\Diamond(\Box q_1 \wedge \Box\Box q_2) \supset \Box\Diamond(q_1 \wedge q_2)$.

If we are happy to compute a tense formula corresponding to (4.19), as discussed in Remark 4.30 we can then replace $\mathcal{R}^1vz$ *and* $\mathcal{R}^2vz$ *respectively, with* $ST_z(\blacklozenge p)$ *and* $ST_z(\blacklozenge\blacklozenge p)$. *Then we can obtain the equivalent type 2 formula*

$$\widetilde{\forall} P_1 \widetilde{\forall} P_2 (\forall u \rhd x)(\forall v \rhd x)\,(ST_v(p) \to (\exists z \rhd u)(ST_z(\blacklozenge p) \wedge ST_z(\blacklozenge\blacklozenge p)))$$

Simplifying as above we get

$$\widetilde{\forall} P_1 \widetilde{\forall} P_2 \neg ST_x(\Diamond p \wedge \Diamond\neg\Diamond(\blacklozenge p \wedge \blacklozenge\blacklozenge p))$$

The corresponding (tense) Sahlqvist formula is $\Diamond p \supset \Box\Diamond(\blacklozenge p \wedge \blacklozenge\blacklozenge p)$.

Example 4.33 *Consider the Kracht formula*

$$(\exists u \rhd x)\mathcal{R}^1 ux \vee (\forall y \rhd x)(\exists v \rhd y)(\forall w \rhd v)\mathcal{R}^1 wx \tag{4.20}$$

The inherently universal variables in the above formula are $\{x, y\}$. *We can pull out the universal restricted quantifier to obtain the equivalent type 1′ formula*

$$(\forall y \rhd x)\,\big((\exists u \rhd x)\mathcal{R}^1 ux \vee (\exists v \rhd y)(\forall w \rhd v)\mathcal{R}^1 wx\big)$$

This is equivalent to the type 2′ formula

$$\widetilde{\forall} P (\forall y \rhd x)\,(ST_x(p) \to (\exists u \rhd x)ST_u(\Diamond p) \vee (\exists v \rhd y)(\forall w \rhd v)ST_w(\Diamond p))$$

Then we simplify

$$\widetilde{\forall} P (\forall y \rhd x)\,(ST_x(p) \to ST_x(\Diamond\Diamond p) \vee (\exists v \rhd y)(\forall w \rhd v)ST_w(\Diamond p))$$
$$\widetilde{\forall} P (\forall y \rhd x)\,(ST_x(p) \to ST_x(\Diamond\Diamond p) \vee (\exists v \rhd y)ST_v(\neg\Diamond\neg\Diamond p))$$
$$\widetilde{\forall} P (\forall y \rhd x)\,(ST_x(p) \to ST_x(\Diamond\Diamond p) \vee ST_y(\Diamond\neg\Diamond\neg\Diamond p))$$
$$\widetilde{\forall} P \neg\,(ST_x(p \wedge \neg\Diamond\Diamond p \wedge \Diamond\neg\Diamond\neg\Diamond\neg\Diamond p))$$

The modal correspondent of the above is

$$p \wedge \neg\Diamond\Diamond p \wedge \Diamond\neg\Diamond\neg\Diamond\neg\Diamond p \supset \bot$$

This simplifies to the Sahlqvist formula

$$p \supset \Diamond\Diamond p \vee \Box\Diamond\Box\Diamond p$$

Let us expand on the errors concerning Definition 3.58 and the proof of Theorem 3.59 in Blackburn *et al.* [7]:

(i) The first step in the algorithm given in [7] for obtaining a modal correspondent from a Kracht formula involves transforming the given Kracht formula into a type 1 formula, where the latter is defined as follows.

Definition 4.34 (type 1 formula [7]) *A* type 1 *formula is a Kracht formula of the following form containing the single free variable x_0:*

$$\forall^{r'} x_1 \ldots \forall^{r'} x_n \, \mathbf{Q}_1^{r'} y_1 \ldots \mathbf{Q}_m^{r'} y_m \beta(x_0, \ldots, x_n, \ldots, y_1, \ldots, y_m) \tag{4.21}$$

such that $n, m \geq 0$ and each variable is restricted by an earlier variable (that is, the restrictor of any x_i is some x_j with $j < i$ and the restrictor of any y_i is either some x_k or some y_j with $j < i$. Furthermore, β is a disjunction of conjunctions of atomic formulae of the form $u \neq u$, $u = u$, $u = v$ and $\mathcal{R}^s uv$ (ie β is in disjunctive normal form DNF).

Comparing (4.13) with (4.21), the only difference is that in the latter case the existential restricted quantifiers have been taken 'outside' β. In order to achieve this, the following equivalence is quoted in [7]:

$$(\exists^{r'} u \, \delta) \vee \gamma = \forall^{r'} u \, (\delta \, \vee \, \gamma) \tag{4.22}$$

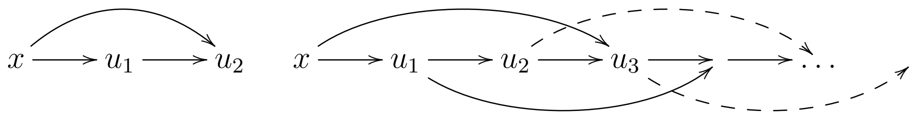
A simple counterexample to the above is the Kracht formula $(\exists u \rhd x)(\mathbf{t}) \vee \mathbf{t}$ where $\mathbf{t}$ should be read as $x = x$. To see this, first observe that this formula is equivalent in first-order logic to the formula $\mathbf{t}$. If the above equivalence was valid, then $(\exists u \rhd x)(\mathbf{t}) \vee \mathbf{t}$ would be equivalent to $(\exists u \rhd x)(\mathbf{t} \vee \mathbf{t})$ and hence to $(\exists u \rhd x)(\mathbf{t})$. Since $(\exists u \rhd x)\mathbf{t}$ and $\mathbf{t}$ are obviously *not* equivalent, it follows that the above equivalence is not valid.

Of course, although the algorithm [7] for transforming a Kracht formula into a type 1 formula is incorrect due to the invalidity of the (4.22), this does not necessarily mean that the type 1 characterisation itself is unattainable. After all, the counterexample $(\exists u \rhd x)(\mathbf{t}) \vee \mathbf{t}$ to the algorithm is equivalent to $\mathbf{t}$, and $\mathbf{t}$ is a type 1 formula.

Since we do not actually need the type 1 characterisation for our work, we do not pursue the question of whether this characterisation is attainable, and if it is, how to obtain an algorithm that witnesses this. Nevertheless we observe that the main obstacle seems to be the simplification of the expression $(\exists^{r'} x\, \delta) \vee \mathcal{R}^s lm$ in a Kracht formula (this is a special case of the lefthand side of (4.22)). As an illustration, consider the following Kracht formula α:

$$(\forall y_1 \rhd x)(\forall y_2 \rhd y_1)(\exists u_1 \rhd x)(\exists u_2 \rhd u_1)\left((\exists u_3 \rhd u_2)\mathcal{R}^1 xu_3 \vee \mathcal{R}^1 xu_2\right)$$

Unlike in the previous example, it is not clear how to rewrite this formula as an equivalent type 1 formula. In particular, observe that $\forall x\alpha$ is true on the frame below left due to the $\mathcal{R}^1 xu_2$ disjunct, and $\forall x\alpha$ is true on the frame below right due to the $(\exists u_3 \rhd u_2)\mathcal{R}^1 xu_3$ disjunct, so neither disjunct can be deleted:



Finally, let α' be the following formula, obtained from α by applying (4.22):

$$(\forall y_1 \rhd x)(\forall y_2 \rhd y_1)(\exists u_1 \rhd x)(\exists u_2 \rhd u_1)(\exists u_3 \rhd u_2)\left(\mathcal{R}^1 xu_3 \vee \mathcal{R}^1 xu_2\right)$$

Notice that α' is false on the frame above left, providing another example of the incorrectness of (4.22).

(ii) According to the definition of modal Kracht formula [7, Definition 3.58] given there (let us name formulae defined according to *that* definition as modal Kracht* formulae) these formulae do not have the liberty of using $\mathcal{R}^s uv$ as atomic, relying on $u = v$ and Ruv terms instead. Specifically, define a modal Kracht* formula as follows:

Definition 4.35 (modal Kracht* formula) *A modal restrictedly positive formula $\alpha(x)$ containing a single free variable x constructed from atomic formulae $u \neq u$, $u = u$, $u = v$ and Ruv is called a* modal Kracht* formula *if α is clean and in atomic formulae of the form $u = v$ and Ruv, either u or v is inherently universal.*

Now consider the formula $\Box\Box p \supset \Diamond p$. Although we only described the computation of modal Kracht formulae for very simple Sahlqvist formulae, following the algorithm [7] we obtain

$$(\exists u \rhd x)R^2 xu$$

This *is* a modal Kracht formula. However, it is *not* a modal Kracht* formula because of the $R^2 xu$ term. If we expand it out and simplify we obtain the formula $(\exists u \rhd x)(\exists v \rhd x)Rvu$. This is not a modal Kracht* formula either, because neither v nor u in Rvu is inherently universal.

Of course, it *may* be possible to rewrite the above as a modal Kracht* formula, but it is not obvious how to do so. To summarise, the (full) algorithm [7] mapping modal Sahlqvist formulae to first-order formulae introduces $\mathcal{R}^s uv$ terms. However, these terms are outside the fragment of modal Kracht* formulae, and expanding the $\mathcal{R}^s uv$ terms may still keep us outside the modal Kracht* fragment. In our description of the algorithm from modal Kracht formulae to modal Sahlqvist formulae, we have shown how to handle the new cases arising from the use of $\mathcal{R}^s uv$ instead of Ruv.

4.2.4 The Sahlqvist completeness theorem

Thus far our discussion has centred around computing the first-order correspondent of a Sahlqvist formula (Section 4.2.2), and computing the Sahlqvist formula corresponding to a (first-order) Kracht formula (Section 4.2.3). In this section we will show how to obtain soundness and completeness results for logics axiomatised over K and Kt by Sahlqvist formulae. It turns out that these logics are sound and weakly complete for the class of frames defined by these formulae.

In this paragraph, let ϕ_i uniformly denote either a tense formula or a formula from the frame languages. We write $\mathcal{F}_{\phi_i}$ to denote the (not necessarily first-order definable) class of frames defined by ϕ_i. Now suppose that $\Delta = \{\phi_1, \ldots, \phi_n\}$ is some set of formulae. From Lemma 4.14 we know that the formula $\bigwedge_i \phi_i$ defines the class $\bigcap_i \mathcal{F}_{\phi_i}$ of frames. We write $\mathcal{F}_\Delta$ to denote the class $\bigcap_i \mathcal{F}_{\phi_i}$ and say that Δ defines $\mathcal{F}_\Delta$.

Of course, if Δ is some set of Sahlqvist formulae, Theorem 4.31 tells us that there is some conjunction α of Kracht formulae such that $\mathcal{F}_\Delta = \mathcal{F}_{\forall x \alpha}$.

Let $K \oplus \Delta$ be the axiomatic extension of basic modal logic K by a finite set Δ of modal Sahlqvist formulae. Certainly we have $\mathcal{F}_\Delta \models K$ (due to Theorem 4.4) and $\mathcal{F}_\Delta \models \Delta$. Recall that $K \oplus \Delta$ is the closure of $K \cup \Delta$ under the rules of modus ponens, necessitation and the uniform substitution of modal formulae for propositional variables. Thus any $A \in K \oplus \Delta$ can be obtained by applications of these rules to formulae in $K \cup \Delta$. Now, for any class $\mathcal{F}$ of frames and formulae A

and B, it is easy to check that

$\mathcal{F} \models A \supset B$ and $\mathcal{F} \models A$ implies $\mathcal{F} \models B$
$\mathcal{F} \models A$ implies $\mathcal{F} \models \Box A$
$\mathcal{F} \models A$ implies $\mathcal{F} \models A'$ where A' is a uniform substitution instance of A

Thus $A \in K \oplus \Delta$ implies that $\mathcal{F}_\Delta \models A$. So $K \oplus \Delta$ is *sound* for $\mathcal{F}_\Delta$.

Weak completeness of $K \oplus \Delta$ for $\mathcal{F}_\Delta$ is the statement:

$$\mathcal{F}_\Delta \models A \text{ implies } A \in K \oplus \Delta$$

Under the assumption that Δ is a finite set of modal Sahlqvist formulae, this statement indeed holds, although the proof requires some work. Here we only sketch the proof following [7]. Sahlqvist's original proof can be found in [62]. Also see Sambin and Vaccaro [63] for a different proof of the main theorem. We observe that analogous results apply for axiomatic extensions of the basic tense logic Kt.

Definition 4.36 (L-consistent sets) *Let L be an extension of the basic modal logic K. A set Γ of formulae is called* L-consistent *if the formula $(\bigwedge \Gamma \supset \bot) \notin L$ and* L-inconsistent *otherwise.*

We have the following lemma (see [7] for a proof).

Lemma 4.37 *A logic L is weakly complete with respect to a class of frames $\mathcal{F}$ iff every L-consistent set is satisfiable on some model for some frame in $\mathcal{F}$.*

To prove that L is complete for $\mathcal{F}$, from the above lemma, it suffices to find a model in $\mathcal{F}$ which makes every L-consistent set satisfiable. What model should we choose? It turns out that there is a natural choice for this purpose, aptly called the *canonical* model for L. The states of this model are the *maximal L-consistent* sets.

Definition 4.38 (maximal L-consistent sets) *A set Γ of formulae is called* maximal L-consistent *if Γ is L-consistent, and any set of formulae properly containing Γ is L-inconsistent. If Γ is a maximal L-consistent set of formulae then for short we say it is a L-MCS.*

Let us define the canonical model and frame for the basic modal language.

Definition 4.39 (canonical model, frame) *The canonical model M^L for a normal modal logic L in the basic modal language is the triple (W^L, R^L, V^L) where:*

(i) W^L is the set of all L-MCSs.

(ii) R^L is the binary relation on W^L defined by $R^L wu$ if for all formulae B, $B \in u$ implies $\Diamond B \in w$.

(iii) V^L is the valuation function defined by

$$V^L(p) = \{w \in W^L \mid p \in w\}$$

The pair $F^L = (W^L, R^L)$ is called the canonical frame *for L.*

The relation R^L is called the *canonical relation*, and V^L the *canonical valuation.*

Next we state without proof the following results [7]:

Lemma 4.40 (Lindenbaum's Lemma) *If Γ is a L-consistent set of formulae, then there is a L-MCS Γ^+ such that $\Gamma \subseteq \Gamma^+$.*

Lemma 4.41 (Truth Lemma) *For any normal modal logic L and set Γ of formulae, $M^L, w \models \Gamma$ iff $\Gamma \subseteq w$.*

Given any set Γ of L-consistent formulae, Lindenbaum's lemma states that there is an L-MCS Γ^+ such that $\Gamma \subseteq \Gamma^+$. From the Truth Lemma we have $M^L, \Gamma^+ \models \Gamma$. What this means is that in order to show that a logic L is weakly complete with respect to some class $\mathcal{F}$ of frames, it suffices to show that the canonical frame $F^L \in \mathcal{F}$. Let us call a formula A *canonical* if, for any normal modal logic L, $A \in L$ implies that A is valid on the canonical frame F^L.

Theorem 4.42 (Sahlqvist completeness theorem) *Every Sahlqvist formula is canonical.*

Thus, for any set Δ of modal Sahlqvist formulae, each formula in Δ must be valid on the canonical frame $F^{K\oplus\Delta}$ for $K \oplus \Delta$. Since $\mathcal{F}_\Delta$ consists of precisely those frames that make Δ valid, we have $F^{K\oplus\Delta} \in \mathcal{F}_\Delta$. By our discussion above, it follows that $K \oplus \Delta$ is *weakly complete* with respect to $\mathcal{F}_\Delta$. Analogous results apply to axiomatic extensions over the basic tense logic Kt. Together with the soundness result we discussed earlier in this section, we have

Corollary 4.43 *Let Δ be a finite set of modal (tense) Sahlqvist formulae. Then $K \oplus \Delta$ ($Kt \oplus \Delta$) is sound and weakly complete with respect to $\mathcal{F}_\Delta$.*

This completes our overview of correspondence theory.

4.3 Introducing the Display Calculus

The Display Calculus [5] is a formal proof system that can be used to present a large class of logics.[2] To motivate the display calculus, let us look at another formal proof system that we have already encountered — Gentzen's sequent calculus [25]. Gentzen's sequent calculus is built from (traditional) sequents of the following form

$$A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$$

where $\{A_i\}_{i \in \mathcal{A}}$ and $\{B_i\}_{i \in \mathcal{B}}$ are logical formulae from some language $\mathcal{L}$ (say). The symbol $\Rightarrow$ in the sequent places the $A_1, \ldots, A_n$ on the left-hand side (the *antecedent*) and and the $B_1, \ldots, B_m$ on the right-hand side (the *succedent*). Moreover, notice how formulae in both the antecedent and succedent are separated by a comma (,). It is important to note that the symbols $\Rightarrow$ and the comma do *not* belong to $\mathcal{L}$. Instead these are meta-logical symbols belonging to the sequent calculus formalism. For this reason, these symbols are called *structural connectives*, to distinguish them from the logical connectives — think $\neg$, $\supset$ and $\Box$ for example — in the language $\mathcal{L}$. Properties of the structural connectives such as associativity and commutativity of the comma are usually defined implicitly.

Although the structural connectives do not actually belong to $\mathcal{L}$, they can often be interpreted in $\mathcal{L}$. For example, in the Gentzen sequent calculus presenting classical propositional logic, the sequent above is intended to mean the formula

$$A_1 \wedge \ldots \wedge A_n \supset B_1 \vee \ldots \vee B_m$$

where the $\Rightarrow$ has been interpreted as the implication connective $\supset$ and the comma has been interpreted as conjunction in the antecedent and disjunction in the succedent. This relationship between the logical connectives and the structural connectives (which lie outside the language of the logic) deserves further examination.

We can examine this relationship by writing a traditional sequent as $X \Rightarrow Y$ where X and Y are *Gentzen structures* composed from logical formulae using the structural connective comma. Gentzen structures can be defined by the following grammar, where $\mathbf{I}$ is a constant (the identity structure) and A denotes a logical formula:

$$X := \mathbf{I} \mid A \mid X_1, X_2$$

[2] Since it is a proof system rather than a logic, we prefer the term "display calculus" to Belnap's original "display logic".

Now we must explicitly define the properties of the comma. There are also other properties concerning the sequent calculus structure that we would typically define, for example, the contraction rule (below)

$$\frac{A, A, X \Rightarrow Y}{A, X \Rightarrow Y}$$

These rules are called *structural rules*, to contrast with the *logical rules* of the sequent calculus which introduce logical and modal connectives into the sequent.

Now let us consider the Display Calculus. A display calculus is built from *display sequents* — each sequent is divided into an antecedent and a succedent by the symbol $\vdash$ (we use $\vdash$ for display sequents and $\Rightarrow$ for Gentzen sequents in order to distinguish between these two objects). Then we can write a display sequent as $X \vdash Y$ where X and Y are *display structures.* Display structures are typically more sophisticated than their Gentzen counterparts as they are enriched with several different types of structural connectives. The properties of these structural connectives are then explicitly specified in such a manner that the expressivity of the structural connectives is 'complete' in the sense that any substructure in a display sequent can be *displayed* as the whole of the antecedent or succedent. This is called the *display property* — the name Display Calculus is due to this property. The display property makes it straightforward to prove a cut-elimination theorem that is applicable to any display calculus whose rules satisfying certain properties (the 'display conditions'). Informally speaking, the greater expressivity that is achieved due to the rich display structures — in particular, the interplay between the logical formulae and the structural connectives — makes it possible to construct display calculi with nice properties (such as cut-elimination and the display property) for a large class of logics.

The generality of the Display Calculus can be seen by the fact that cutfree display calculi have been presented for a broad class of logics including substructural logics [60, 29], modal and polymodal logics [78, 39] and intuitionistic logics (see [28] and Chapter 6 of this thesis). Mints [50] and Wansing [79] have demonstrated an embedding, respectively, from labelled sequent calculi and hypersequent calculi into the display calculus, and Restall [59] has presented an embedding of the display calculus into the labelled sequent system of Negri [52].

In the following subsection, we present the display calculus DLM for the basic tense logic Kt and then introduce Belnap's cut-elimination theorem for display calculi. In Section 5.1 we present Kracht's elegant result characterising the axiomatic extensions of Kt that can be presented as structural rules extensions

$$\frac{\mathbf{I} \vdash X}{\top \vdash X}\ (\top \vdash) \qquad \frac{X \vdash \mathbf{I}}{X \vdash \bot}\ (\vdash \bot)$$

$$\frac{X \vdash *A}{X \vdash \neg A}\ (\vdash \neg) \qquad \frac{*A \vdash X}{\neg A \vdash X}\ (\neg \vdash)$$

$$\frac{X \vdash A \qquad Y \vdash B}{X \circ Y \vdash A \wedge B}\ (\vdash \wedge) \qquad \frac{A \circ B \vdash X}{A \wedge B \vdash X}\ (\wedge \vdash)$$

$$\frac{X \vdash A \circ B}{X \vdash A \vee B}\ (\vdash \vee) \qquad \frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash X \circ Y}\ (\vee \vdash)$$

$$\frac{X \circ A \vdash B}{X \vdash A \supset B}\ (\vdash \supset) \qquad \frac{X \vdash A \qquad B \vdash Y}{A \supset B \vdash *X \circ Y}\ (\supset \vdash)$$

$$\frac{\bullet X \vdash A}{X \vdash \Box A}\ (\vdash \Box) \qquad \frac{A \vdash X}{\Box A \vdash \bullet X}\ (\Box \vdash)$$

$$\frac{X \vdash A}{* \bullet * X \vdash \Diamond A}\ (\vdash \Diamond) \qquad \frac{* \bullet * A \vdash X}{\Diamond A \vdash X}\ (\Diamond \vdash)$$

$$\frac{A \vdash \bullet X}{\blacklozenge A \vdash X}\ (\blacklozenge \vdash) \qquad \frac{X \vdash A}{\bullet X \vdash \blacklozenge A}\ (\vdash \blacklozenge)$$

$$\frac{A \vdash X}{\blacksquare A \vdash * \bullet * X}\ (\blacksquare \vdash) \qquad \frac{X \vdash * \bullet * A}{X \vdash \blacksquare A}\ (\vdash \blacksquare)$$

Table 4.1: Logical rules for the display calculus DLM

of DLM. There is a good reason for focusing on structural rule extensions — it is particularly easy to verify the display conditions for structural rules. An analogous characterisation for axiomatic extensions of the basic modal logic K is claimed in Kracht [39] but R. Goré has recently observed that this claim is incorrect. In Section 5.2 we examine a new characterisation.

4.3.1 The display calculus DLM

Let us now introduce the display calculus DLM [39].

The class of *display structures* for DLM is defined over the binary structural connective $\circ$, the unary structural connectives $*$ and $\bullet$, and the constant $\mathbf{I}$ (the 'identity structure') by the following grammar, where A denotes a formula from $\mathbf{For}\mathcal{TL}$:

$$X := \mathbf{I} \mid A \mid X_1 \circ X_2 \mid *X \mid \bullet X$$

We will use the letters $X, Y, \ldots$ (and later U, V, L, M, possibly with subscripts) to denote display structures. If Y is a structure occurring in the structure X, we say that Y is a *substructure* of X and write $X[Y]$. A *proper structure* is a structure that is not a formula. The structure Z appears *positively* (resp. *negatively*) in X if Z occurs in the scope of an even (odd) number of $*$ symbols in $X[Z]$.

A *display sequent* has the form $X \vdash Y$ where X and Y are display structures. A structure Z is said to be an antecedent (resp. succedent) part of the sequent $X \vdash Y$ if Z occurs positively (negatively) in X or negatively (positively) in Y.

A *rule* in the display calculus consists of some number of premise sequents and a conclusion sequent. We represent a rule in the usual way, by drawing a line to separate the premises from the conclusion. We use a double line to indicate that a (single-premise) rule can be read both upwards *and* downwards (so each item represents a pair of rules).

The rules are often presented as *rule schemata*. In this context, the letters $X, Y, \ldots$ should be treated as *schematic structure variables*, the letters $p, q, \ldots$ as *schematic propositional variables* and $A, B, \ldots$ as *schematic formula variables*. A schematic structure is constructed from these constituents and $\mathbf{I}$ using appropriate structural and formula connectives. A schematic formula is constructed from schematic formula variables and propositional variables using the logical connectives and constants. A *rule instance* is obtained from a rule schema by the uniform substitution of structures for schematic structure variables, propositional variables for schematic propositional variables and tense formulae for schematic formula variables. In practice, we use the term "rule" to refer to both a rule schema and its instance. The use of rule schemata allows us to do without a substitution rule (compare with the Hilbert calculus).

A rule is called a *logical* rule if it introduces a logical connective or logical constant into the conclusion sequent. Clearly, each rule in Table 4.1 is a logical rule. A rule is called a *structural* rule if its rule schema consists of schematic structures built from schematic structure variables $X, Y, \ldots$ and structural connectives (so no schematic formula variables or logical connectives are permitted). For example, consider the following rule schemata:

$$\frac{X \circ X \vdash Y}{X \vdash Y}\, Cl \qquad\qquad \frac{A \circ A \vdash Y}{A \vdash Y}\, Cl_{form}$$

The rule schema Cl is clearly a structural rule. However, the rule schema Cl_{form} is *not* a structural rule (under the definition of display structural rule given above) because it contains the schematic formula variable A. Furthermore, when viewed as an instance of a rule schema (rather than as a rule schema itself), notice that

$$\begin{array}{c} X \circ Y \vdash Z \\ \hline\hline X \vdash Z \circ *Y \end{array} \qquad \begin{array}{c} X \circ Y \vdash Z \\ \hline\hline Y \vdash *X \circ Z \end{array}$$

$$\begin{array}{c} X \vdash Y \circ Z \\ \hline\hline X \circ *Z \vdash Y \end{array} \qquad \begin{array}{c} X \vdash Y \circ Z \\ \hline\hline *Y \circ X \vdash Z \end{array}$$

$$\begin{array}{c} *X \vdash Y \\ \hline\hline *Y \vdash X \end{array} \qquad \begin{array}{c} X \vdash *Y \\ \hline\hline Y \vdash *X \end{array}$$

$$\begin{array}{c} **X \vdash Y \\ \hline\hline X \vdash Y \end{array} \qquad \begin{array}{c} X \vdash **Y \\ \hline\hline X \vdash Y \end{array}$$

$$\begin{array}{c} X \vdash \bullet Y \\ \hline\hline \bullet X \vdash Y \end{array}$$

Table 4.2: The display rules for the display calculus DLM

$$\frac{A \circ A \vdash Y}{A \vdash Y}$$

is an instance of the rule schema Cl. Clearly, each rule in Table 4.3 and 4.2 is a structural rule. Observe that no rule can be both a logical rule and a structural rule.

The rules for the display calculus DLM are given below.

(i) initial sequents: $p \vdash p$ for every propositional variable p, and the sequents $\mathbf{I} \vdash \top$ and $\bot \vdash \mathbf{I}$;

(ii) the logical rules in Table 4.1;

(iii) the structural rules in Tables 4.2 and 4.3; and

(iv) the cutrule $\dfrac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\,(cut)$

Notice that the cutrule is neither a logical rule (because it does not introduce a logical connective or a constant) nor a structural rule (because the rule schema contains a formula).

A *derivation* in the display calculus DLM is defined recursively in the usual way as either an initial sequent $p \vdash p$ or an application of the rules to derivations concluding its premises. The last sequent in the derivation is called the *end-sequent*. The sequent $X \vdash Y$ is *derivable* if there is some derivation with end-sequent $X \vdash Y$.

A rule in the display calculus is called *invertible* if the premise sequents of a rule instance are derivable whenever the conclusion sequent is derivable. By

$$\frac{X \vdash Z}{\mathbf{I} \circ X \vdash Z}\ (\mathbf{I}l) \qquad \frac{X \vdash Z}{X \vdash \mathbf{I} \circ Z}\ (\mathbf{I}r)$$

$$\frac{\mathbf{I} \vdash Y}{*\mathbf{I} \vdash Y}\ (Ql) \qquad \frac{X \vdash \mathbf{I}}{X \vdash *\mathbf{I}}\ (Qr)$$

$$\frac{X \vdash Z}{Y \circ X \vdash Z}\ (Wl) \qquad \frac{X \vdash Z}{X \circ Y \vdash Z}\ (Wr)$$

$$\frac{X_1 \circ (X_2 \circ X_3) \vdash Z}{(X_1 \circ X_2) \circ X_3 \vdash Z}\ (Al) \qquad \frac{Z \vdash X_1 \circ (X_2 \circ X_3)}{Z \vdash (X_1 \circ X_2) \circ X_3}\ (Ar)$$

$$\frac{X \circ Y \vdash Z}{Y \circ X \vdash Z}\ (Pl) \qquad \frac{Z \vdash X \circ Y}{Z \vdash Y \circ X}\ (Pr)$$

$$\frac{X \circ X \vdash Z}{X \vdash Z}\ (Cl) \qquad \frac{Z \vdash X \circ X}{Z \vdash X}\ (Cr)$$

$$\frac{\mathbf{I} \vdash Y}{\bullet\mathbf{I} \vdash Y}\ (Ml) \qquad \frac{X \vdash \mathbf{I}}{X \vdash \bullet\mathbf{I}}\ (Mr)$$

Table 4.3: Proper structural rules for the display calculus DLM

our previous comment, we can equally say that a rule is invertible if there is a derivation of the premise sequents of a rule instance from the conclusion sequent. Because every rule in Table 4.2 has double lines, it is obvious that each of these rules is invertible.

For brevity, we will often omit labelling an application of the structural rules from Tables 4.2 or 4.3 or apply some number of these rules in a single step of the derivation.

Notice that the rules in Table 4.3 specify the properties of the structural connectives in the language. For example, the rules (Pl) and (Pr) respectively specify commutativity of $\circ$ in the antecedent and succedent of a sequent. The rules in Table 4.2 are called the *display rules* because these rules enable us to 'display' a substructure occuring in a sequent as the entire antecedent or succedent. The sequents $X_1 \vdash Y_1$ and $X_2 \vdash Y_2$ are called *display equivalent* if each sequent is derivable from the other using just the display rules.

Theorem 4.44 (display property) *Suppose that the proper structure Z is an antecedent (resp. succedent) part of the sequent $X \vdash Y$. Then there is a display equivalent display sequent $Z \vdash Y'$ ($X' \vdash Z$). Structure Z is said to be* displayed *in the resulting sequent.*

Proof. See [5]. Q.E.D.

For example, consider the sequent $\bullet * (*(X \circ M) \circ Y) \vdash Z$. Since the structure M appears positively in the antecedent of this sequent, M is an antecedent part of the sequent. Using the display rules we can display M as follows:

$$\begin{array}{c} \bullet * (*(X \circ M) \circ Y) \vdash Z \\ \hline *(*(X \circ M) \circ Y) \vdash \bullet Z \\ \hline * \bullet Z \vdash *(X \circ M) \circ Y \\ \hline * \bullet Z \circ *Y \vdash *(X \circ M) \\ \hline X \circ M \vdash *(* \bullet Z \circ *Y) \\ \hline M \vdash *X \circ *(* \bullet Z \circ *Y) \end{array}$$

We end this section with the following basic results.

Lemma 4.45 *For every formula A, the sequent $A \vdash A$ is derivable in DLM.*

Proof. The proof is by induction on the size of A. Q.E.D.

Lemma 4.46 *The single-premise logical rules for the connectives $\neg$, $\wedge$, $\vee$ as well the modal rules $\blacklozenge \vdash$ and $\vdash \Box$ (Tables 4.1) are invertible.*

Proof. This result can be proved easily by making use of the cutrule. For example, a conclusion sequent of a $\blacklozenge \vdash$ rule instance has the form $\blacklozenge A \vdash X$. We can obtain a derivation of the premise sequent $A \vdash \bullet X$ from $\blacklozenge A \vdash X$ as follows:

$$\dfrac{\dfrac{A \vdash A}{\bullet A \vdash \blacklozenge A}\ {\vdash \blacklozenge} \qquad \blacklozenge A \vdash X}{\dfrac{\bullet A \vdash X}{A \vdash \bullet X}}\ \mathit{cut}$$

For the $\vdash \Box$ rule, suppose that we are given a derivation of $X \vdash \Box A$. Then we can obtain a derivation of $\bullet X \vdash A$ as follows:

$$\dfrac{X \vdash \Box A \qquad \dfrac{A \vdash A}{\Box A \vdash \bullet A}\ {\Box \vdash}}{\dfrac{X \vdash \bullet A}{\bullet X \vdash A}}\ \mathit{cut}$$

As another example, we can obtain the premise sequent $X \vdash A \circ B$ of a $\vdash \vee$ rule instance from the conclusion sequent $X \vdash A \vee B$ as follows:

$$\dfrac{X \vdash A \vee B \qquad \dfrac{A \vdash A \qquad B \vdash B}{A \vee B \vdash A \circ B}\ \mathit{cut}}{X \vdash A \circ B}\ \mathit{cut}$$

The other cases are similar. Q.E.D.

4.3.2 Motivation for the modal and tense rules

We noted that in Gentzen's sequent calculus, the structural connective comma stands for conjunction in the antecedent and disjunction in the succedent. Thus, the meaning of the comma *toggles* between conjunction and disjunction. In the display calculus, Belnap [5] generalises this idea by ensuring that every structural connective toggles between two logical connectives. The structural connective $\circ$ in DLM plays a similar role to the comma, toggling between conjunction in the antecedent and disjunction in the succedent. For the case of the structural connective $*$ in DLM, the rules $\neg\vdash$ and $\vdash\neg$ indicate that $*$ stands for negation in both the antecedent and the succedent. Because $\supset$ can be defined in terms of $\vee$ and $\neg$, the rules for $\supset$ involve both $*$ and $\bullet$. Moreover, the structural constant **I** toggles between the logical constant $\top$ in the antecedent and $\bot$ in the succedent.

Together with the obvious correspondence between $\vdash$ in the display calculus and $\Rightarrow$ in the Gentzen sequent calculus, the logical rules for $\neg$ $\wedge$, $\vee$ and $\supset$ in Table 4.1 are straightforward analogues of the corresponding rules for Gentzen sequent calculi [25] for classical logic. This leaves the question: how do we formulate the rules for the modal and tense operators? In this section we will provide a motivation for the modal and tense rules. We are aiming for the calculus DLM to present the basic tense logic Kt (see Section 4.3.4), so the motivation is guided by this logic.

First consider the following result.

Lemma 4.47 *For all tense formulae A and B, $\blacklozenge A \supset B \in Kt$ iff $A \supset \Box B \in Kt$.*

Proof. Let us make use of the Hilbert calculus for Kt. Although the argument presented here is informal, it should be clear that it can be made into a formal derivation in the Hilbert calculus.

First suppose that $\blacklozenge A \supset B \in Kt$. From Necessitation we know that $\Box(\blacklozenge A \supset B)$ is a theorem. From $(Ax-\Box)$ and modus ponens, we obtain $\Box\blacklozenge A \supset \Box B$. Since $A \supset \Box\blacklozenge A$ is an instance of the (Converse1) axiom, together with $\Box\blacklozenge A \supset \Box B$ it follows that $A \supset \Box B \in Kt$ as required.

Now suppose that $A \supset \Box B \in Kt$. From a standard classical equivalences we know that $\neg\Box B \supset \neg A$ is a theorem. From Necessitation we obtain $\blacksquare(\neg\Box B \supset \neg A)$, and hence from $(Ax - \blacksquare)$ and modus ponens we get $\blacksquare\neg\Box B \supset \blacksquare\neg A$, and then $\neg\blacksquare\neg A \supset \neg\blacksquare\neg\Box B$. From $(Dual - \blacksquare)$ this is simply $\blacklozenge A \supset \blacklozenge\Box B$. Also, $\neg B \supset \blacksquare\Diamond\neg B$ is an instance of (Converse2), and hence we have $\neg\blacksquare\Diamond\neg B \supset B$ and $\blacklozenge\Box B \supset B$. Finally, from $\blacklozenge A \supset \blacklozenge\Box B$ and $\blacklozenge\Box B \supset B$ it follows that $\blacklozenge A \supset B$ as required. Q.E.D.

Lemma 4.47 suggests that if we assigned a structural connective for $\blacklozenge$ in the antecedent, the connective could toggle as $\Box$ in the succedent. Let us pursue this line of enquiry by assigning the symbol $\bullet$ to $\blacklozenge$ to obtain:

$$\frac{\bullet A \vdash X}{\blacklozenge A \vdash X} \qquad\qquad \frac{X \vdash \bullet A}{X \vdash \Box A}$$

We have reproduced, respectively, the rules $\blacklozenge\vdash$ and $\vdash\Box$ in DLM (upto display equivalence). Whence come the other rules? It turns out that all the remaining rules can be deduced from the above rules. For example, let us motivate the rule $\blacksquare\vdash$:

$$\frac{P \vdash X}{\blacksquare P \vdash * \bullet * X}\ \blacksquare\vdash$$

This rule says that if $A \vdash X$ is derivable in DLM, then so is $\blacksquare A \vdash * \bullet * X$. Reading $\vdash$ as $\supset$ as usual, from Necessitation, the $(Ax - \blacksquare)$ axiom and modus ponens we can imagine introducing $\blacksquare$ simultaneously in the antecedent and succedent. Compare $\blacksquare P \vdash \blacksquare X$ with the conclusion of $\blacksquare\vdash$. We have introduced $\blacksquare$ in the antecedent as required. *We stress that this is an informal discussion* — in general, $\blacksquare X$ is not a legal structure. What we really need to do is write the $\blacksquare X$ in the succedent in some way that makes it a legal structure for an arbitrary structure X. Since $*$ is the metalevel symbol for negation, $*\bullet$ in the succedent stands for $\bullet$ in the antecedent which we have taken to stand for $\blacklozenge$ in the antecedent. Hence it follows that $* \bullet *$ in the succedent stands for $\blacklozenge\neg$ in the antecedent, and thus $\neg\blacklozenge\neg$ in the succedent. From $(Dual - \blacksquare)$ this is precisely $\blacksquare$. If we replace $\blacksquare P \vdash \blacksquare X$ with $\blacksquare A \vdash * \bullet * X$ we have obtained a sequent that is legal for all structures X. This is precisely the conclusion sequent of $\blacksquare\vdash$, and this constitutes an informal argument for the soundness of that rule.

Incidentally, it is the 'complete coverage' of the logical connectives and the modal and tense operators at the structural level that enables the powerful display property (Theorem 4.44).

4.3.3 Belnap's cut-elimination theorem

We say that a display calculus has cut-elimination if any derivation containing the cutrule can be effectively transformed to a cutfree derivation of the identical sequent. If a display calculus has cut-elimination, then the cutrule is redundant in the sense that removing it from the calculus does not reduce the set of derivable sequents. Belnap [5] shows that any display calculus whose rules satisfy the *display conditions* has cut-elimination.

Before we present the conditions, let us introduce some terminology. An occurrence of a schematic structure variable in a rule schema is called a *parameter*. Every other variable or structure occurring in the rule is called *nonparametric*. For example, in the rule schema:

$$\frac{X \vdash A \qquad B \vdash Y}{A \supset B \vdash *X \circ Y} \supset\vdash$$

the parameters are X and Y. The remaining variables — in this case the schematic formula variables A and B — and the formula $A \supset B$ and structures $*X$ and $*X \circ Y$ are nonparametric. The motivation for definition is that parameters are structures that 'go through unchanged' when passing from the premises to the conclusion. In the above example, it is easy to see that the conclusion sequent is display equivalent to $X \vdash Y \circ *(A \supset B)$ and $X \circ (A \supset B) \vdash Y$. Comparing these two sequents, respectively, to the the left and right premise of the above rule we see that the X and Y have passed unchanged from the premise to the conclusion sequent.

The set of display conditions appears in various guises in the literature. Here we give the conditions for rules presented as rule schema, following [39].

(C2) *Congruent parameters* is a relation between parameters of the identical schematic structure variable occurring in the premise and conclusion sequents. [In the example above, the parameter X in the left premise and conclusion of $\supset\vdash$ are congruent parameters. Similarly, the parameter Y in the right premise and conclusion of $\supset\vdash$ are congruent parameters.]

(C3) Each parameter is congruent to at most one schematic structure variable in the conclusion. Equivalently, no two schematic structure variables in the conclusion are congruent to each other.

(C4) Congruent parameters are either all antecedent or all succedent parts of their respective sequent.

(C5) A schematic formula in the conclusion of a rule ρ is either the entire antecedent or the entire succedent. Such a formula is called a **principal formula** of ρ.

(C6/7) Each rule is closed under simultaneous substitution of arbitrary structures for congruent parameters.

(C8) If there are rules ρ and σ with respective conclusions $X \vdash A$ and $A \vdash Y$ with formula A principal in both inferences (in the sense of C5) and if *cut* is applied to yield $X \vdash Y$, then either $X \vdash Y$ is identical to either $X \vdash A$ or $A \vdash Y$; or it is possible to pass from the premises of ρ and σ to $X \vdash Y$ by means of inferences falling under *cut* where the cut-formula always is a proper subformula of A.

Theorem 4.48 (Belnap) *Any display calculus satisfying (C2)–(C8) has cut-elimination.*

Proof. See Belnap [5] for the proof, based on transformations in Curry [18]. Although the theorem in [5] states a weaker statement, namely that there is a derivation of the conclusion sequent of the *cut* rule whenever the premise sequents are derivable ('cut-admissibility'), the proof specifies an effective procedure thus witnessing the stronger statement of cut-elimination. Q.E.D.

Suppose that the formula A occurs in some sequent $\mathcal{S}$ in some derivation δ. The *parametric ancestors* of A is the set $\mathcal{G}$ of occurrences of A defined as follows: put the given occurrence of A in $\mathcal{G}$. For every rule instance ρ in the subderivation above $\mathcal{S}$, add each occurrence of A that is a part of some structure X in the premise of ρ whose congruent structure in the conclusion of ρ (with respect to the rule schema for ρ) contains an occurrence of A that is already in $\mathcal{G}$. Belnap uses this concept of parametric ancestor to elegantly capture the notion of 'tracing a formula upwards' in his proof of the cut-elimination theorem.

Belnap also described another condition

(C1) Each schematic formula variable occurring in some premise of a rule ρ is a subformula of some schematic formula in the conclusion of ρ.

This condition is not required for cut-elimination. However, it is easy to see that

Theorem 4.49 *Any display calculus where every rule aside from the cutrule obeys (C1) has the* subformula property*, that is, every sequent in a cutfree derivation of the sequent $X \vdash Y$ is built from structures over the subformulae of the formulae occurring in X and Y (as opposed to arbitrary formulae).*

If every rule in a display calculus obeys (C2)–(C8) and every rule aside from the cutrule obeys (C1) then we say that the display calculus obeys (C1)–(C8). The above results tell us that if a display calculus obeys (C1)–(C8) then the calculus has cut-elimination and the subformula property.

4.3.4 Structural rule extensions of *DLM*

The τ-translation

Define the functions τ_1 and τ_2 from display structures to tense formulae as follows [5,39]:

$$
\begin{array}{rlrl}
\tau_1(A) &= A & \qquad \tau_2(A) &= A \\
\tau_1(\mathbf{I}) &= \top & \tau_2(\mathbf{I}) &= \bot \\
\tau_1(*X) &= \neg\tau_2(X) & \tau_2(*X) &= \neg\tau_1(X) \\
\tau_1(X \circ Y) &= \tau_1(X) \wedge \tau_1(Y) & \tau_2(X \circ Y) &= \tau_2(X) \vee \tau_2(Y) \\
\tau_1(\bullet X) &= \blacklozenge\tau_1(X) & \tau_2(\bullet X) &= \Box\tau_2(X)
\end{array}
$$

The τ-translation $\tau(X \vdash Y)$ of the sequent $X \vdash Y$ is the tense formula $\tau_1(X) \supset \tau_2(Y)$.

Some properties of the calculus *DLM*

The following results identify the relationship between the display calculus *DLM* and the basic tense logic *Kt*. The results are constructive in the sense that there is an algorithm witnessing each result.

Lemma 4.50 *For any sequent $X \vdash Y$, if $X \vdash Y$ is derivable in DLM then $\tau(X \vdash Y) \in Kt$.*

Proof. See Wansing [80, Theorem 3.14]. Q.E.D.

Lemma 4.51 *For any tense formula A, $A \in Kt$ iff $\mathbf{I} \vdash A$ is derivable in DLM.*

Proof. For the forward direction it suffices to show that (i) for each axiom A in the Hilbert calculus for Kt, $\mathbf{I} \vdash A$ is derivable in DLM, and (ii) all the rules in the Hilbert calculus preserve derivability in DLM. See Wansing [80, Theorem 3.14]. The reverse direction follows from Lemma 4.50. Q.E.D.

Lemma 4.52 *For any sequent $X \vdash Y$, $X \vdash Y$ is derivable in DLM iff $\mathbf{I} \vdash \tau(X \vdash Y)$.*

Proof. See Wansing [80, Theorem 3.17]. Q.E.D.

As an illustration of the above lemma, let us construct a derivation of $\tau(\bullet * \bullet * p \vdash * \bullet * \bullet p) = \blacklozenge\neg\Box\neg p \supset \neg\blacklozenge\neg\Box p$ from the sequent $\bullet * \bullet * p \vdash * \bullet * \bullet p$.

$$
\begin{array}{c}
\bullet \ast \bullet \ast p \vdash \ast \bullet \ast \bullet p \\ \hline
\bullet \ast \bullet \ast \bullet \ast \bullet p \vdash \ast p \\ \hline
\bullet \ast \bullet \ast \bullet \ast \bullet p \vdash \neg p \\ \hline
\ast \bullet \ast \bullet \ast \bullet p \vdash \bullet \neg p \\ \hline
\ast \bullet \ast \bullet \ast \bullet p \vdash \Box \neg p \\ \hline
\ast \Box \neg p \vdash \bullet \ast \bullet \ast \bullet p \\ \hline
\neg \Box \neg p \vdash \bullet \ast \bullet \ast \bullet p \\ \hline
\bullet \neg \Box \neg p \vdash \ast \bullet \ast \bullet p \\ \hline
\blacklozenge \neg \Box \neg p \vdash \ast \bullet \ast \bullet p \\ \hline
\ast \bullet \ast \blacklozenge \neg \Box \neg p \vdash \bullet p
\end{array}
\begin{array}{l}
\\ \vdash \neg \\ \\ \vdash \Box \\ \\ \neg \vdash \\ \\ \blacklozenge \vdash \\ \\
\end{array}
\qquad
\begin{array}{c}
\text{(cont.)} \\ \hline
\ast \bullet \ast \blacklozenge \neg \Box \neg p \vdash \Box p \\ \hline
\ast \Box p \vdash \bullet \ast \blacklozenge \neg \Box \neg p \\ \hline
\neg \Box p \vdash \bullet \ast \blacklozenge \neg \Box \neg p \\ \hline
\bullet \neg \Box p \vdash \ast \blacklozenge \neg \Box \neg p \\ \hline
\blacklozenge \neg \Box p \vdash \ast \blacklozenge \neg \Box \neg p \\ \hline
\blacklozenge \neg \Box \neg p \vdash \ast \blacklozenge \neg \Box p \\ \hline
\blacklozenge \neg \Box \neg p \vdash \neg \blacklozenge \neg \Box p \\ \hline
\mathbf{I} \vdash \blacklozenge \neg \Box \neg p \supset \neg \blacklozenge \neg \Box p
\end{array}
\begin{array}{l}
\vdash \Box \\ \\ \neg \vdash \\ \\ \blacklozenge \vdash \\ \\ \vdash \neg \\ \vdash \supset
\end{array}
$$

Lemma 4.53 *For sequents $X_1 \vdash Y_1$ and $X_2 \vdash Y_2$, (i) if $\tau(X_1 \vdash Y_1) \approx \tau(X_2 \vdash Y_2) \in Kt$ then $X_2 \vdash Y_2$ is derivable in DLM whenever $X_1 \vdash Y_1$ is derivable in DLM, and (ii) if $X_1 \vdash Y_1$ and $X_2 \vdash Y_2$ are display equivalent, then $\tau(X_1 \vdash Y_1) \in Kt$ whenever $\tau(X_2 \vdash Y_2) \in Kt$.*

Proof. For (i), if $X_1 \vdash Y_1$ is derivable in DLM, we know that $\tau(X_1 \vdash Y_1) \in Kt$. From $\tau(X_1 \vdash Y_1) \approx_{Kt} \tau(X_2 \vdash Y_2)$ by modus ponens we have $\tau(X_2 \vdash Y_2) \in Kt$, and thus $X_2 \vdash Y_2$ is derivable in DLM.

To prove (ii), first observe that if $\tau(X_2 \vdash Y_2) \in Kt$, then from Lemma 4.51, we have that $\mathbf{I} \vdash \tau(X_2 \vdash Y_2)$ is derivable in DLM. Then from Lemma 4.52 we have that $X_2 \vdash Y_2$ is derivable. Since $X_1 \vdash Y_1$ and $X_2 \vdash Y_2$ are display equivalent, we have that $X_1 \vdash Y_1$ is derivable. Hence it follows that $\tau(X_1 \vdash Y_1)$ is derivable.

Q.E.D.

Structural rule extensions of DLM

The display calculus obtained by the addition of a set $\{\rho_i\}_{i \in I}$ of structural rules to DLM is denoted $DLM + \{\rho_i\}_{i \in I}$. In practice, the index set I will always be finite. Moreover, we often drop the reference to the index set, writing $DLM + \{\rho_i\}_{i \in I}$. Similarly, we write $Kt + \{r_i\}_{i \in R}$ to denote the Hilbert calculus obtained by the addition of the Hilbert calculus rules $\{r_i\}_{i \in R}$ to Kt.

Theorem 4.54 *Let $\{\rho_i\}_{i \in I}$ be a set of structural rules. Then,*

(i) $X \vdash Y$ is derivable in $DLM + \{\rho_i\}_{i \in I}$ iff $\mathbf{I} \vdash \tau(X \vdash Y)$ is derivable in $DLM + \{\rho_i\}_{i \in I}$; and

(ii) for sequents $X_1 \vdash Y_1$ and $X_2 \vdash Y_2$, if $\tau(X_1 \vdash Y_1) \approx \tau(X_2 \vdash Y_2) \in Kt$ then $X_2 \vdash Y_2$ is derivable in $DLM + \{\rho_i\}_{i \in I}$ whenever $X_1 \vdash Y_1$ is derivable in DLM; and

(iii) if $X_1 \vdash Y_1$ and $X_2 \vdash Y_2$ are display equivalent, then $\tau(X_1 \vdash Y_1) \in Kt$ whenever $\tau(X_2 \vdash Y_2) \in Kt$.

Proof. The proofs are analogous to those for the calculus DLM. In particular, observe that the presence of the rules $\{\rho_i\}_{i \in I}$ does not complicate the proof in any way. Q.E.D.

Definition 4.55 (properly displays) *Let $\{\rho_i\}_{i\in I}$ be a set of structural rules. We say that $DLM + \{\rho_i\}_{i\in I}$* properly displays *the tense logic L if*

(i) all the rules of $DLM + \{\rho_i\}_{i\in I}$ satisfy (C1)–(C8), and

(ii) for any formula $A \in \mathbf{For}\mathcal{TL}$, $A \in L$ iff $\mathbf{I} \vdash A$ is derivable in $DLM + \{\rho_i\}_{i\in I}$.

We say that the logic L is *properly displayable* over DLM if there is a set $\{\rho_i\}_{i\in I}$ of structural rules such that $DLM + \{\rho_i\}_{i\in I}$ properly displays L.

Notice that in the calculus $DLM + \{\rho_i\}_{i\in I}$, the condition (C8) needs only to be checked for the logical rules. This is because, by the definition (see pg 111) of structural rule (schema), these rules only contain parameters and structural connectives. Specifically, no nonparametric formulae such as logical formulae can occur in the structural rule schema, and so the hypothesis of C8 can never be satisfied for structural rules. Incidentally, the transformations [5] witnessing condition (C8) for the logical rules generalise the 'principal transformations' [25] for classical sequent calculi.

By inspection it is easy to verify that all the rules of DLM satisfy the display conditions. Together with Theorem 4.51 we then have

Theorem 4.56 *The display calculus DLM properly displays Kt.*

Equivalent definitions of "properly displays"

We close by showing that the definition of "properly displays" given in Kracht [39] and Wansing [80] is equivalent to our definition.

Definition 4.57 (τ-translation of a rule) *Let ρ be the rule*

$$\frac{X_1 \vdash Y_1 \quad \dots \quad X_n \vdash Y_n}{X \vdash Y}$$

*Then the τ-*translation *of ρ is the rule*

$$\frac{\tau(X_1 \vdash Y_1) \quad \dots \quad \tau(X_n \vdash Y_n)}{\tau(X \vdash Y)}$$

Definition 4.58 (admissible rule) *A rule*

$$\frac{\phi_1 \quad \dots \quad \phi_n}{\phi}$$

in some Hilbert (resp. display) calculus $\mathcal{C}$ *where* $\{\phi, \phi_1, \dots, \phi_n\}$ *is a set of formulae (sequents) is said to be* admissible *if whenever instances of the premise formulae (sequents) are derivable then the corresponding instance of the conclusion formula (sequent) is derivable in* $\mathcal{C}$*.*

Kracht [39] and Wansing [80] define the term *properly displays* for Hilbert calculi $Kt + \{r_i\}_{i \in R}$ (finite R) by substituting condition (ii) in Definition 4.55 with the following two conditions:

(i*) the τ-translation of every admissible rule in $DLM + \{\rho_i\}_{i \in I}$ is admissible in $Kt + \{r_i\}_{i \in R}$.

(ii*) every admissible rule in $Kt + \{r_i\}_{i \in R}$ is logically equivalent in Kt to the τ-translation of some admissible rule in $DLM + \{\rho_i\}_{i \in I}$.

Condition (ii*) means that if the rule

$$\frac{A_1 \quad \dots \quad A_n}{B}$$

is admissible in $Kt + \{r_i\}_{i \in R}$, then there is some admissible rule in $DLM + \{\rho_i\}_{i \in I}$

$$\frac{X_1 \vdash Y_1 \quad \dots \quad X_n \vdash Y_n}{X \vdash Y}$$

such that $\tau(X_i \vdash Y_i) \approx A_i \in Kt$ for each i, and $\tau(X \vdash Y) \approx B \in Kt$.

Let us use the term "*properly displays**" to mean the definition obtained from Definition 4.55 by substituting condition (ii) with (i)* and (ii*). In the following, we show that the two definitions are equivalent.

Lemma 4.59 $DLM + \{\rho_i\}_{i \in I}$ *properly displays* $Kt + \{r_i\}_{i \in R}$ *iff* $DLM + \{\rho_i\}_{i \in I}$ *properly displays** $Kt + \{r_i\}_{i \in R}$*.*

Proof. ($\Rightarrow$) Assume that $DLM + \{\rho_i\}_{i \in I}$ properly displays $Kt + \{r_i\}_{i \in R}$. We must show that (i*) and (ii*) hold. To show (i*), suppose that an arbitrary rule

$$\frac{X_1 \vdash Y_1 \quad \dots \quad X_n \vdash Y_n}{X \vdash Y}\,\rho$$

is admissible in $DLM + \{\rho_i\}_{i\in I}$ and suppose that the τ-translation $\tau(X_i \vdash Y_i)$ of each of the premises of ρ is a theorem of $Kt + \{r_i\}_{i\in R}$ for each i, $1 \leq i \leq n$. Since $DLM + \{\rho_i\}_{i\in I}$ properly displays $Kt + \{r_i\}_{i\in R}$ we know that $\mathbf{I} \vdash \tau(X_i \vdash Y_i)$ is derivable, and hence by Theorem 4.54, $X_i \vdash Y_i$ is derivable in $DLM + \{\rho_i\}_{i\in I}$. From the admissibility of ρ we conclude that $X \vdash Y$ is derivable. Use Theorem 4.54 to get that $\mathbf{I} \vdash \tau(X \vdash Y)$ is derivable, and then from the definition of properly displays we get that $\tau(X \vdash Y) \in Kt + \{r_i\}_{i\in R}$. Thus we have proved that the τ-translation of ρ is admissible in $Kt + \{r_i\}_{i\in R}$.

To show (ii*), suppose that an arbitrary rule

$$\frac{A_1 \quad \ldots \quad A_n}{B}\, r$$

is admissible in $Kt + \{r_i\}_{i\in R}$ and suppose that there are sequents $\{X_i \vdash Y_i\}_{i\in\Lambda}$ derivable in $DLM + \{\rho_i\}_{i\in I}$ such that $\tau(X_i \vdash Y_i)$ is logically equivalent in Kt to A_i for each i, $1 \leq i \leq n$. Certainly, there is no difficulty with the *existence* sequents $\{X_i \vdash Y_i\}_{i\in\Lambda}$ with this property, since it is the case for each i that $\mathbf{I} \vdash A_i$ under the τ-translation is logically equivalent in Kt to A_i. From Theorem 4.54 it follows that $\mathbf{I} \vdash \tau(X_i \vdash Y_i)$ is derivable, and then by the definition of properly displays we have $\tau(X_i \vdash Y_i) \in Kt + \{r_i\}_{i\in R}$. Since $\tau(X_i \vdash Y_i) \approx A_i \in Kt + \{r_i\}_{i\in R}$ it follows that $A_i \in Kt + \{r_i\}_{i\in R}$ for each i. Because the rule r is admissible, it follows that $B \in Kt + \{r_i\}_{i\in R}$ and thus $\mathbf{I} \vdash B$ is derivable. Moreover $\tau(\mathbf{I} \vdash B) = \top \supset B$ is logically equivalent in Kt to B. Thus we have proved that the the τ-translation of the rule

$$\frac{X_1 \vdash Y_1 \quad \ldots \quad X_n \vdash Y_n}{\mathbf{I} \vdash B}$$

is logically equivalent in Kt to r.

($\Leftarrow$) Assume that $DLM + \{\rho_i\}_{i\in I}$ properly displays* $Kt + \{r_i\}_{i\in R}$. We need to show that $A \in Kt + \{r_i\}_{i\in R}$ iff $\mathbf{I} \vdash A$ is derivable in $DLM + \{\rho_i\}_{i\in I}$.

First suppose that $\mathbf{I} \vdash A$ is derivable in $DLM + \{\rho_i\}_{i\in I}$. Therefore the rule

$$\frac{p \vdash p}{\mathbf{I} \vdash A}$$

is admissible in $DLM + \{\rho_i\}_{i\in I}$. By (i*) its τ-translation must be admissible in $Kt + \{r_i\}_{i\in R}$:

$$\frac{p \supset p}{\tau(I \vdash A)}$$

Since $p \supset p \in Kt + \{r_i\}_{i \in R}$ it follows that $\tau(\mathbf{I} \vdash A) = \top \supset A \in Kt + \{r_i\}_{i \in R}$ and thus $A \in Kt + \{r_i\}_{i \in R}$.

Now suppose that $A \in Kt + \{r_i\}_{i \in R}$. We need to show that $\mathbf{I} \vdash A$ is derivable in $DLM + \{\rho_i\}_{i \in I}$. Notice that the rule

$$\frac{p \supset p}{A}$$

must be admissible in $Kt + \{r_i\}_{i \in R}$. By (ii*) there are sequents $X \vdash Y$ and $U \vdash V$ satisfying $\tau(X \vdash Y) \approx p \supset p \in Kt$ and $\tau(U \vdash V) \approx A \in Kt$ such that the rule ρ

$$\frac{X \vdash Y}{U \vdash V} \rho$$

is admissible in $DLM + \{\rho_i\}_{i \in I}$. Since $p \supset p \in Kt$ it follows that $\tau(X \vdash Y) \in Kt$ and hence $X \vdash Y$ is derivable in DLM. Therefore $X \vdash Y$ is derivable in $DLM + \{\rho_i\}_{i \in I}$ too, and by admissibility of the rule ρ, so is the sequent $U \vdash V$. From Theorem 4.54 we can show that $\mathbf{I} \vdash \tau(U \vdash Y)$ is derivable in $DLM + \{\rho_i\}_{i \in I}$. Since $\tau(U \vdash V) \approx A \in Kt$, we can show that $\tau(U \vdash V) \vdash A$ is derivable in DLM. Then obviously $\tau(U \vdash V) \vdash A$ is derivable in $DLM + \{\rho_i\}_{i \in I}$. Since we know that $\mathbf{I} \vdash \tau(U \vdash V)$ is derivable in $DLM + \{\rho_i\}_{i \in I}$, by the cutrule $\mathbf{I} \vdash A$ is derivable in $DLM + \{\rho_i\}_{i \in I}$. Q.E.D.

Although our definition closely reflects the notion of a calculus being sound and complete for a logic, we observe that it is often more convenient to use the equivalent definition of properly displays* when dealing with rule extensions of the Hilbert calculus Kt.

Chapter 5

Displaying tense and modal logics

Kracht's Display Theorem I (Section 5.1.1) identifies the axiomatic extensions over Kt that are properly displayable by the addition of structural rules (satisfying (C1)–(C8)) to DLM, as the class of primitive tense axiomatic extensions over Kt. A semantic characterisation of primitive tense formulae is presented in Section 5.1.2. Kracht also claims a characterisation of the properly displayable axiomatic extensions over K. In Section 5.2 we show why this characterisation is incorrect and examine how we might obtain a new characterisation. In particular, our work here extends the class of modal logics that are properly displayable from the primitive modal axiomatic extensions of K studied by Kracht.

Our contribution in this chapter is as follows.

We present two new proofs — based on a second-order representation of tense formulae, and using model-theoretic arguments — of Kracht's semantic characterisation of primitive tense formulae. The original proof relies on Kracht's Calculus of Internal Descriptions which we do not use in this work.

Although the logic $K \oplus \Diamond\Box p \supset \Box\Diamond p$ has been identified as a counterexample to Kracht's characterisation for modal logics, we are not aware of any existing proof of the (critical and non-trivial) statement that this logic cannot be written as a primitive modal axiomatic extension of K. Here we give a proof of this statement using proof-theoretic methods.

Next, we identify the error in Kracht's claim and present a class of modal formulae called M-formulae, and show that any axiomatic extension via M-formulae is properly displayable. Then we identify the form of the structural rules ('basic rules') corresponding to M-formulae. This leads to the question — what, if any, other axiomatic extensions over K are properly displayable by structural rule extensions? We present a conjecture that answers this question — indeed, Kracht has given a 'proof' for this conjecture, but we have shown that his proof is incomplete.[3] Validity of the conjecture would imply that every structural rule that corresponds to a tense formula defining a modally definable class of frames can be rewritten as a basic rule, thus leading to a complete characterisation of properly displayable modal logics.

5.1 Displaying tense logics

5.1.1 Syntactic characterisation of properly displayable tense logics

Kracht [38] has given a complete syntactic characterisation of the tense logics properly displayable over DLM. This is the Display Theorem I.

Definition 5.1 (primitive tense formula) *A tense formula is called a* primitive tense formula *if it has the form $A \supset B$, where A contains each propositional variable at most once, and A and B are built from $\top$, $\wedge$, $\vee$, $\blacklozenge$ and $\lozenge$.*

An axiomatic extension of Kt by a primitive tense formula is called a *primitive tense axiomatic extension.*

Theorem 5.2 (Display theorem I) *Let $Kt \oplus \Delta$ be an axiomatic extension of Hilbert-style tense logic. Then $Kt \oplus \Delta$ can be properly displayed over DLM iff it is axiomatizable by a set of primitive tense formulae.*

Proof. See Kracht [39]; Wansing [80]. Q.E.D.

The reverse direction of Display Theorem I states that for any set $\{t_i\}_{i \in T}$ of primitive tense formulae, the logic $Kt \oplus \{t_i\}_{i \in T}$ is properly displayable over DLM. Let us describe the algorithm for constructing the display calculus as we will use this later on.

Suppose that δ is the primitive tense formula $A \supset B$. Since A and B are built up from only $\top$, $\wedge$, $\vee$, $\blacklozenge$ and $\lozenge$, using the following equivalences in Kt for distribution of $\blacklozenge$ and $\lozenge$ over disjunction:

$$\begin{aligned} &\lozenge(C \vee D) \approx \lozenge C \vee \lozenge D \\ &\blacklozenge(C \vee D) \approx \blacklozenge C \vee \blacklozenge D \\ &(C \vee D) \wedge E \approx (C \wedge E) \vee (D \wedge E) \end{aligned} \tag{5.1}$$

[3]M. Kracht completely agrees with our analysis regarding this problem and concedes that it is not clear how to obtain the result: personal correspondence by email dated 13/Dec/2010.

we can write $A \approx \bigvee_{i \leq m} C_i$ and $B \approx \bigvee_{j \leq n} D_j$ where every C_i and D_j is built up from only $\top$, $\wedge$, $\blacklozenge$ and $\lozenge$. Now $\bigvee_{i \leq m} C_i \supset \bigvee_{j \leq n} D_j$ is a theorem of Kt iff $C_i \supset \bigvee_{j \leq n} D_j$ is a theorem for all i, $1 \leq i \leq m$. The axiom $A \supset B$ is equivalent in deductive power to the addition of the rule r for some propositional variable q not appearing in $\{A, B\}$:

$$\frac{B \supset q}{A \supset q}$$

which in turn is equivalent to the addition of the rules $\{r_i\}_{i \in R}$ (so $Kt \oplus A \supset B = Kt + \{r_i\}_{i \in R}$), where r_i is the following rule:

$$\frac{D_1 \supset q \qquad \ldots \qquad D_n \supset q}{C_i \supset q}$$

(q does not appear in either $\{D_i\}_{i \in \mathcal{D}}$ or C_i). Notice that each formula that occurs in r_i is free of the disjunction and implication connectives. Now we translate the rules $\{r_i\}_{i \in R}$ into structural rule schemata by replacing $\supset$ with the symbol $\vdash$, and mapping the formulae (which are built from $\top$, $\wedge$, $\blacklozenge$ and $\lozenge$) into display structures using the map σ:

$$\begin{aligned}
&\sigma(\top) = \mathbf{I} \\
&\sigma(p) = X_p \\
&\sigma(A \wedge B) = \sigma(A) \circ \sigma(B) \\
&\sigma(\blacklozenge B) = \bullet\sigma(B) \\
&\sigma(\lozenge B) = * \bullet *\sigma(B)
\end{aligned}$$

where X_p is a schematic structure variable that is uniquely assigned to the propositional variable p. The resulting structural rules schemata $\{\rho_1, \ldots, \rho_m\}$ have the following form:

$$\frac{\sigma(D_1) \vdash X_q \qquad \ldots \qquad \sigma(D_n) \vdash X_q}{\sigma(C_i) \vdash X_q}\ \rho_i$$

It is easy to verify that the $\{\rho_i\}_{i \in I}$ satisfy (C1)–(C8). Next it is shown that the addition of the rule ρ_i to DLM is equivalent in deductive power to the addition of the rule r_i to Kt, so $DLM + \{\rho_i\}_{i \in I}$ properly displays $Kt + \{r_i\}_{i \in R}$.

The forward direction of Display Theorem I states that if the structural rule extension $DLM + \{\rho_i\}_{i \in I}$ properly displays $Kt \oplus \Delta$ then there is some set $\{t_i\}_{i \in T}$ of primitive tense formulae such that $Kt \oplus \Delta = Kt \oplus \{t_i\}_{i \in T}$. The idea here is to show that each ρ_i can be written in the following form for some schematic structure variable Y not occurring in $\{X, X_1, \ldots, X_n\}$:

$$\frac{X_1 \vdash Y \qquad \ldots \qquad X_n \vdash Y}{X \vdash Y}$$

The addition of a structural rule of the above form to DLM corresponds to an axiomatic extension over Kt by a primitive tense formula so the result follows.

Define the following classes of logics:

$$\mathcal{D}_{tense} = \{L \text{ is properly displayable over } DLM \mid L = Kt \oplus \Delta \text{ for some set } \Delta \text{ of tense formulae}\}$$

$$\mathcal{P}_{tense} = \{L \mid L = Kt \oplus \{t_i\}_{i \in T} \text{ for some set } \{t_i\}_{i\in T} \text{ of primitive tense formulae }\}$$

Then Kracht's Display Theorem I states that $\mathcal{D}_{tense} = \mathcal{P}_{tense}$.

Example 5.3 *Consider the primitive tense formula Q_1: $\blacklozenge\Diamond p \supset \Diamond\blacklozenge p$. Since*

$$\sigma(\blacklozenge\Diamond p) = \bullet * \bullet * X_p \text{ and } \sigma(\Diamond\blacklozenge p) = * \bullet * \bullet X_p$$

from $\blacklozenge\Diamond p \supset \Diamond\blacklozenge p$ we obtain the rule schema (dropping the subscript p)

$$\frac{* \bullet * \bullet X \vdash Y}{\bullet * \bullet * X \vdash Y} q_1$$

such that $DLM + q_1$ properly displays $Kt \oplus Q_1$.

Next, consider the primitive tense formula Q_2:

$$\Diamond p \wedge \Diamond q \wedge \Diamond r \supset \Diamond((\blacklozenge p \wedge \blacklozenge q) \vee (\blacklozenge q \wedge \blacklozenge r) \vee (\blacklozenge p \wedge \blacklozenge r))$$

Using the classical equivalences $\Diamond(A \vee B) = \Diamond A \vee \Diamond B$ to push the disjunctions outwards, the above formula is equivalent to

$$\Diamond p \wedge \Diamond q \wedge \Diamond r \supset \Diamond(\blacklozenge p \wedge \blacklozenge q) \vee \Diamond(\blacklozenge q \wedge \blacklozenge r) \vee \Diamond(\blacklozenge p \wedge \blacklozenge r)$$

By inspection, the above formula is a primitive tense formula. Using σ we obtain the rule schema q_2

$$\frac{* \bullet *(\bullet L \circ \bullet M) \vdash Y \qquad * \bullet *(\bullet M \circ \bullet N) \vdash Y \qquad * \bullet *(\bullet L \circ \bullet N) \vdash Y}{* \bullet * L \circ * \bullet * M \circ * \bullet * N \vdash Y} q_2$$

such that $DLM + q_2$ properly displays $Kt \oplus Q_2$.

5.1.2 A semantic characterisation for primitive tense formulae

In the previous section we saw that the properly displayable axiomatic extensions of Kt is exactly the class of axiomatic extensions over Kt by primitive tense formulae. In this section we will see that these formulae can be characterised by a certain class of tense Kracht formulae from the frame language $\mathcal{L}^f$.

Definition 5.4 ($\mathbf{A^r f \exists^r x}$) *Let* $\mathbf{A^r f \exists^r x}$ *denote the class of tense restrictedly positive formulae (Definition 4.16) with a single free variable x having the form $\forall^r \overline{x}\, \phi(\exists^r \overline{y}\,; x, \overline{x})$ where $\phi(x)$ is constructed from existential restricted quantifiers and (positive) atomic formulae of the form $u = u$, $u = v$ and $\mathcal{R}^s uv$ using $\wedge$ and $\vee$, and in an atomic formula $u = v$ and $\mathcal{R}^s uv$ at least one of u and v is inherently universal (Definition 4.17).*

Kracht [39] has identified the relationship between primitive tense formulae and the class $\mathbf{A^r f \exists^r x}$[4] — namely that (finite sets of) primitive tense formulae and conjunctions of formulae from $\mathbf{A^r f \exists^r x}$ define exactly the same class of frames. Kracht makes use of his Calculus of Internal Descriptions [38, 40] to obtain this result. In this work, instead of using that calculus, we make use of the algorithm given in [7] and described in Section 4.2.2. Since we will make use of the result to characterise properly display axiomatic extensions of modal logic in later sections, we will present two new proofs of this result that do not rely on the Calculus of Internal Descriptions.

Define a $\blacklozenge\Diamond\wedge$-*formula* $f(X_0, \dots, X_n)$ to be a formula constructed from distinct symbols $X_1, \dots, X_n$, each appearing exactly once, using $\wedge$, $\Diamond$ and $\blacklozenge$. A $\Diamond\wedge$-*formula* is a $\blacklozenge\Diamond\wedge$-formula containing no occurrences of $\blacklozenge$. We write $f(Y_0, \dots, Y_n)$ to mean the formula obtained from $f(X_0, \dots, X_n)$ by substituting Y_i for each X_i. For brevity we will write f for $f(X_0, \dots, X_n)$ and $f(Y_i)$ for $f(Y_0, \dots, Y_n)$.

Let $\overset{\vee}{f}$ be the formula obtained from f by replacing each conjunction symbol $\wedge$ with the disjunction symbol $\vee$. The formula f^b is obtained from f by deleting all occurrences of $\Diamond$ and $\blacklozenge$. When f is a $\blacklozenge\Diamond\wedge$-formula, f^b is equivalent to the

[4] Kracht [39] actually relates the primitive tense formulae with type 1 formulae of the form $\forall^r \overline{x}\, \exists^r \overline{y}\, \phi(x)$ such that ϕ is free of quantifiers, as opposed to the type $1'$ formulae in $\mathbf{A^r f \exists^r x}$. This characterisation has persisted in the subsequent literature. M. Kracht agrees with the observation in the text following Example 4.33 that it is not clear how to obtain a type 1 formula (personal correspondence). None of the results in [39] is affected by the use of type $1'$ formulae.

conjunction $X_0 \wedge \ldots \wedge X_n$ and $\overset{\vee}{f}{}^b$ is equivalent to the formula $X_0 \vee \ldots \vee X_n$. Finally, we attach the subscript ST to a function (eg. f_{ST}, $\overset{\vee}{f}_{ST}$) to mean the image of that function under ST_{x_0}.

The following result indicates that a $\blacklozenge\Diamond\wedge$-formula arises when we apply the algorithm in Theorem 4.31 to compute the tense formula correspondent of a formula in $\mathbf{A^r f \exists^r x}$.

Lemma 5.5 *Suppose that $\alpha \in \mathbf{A^r f \exists^r x}$. Then the tense formula correspondent of α obtained from the algorithm in Theorem 4.31 is frame equivalent to a formula of the form*

$$f(p_0 \wedge \neg D_0, \ldots, p_n \wedge \neg D_n) \supset \bot \tag{5.2}$$

where $f(X_0, \ldots, X_n)$ is a $\blacklozenge\Diamond\wedge$-formula and each p_i is a distinct propositional variable and each D_i is either $\bot$ or constructed from variables in $\{p_i\}$ and $\top$ using $\Diamond$, $\blacklozenge$, $\wedge$ and $\vee$.

Proof. Inspection of the algorithm in Theorem 4.31. It is easy to check that the resulting tense formula has the form $A \supset \bot$, where A is a substitution instance of a $\blacklozenge\Diamond\wedge$-formula $f(X_0, \ldots, X_n)$, where each X_i is substituted with a formula of one of the following forms: $p_i \wedge \neg D_i$, p_i or $\neg D_i$, where the $\{p_i\}_{i \in I}$ ($I \subseteq \{0, \ldots, n\}$) are distinct, and the tense formulae $\{D_i\}_{i \in J}$ ($J \subseteq \{0, \ldots, n\}$) are constructed from the propositional variables $\{p_i\}_{i \in I}$ using $\top, \Diamond, \blacklozenge, \wedge$ and $\vee$.

Form the sequence $p_1, \ldots, p_n$ from $\{p_i\}_{i \in I}$ in the obvious way by introducing new propositional variables p_j for each missing index j. Similarly, form the sequence $D_1, \ldots, D_n$ from $\{D_i\}_{i \in J}$ by setting $D_j = \bot$ for each missing index j. It is easy to see that for any frame F, $F \models A$ iff $F \models f(p_i \wedge \neg D_i)$. Hence the claim is proved. Q.E.D.

Example 5.6 *Consider the formula $(\forall y \rhd x)(\forall u \lhd y) Rux$. Applying the algorithm in Theorem 4.31, we obtain the correspondent formula $p_0 \wedge \Diamond\blacklozenge\neg\Diamond p_0 \supset \bot$. This formula is frame equivalent to $(p_0 \wedge \neg\bot) \wedge \Diamond\blacklozenge(p_1 \wedge \neg\Diamond p_0) \supset \bot$. Then this formula can be written as $f(p_0 \wedge \neg\bot, p_1 \wedge \neg\Diamond p_0)$ where f is the $\blacklozenge\Diamond\wedge$-formula $X_0 \wedge \Diamond\blacklozenge X_1$.*

Because of the presence of the negation symbols, the formula (5.2) is *not* a primitive tense formula. However this formula *is* frame-equivalent to a primitive tense formula as the following result shows.

Lemma 5.7 *A formula of the form (5.2) under the restrictions for $\{p_i\}_{i \in P}$ and $\{D_i\}_{i \in \mathcal{D}}$ stated in Lemma 5.5 is frame-equivalent to the primitive tense formula $f(p_i) \supset \overset{\vee}{f}(p_i \wedge D_i)$.*

Using this result we can prove Kracht's [39] characterisation of primitive tense formulae.

Theorem 5.8 *Let $\mathcal{F}$ be some class of frames. Then, $\mathcal{F}$ is defined by a set of primitive tense formulae iff $\mathcal{F}$ can be defined by a formula of the form $\forall x(\wedge_i \alpha_i)$ for $\{\alpha_i\}_{i \in J} \subset$ $\mathbf{A^r f \exists^r x}$.*

Proof. The direction from left-to-right follows from an inspection of the algorithm in Theorem 4.24 for computing the first-order formula corresponding of a given very simple Sahlqvist formula. In particular, it is easy to verify that the first-order correspondent α_i of a primitive tense formula t_i is a formula in $\mathbf{A^r f \exists^r x}$. Since $\mathcal{F}_{t_i} = \mathcal{F}_{\forall x \alpha_i}$ for each i, it follows that

$$\begin{aligned} \mathcal{F}_{\{t_i\}_{i \in T}} &= \mathcal{F}_{\wedge_i (\forall x \alpha_i)} \\ &= \mathcal{F}_{\forall x (\wedge_i \forall x \alpha_i)} \end{aligned}$$

For the right-to-left direction, by Lemma 5.5 and Lemma 5.7 we know that for each $\alpha_i \in \mathbf{A^r f \exists^r x}$, $\forall x \alpha_i$ corresponds to some primitive tense formula t_i. It follows that $\mathcal{F}_{\forall x(\wedge_i \alpha_i)} = \mathcal{F}_{\{t_i\}_{i \in T}}$ as required. Q.E.D.

Thus it remains to prove Lemma 5.7. The claim that $f(p_i) \supset \overset{\vee}{f}(p_i \wedge D_i)$ is a primitive tense formula is easily verified. We present two proofs of the frame-equivalence portion of the claim. The first utilises the second-order representation of a tense formula. The second proof (see Appendix A) uses a model-theoretic argument.

Proof.[First proof of Lemma 5.7] In the following we will implicitly make use of the following properties for a $\blacklozenge\Diamond\wedge$-formula f. Because $\exists^r u\,(A \wedge B) = (\exists^r u\, A \wedge B)$ when there is no free occurrence of u in B, we have $ST_{x_0}(f(X_i)) = f_{ST}(ST_{x_i}(X_i))$ for some distinct set $\{x_1, \ldots, x_n\}$. Also, $f_{ST}(ST_{x_i}(X_i)) = \exists^r \bar{x}\, f^b(X_i)$ where $\exists^r \bar{x}$ is an abbreviation for $\exists^r x_1 \ldots \exists^r x_n$. Finally, observe that when $\bar{x}$ is not free in B, then $(\exists^r \bar{x}\, A) \supset B = \forall^r \bar{x}\,(A \supset B)$.

We will prove the claim in three steps.

STEP 1.

We know that $f(p_i \wedge \neg D_i) \supset \bot$ is frame-equivalent to the following (see text around (4.3)):

$$
\begin{aligned}
\widetilde{\forall}\overline{P}\forall x_0 ST_x(f(p_i \wedge \neg D_i)) \rightarrow \mathbf{f} &= \widetilde{\forall}\overline{P}\forall x_0 f_{ST}(ST_{x_i}(p_i) \wedge ST_{x_i}(\neg D_i)) \rightarrow \mathbf{f} \\
&\quad \text{(for some distinct set } \{x_1, \ldots, x_n\}) \\
&= \widetilde{\forall}\overline{P}\forall x_0 \forall^r \bar{x}\ \big(f^b(ST_{x_i}(p_i) \wedge ST_{x_i}(\neg D_i)) \rightarrow \mathbf{f}\big) \\
&= \widetilde{\forall}\overline{P}\forall x_0 \forall^r \bar{x}\ \left(f^b(ST_{x_i}(p_i)) \rightarrow \overset{\vee}{f}{}^{b}\ (ST_{x_i}(D_i))\right) \\
&= \widetilde{\forall}\overline{P}\forall x_0 \forall^r \bar{x}\ \left(f^b(P_i x_i) \rightarrow \overset{\vee}{f}{}^{b}\ (ST_{x_i}(D_i))\right) \qquad (5.3)
\end{aligned}
$$

We claim that (5.3) is frame-equivalent to

$$\forall x_0 \forall^r \bar{x}\ \left(\overset{\vee}{f}{}^{b}\ (ST_{x_i}(D_i))\right)[\{\sigma(P_i)/P_i\}] \tag{5.4}$$

Define the predicate $\sigma(P_i)(\omega) = (x_i = \omega)$. It is easy to check that (5.4) can be obtained from (5.3) by instantiating each predicate variable P_i with $\sigma(P_i)$ so $M \models (5.3)$ implies $M \models (5.4)$ for any model M. To show that (5.4) implies (5.3), for an arbitrary model M and arbitrary states $w, w_1, \ldots, w_n$, assuming that $M \models$ (5.4) and

$$M \models f^b(P_i x_i)[w_0/x_0, w_1/x_1, \ldots, w_n/x_n]$$

we need to show that

$$M \models \overset{\vee}{f}{}^{b}\ (ST_{x_i}(D_i))[w_0 w_1 \ldots w_n] \tag{5.5}$$

It follows from the assumptions that

$$M \models \overset{\vee}{f}{}^{b}\ (ST_{x_i}(D_i))[w_0 w_1 \ldots w_n][\{\sigma(P_i)/P_i\}] \tag{5.6}$$

Next, since $f^b(P_i x_i)[w_0 w_1 \ldots w_n]$ is a conjunction of formulae $P_i w_i$ for each i, from our assumption we have that

$$M \models \forall y(\sigma(P_i)(y) \rightarrow Py)[w_0 w_1 \ldots w_n]$$

This tells us that each predicate P_i *extends* the valuation $\sigma(P_i)$. Since (5.5) is positive in every predicate variable, the result follows from (5.6) by appealing to monotonicity.

STEP 2.

Now consider the formula $f(p_i) \supset \overset{\vee}{f}(p_i \wedge D_i)$. We know that this formula is frame-equivalent to the following second-order formula

$$\widetilde{\forall}\overline{P}\forall x_0 f_{ST}(P_i x_i) \rightarrow \overset{\vee}{f}_{ST}(ST_{z_i}(p_i \wedge D_i))$$

where, without loss of generality we may assume that the sets $\{x_1, \ldots, x_n\}$ and $\{z_1, \ldots, z_n\}$ are disjoint. This simplifies to

$$\widetilde{\forall}\overline{P}\forall x_0 \forall^r \bar{x} \left(f^b(P_i x_i) \rightarrow \overset{\vee}{f}_{ST}(P_i z_i \wedge ST_{z_i}(D_i)) \right) \tag{5.7}$$

Instantiating each P_i with the predicate $\sigma(P_i)$ we defined in Step 1, we get

$$\forall x_0 \forall^r \bar{x}\, f_{ST}(z_i = x_i \wedge ST_{z_i}(D_i))[\{\sigma(P_i)/P_i\}]$$

This formula is equivalent to

$$\forall x_0 \forall^r \bar{x} \left(\overset{\vee}{f}^b (ST_{x_i}(D_i)) \right) [\{\sigma(P_i)/P_i\}] \tag{5.8}$$

(because of the $z_i = x_i$ terms, the existential restricted quantifiers $\{\exists^r z_i\}$ in f_{ST} are redundant and hence can be deleted). So $M \models (5.7)$ implies $M \models (5.8)$ for any model M. We claim that (5.8) implies (5.7) (so these formulae are frame-equivalent). For an arbitrary model M and arbitrary states $w_0, w_1, \ldots, w_n$, assuming that $M \models (5.8)$ and

$$M \models f^b(P_i x_i)[w_0 w_1 \ldots w_n]$$

we need to show that

$$M \models \overset{\vee}{f}_{ST}(P_i z_i \wedge ST_{z_i}(D_i))[w_0 w_1 \ldots w_n] \tag{5.9}$$

It follows from the assumptions that

$$M \models \overset{\vee}{f}_{ST}(P_i z_i \wedge ST_{z_i}(D_i))[w_0 w_1 \ldots w_n][\{\sigma(P_i)/P_i\}] \tag{5.10}$$

Next, since $f^b(P_i x_i)[w_0 w_1 \ldots w_n]$ is a conjunction of formulae $P_i w_i$ for each i, from our assumption we have that

$$M \models \forall y(\sigma(P_i)(y) \rightarrow Py)[w_0 w_1 \ldots w_n]$$

The above tells us that each predicate P_i extends the valuation $\sigma(P_i)$. Since (5.9) is positive in every predicate variable, the result follows from (5.10) by appealing to monotonicity.

STEP 3.

Finally, since (5.4) and (5.8) are identical, it follows that the formulae $f(p_i \wedge \neg D_i) \supset \bot$ and $f(p_i) \supset \overset{\vee}{f}(p_i \wedge D_i)$ are frame-equivalent. Q.E.D.

We observe that the stronger result that every formula in $\mathbf{A^r f \exists^r x}$ locally corresponds to a primitive tense formula also holds. To prove this result, simply remove the quantification over x_0 in the above proof.

5.2 Displaying modal logics

A *primitive modal formula* is a primitive tense formula that does not contain $\blacklozenge$. In other words, a modal formula is called a primitive modal formula if it has the form $A \supset B$, where A contains each propositional variable at most once, and A and B are built from $\top$, $\wedge$, $\vee$ and $\Diamond$. Also let $DLM.K$ denote the display calculus obtained from DLM by deleting the introducing the tense operators (ie the rules $\blacklozenge \vdash$, $\vdash \blacklozenge$, $\blacksquare \vdash$ and $\vdash \blacksquare$). All the definitions for DLM (such as the definition of "derivation", "invertible", "admissible" and "structural rules") apply to $DLM.K$ in the obvious way. For example, compare the following definition of *properly displays* for $DLM.K$ with Definition 4.55.

Definition 5.9 (properly displays for $DLM.K$) *Let $\{\rho_i\}_{i\in I}$ be a set of structural rules. We say that $DLM.K + \{\rho_i\}_{i\in I}$* properly displays *the modal logic L if*

(i) all the rules of $DLM.K + \{\rho_i\}_{i\in I}$ satisfy (C1)–(C8), and

(ii) for any formula $A \in \mathbf{For}\mathcal{ML}$, $A \in L$ iff $\mathbf{I} \vdash A$ is derivable in $DLM.K + \{\rho_i\}_{i\in I}$.

Given a modal logic L, if there is a set $\{\rho_i\}_{i\in I}$ of structural rules such that $DLM.K + \{\rho_i\}_{i\in I}$ properly displays L, we say that L is properly displayable over $DLM.K$. We also have at our disposal the Display Property and Belnap's cut-elimination theorem. The obvious question is whether there is an analogue of Display Theorem I for modal logics. Kracht has claimed such an analogue (Section 5.2.1). However, a counterexample to the claim has been presented (Section 5.2.2). In Section 5.2.3 we identify the error in Kracht's claim. Then in Section 5.2.4 we introduce the M-formulae and show that any axiomatic extension of K via M-formulae is properly displayable. Finally, in Section 5.2.5 we investigate what other logics might be properly displayable.

5.2.1 Kracht's claim

Define the following classes of logics:

$$\mathcal{D}_{modal} = \{L \text{ is properly displayable over } DLM.K \mid L = K \oplus \Delta \text{ for some set } \Delta \text{ of modal formulae}\}$$

$$\mathcal{P}_{modal} = \{L \mid L = K \oplus \{m_i\}_{i\in M} \text{ for some set } \{m_i\}_{i\in M} \text{ of primitive modal formulae }\}$$

Consider the following lemma.

Lemma 5.10 *Let $\{m_i\}_{i \in M}$ be a set of primitive modal formulae. Then, using Display Theorem I we can compute a set $\{\rho_i\}_{i \in I}$ of structural rules such that $DLM + \{\rho_i\}_{i \in I}$ properly displays $Kt \oplus \{m_i\}_{i \in M}$ and $DLM.K + \{\rho_i\}_{i \in I}$ properly displays $K \oplus \{m_i\}_{i \in M}$.*

Proof. Since every primitive modal formula is a primitive tense formula, by Theorem 5.2, for an arbitrary set $\{m_i\}_{i \in M}$ of primitive modal formulae we can compute a set $\{\rho_i\}_{i \in I}$ of structural rules such that $DLM + \{\rho_i\}_{i \in I}$ properly displays $Kt \oplus \{m_i\}_{i \in M}$.

We will show that $DLM.K + \{\rho_i\}_{i \in I}$ properly displays $K \oplus \{m_i\}_{i \in M}$.

Suppose that $A \in K \oplus \{m_i\}_{i \in M}$. Then $A \in Kt \oplus \{m_i\}_{i \in M}$ and so the sequent $\mathbf{I} \vdash A$ is derivable in $DLM + \{\rho_i\}_{i \in I}$. As a consequence of the cut-elimination theorem and (C1), if a tense introduction rule occurs in cutfree derivation then the endsequent would contain either $\blacklozenge$ or $\blacksquare$. Hence there is a derivation of $\mathbf{I} \vdash A$ that does not use the tense introduction rules. Thus $\mathbf{I} \vdash A$ is derivable in $DLM.K + \{\rho_i\}_{i \in I}$.

Next, suppose that $\mathbf{I} \vdash A$ is derivable in $DLM.K + \{\rho_i\}_{i \in I}$. Certainly then $\mathbf{I} \vdash A$ is derivable in $DLM + \{\rho_i\}_{i \in I}$ and thus $A \in Kt \oplus \{m_i\}_{i \in M}$. By the Sahlqvist Completeness Theorem, $Kt \oplus \{m_i\}_{i \in M}$ is sound and complete for (tense frames based on) the class $\mathcal{F}_{\{m_i\}_{i \in M}}$ of modal frames, so $\mathcal{F}_{\{m_i\}_{i \in M}} \models A$. By the same theorem, $K \oplus \{m_i\}_{i \in M}$ is sound and complete for $\mathcal{F}_{\{m_i\}_{i \in M}}$. It follows that $A \in K \oplus \{m_i\}_{i \in M}$.

So $A \in K \oplus \{m_i\}_{i \in M}$ iff $\mathbf{I} \vdash A$ is derivable in $DLM.K + \{\rho_i\}_{i \in I}$. Obviously every rule in the calculus obeys (C1)–(C8). Hence it follows that $DLM.K + \{\rho_i\}_{i \in I}$ properly displays $K \oplus \{m_i\}_{i \in M}$. Q.E.D.

From the above lemma it follows that

$$\mathcal{P}_{modal} \subseteq \mathcal{D}_{modal}$$

So every axiomatic extension of K by primitive modal formulae is properly displayable. What can we say about the other direction? Kracht [39, pg 144] has claimed that the other direction is also true (he calls this result "Display Theorem II").

Claim 5.11 *An axiomatic extension of Hilbert-style modal logic can be properly displayed (by structural rules over $DLM.K$) iff it is axiomatizable by a set of primitive modal formulae.*

In our notation, Claim 5.11 states that $\mathcal{D}_{modal} = \mathcal{P}_{modal}$. Recently however, the logic $K \oplus \Diamond\Box p \supset \Box\Diamond p$ has been proposed as an example of a logic that is in $\mathcal{D}_{modal}$ and not in $\mathcal{P}_{modal}$ (this counterexample has been known at least as far back as Wansing [80], where he credits Rajeev Goré).

5.2.2 A counterexample to Kracht's claim

We need to show that $K \oplus \Diamond\Box p \supset \Box\Diamond p \in \mathcal{D}_{modal}$ and $K \oplus \Diamond\Box p \supset \Box\Diamond p \notin \mathcal{P}_{modal}$. So there are two things to prove.

(i) the logic $K \oplus \Diamond\Box p \supset \Box\Diamond p$ is properly displayable (Lemma 5.12); and

(ii) $K \oplus \Diamond\Box p \supset \Box\Diamond p$ cannot be expressed as a primitive modal axiomatic extension over K (Lemma 5.16)

Because $\Diamond\Box p \supset \Box\Diamond p$ and $\blacklozenge\Diamond p \supset \Diamond\blacklozenge p$ are frame-equivalent, the proof of (i) is essentially an application of the work in Section 5.1. To prove (ii) requires more work. Clearly $\Diamond\Box p \supset \Box\Diamond p$ is not a primitive modal formula but we need to make sure that there is no set $\{m_i\}_{i\in M}$ of primitive modal formulae such that $K \oplus \Diamond\Box p \supset \Box\Diamond p = K \oplus \{m_i\}_{i\in M}$. We are not aware of any proof of (ii) (although the result has obviously been accepted as true in Wansing [80]).

Lemma 5.12 *(i) The logic $Kt \oplus \blacklozenge\Diamond p \supset \Diamond\blacklozenge p$ is properly displayable over DLM.*

(ii) For any frame F, $F \models \blacklozenge\Diamond p \supset \Diamond\blacklozenge p$ iff $F \models \Diamond\Box p \supset \Box\Diamond p$.

(iii) The logic $K \oplus \Diamond\Box p \supset \Box\Diamond p$ is properly displayable over $DLM.K$.

Proof. In Example 5.3 we saw that $DLM+\rho$ properly displays $Kt \oplus \blacklozenge\Diamond p \supset \Diamond\blacklozenge p$ where ρ is the structural rule

$$\frac{*\bullet*\bullet X \vdash Y}{\bullet*\bullet* X \vdash Y}$$

Proof of (ii). Let us compute the first-order correspondents of $\blacklozenge\Diamond p \supset \Diamond\blacklozenge p$ and $\Diamond\Box p \supset \Box\Diamond p$. We have already seen that the first-order correspondent of the very simple Sahlqvist formula $\blacklozenge\Diamond p \supset \Diamond\blacklozenge p$ is the following Kracht tense formula (see Example 4.25):

$$(\forall u \lhd x)(\forall v \rhd u)(\exists l \rhd x)Rvl \tag{5.11}$$

Since $\Diamond\Box p \supset \Box\Diamond p$ is a Sahlqvist formula but not a very simple Sahlqvist formula, we would need to use the full algorithm here (in Theorem 4.24 we only described the procedure for very simple Sahqlvist formulae). However this formula is an example of the so-called *incestual* axioms discussed by Lemmon and Scott [43] who show that $\Diamond\Box p \supset \Box\Diamond p$ is a first-order correspondent of the formula $\forall uv(Rxu \wedge Rxv \to \exists l(Rul \wedge Rvl)$ which is equivalent to the Kracht modal formula

$$(\forall u \rhd x)(\forall v \rhd x)(\exists l \rhd u)(Rvl) \tag{5.12}$$

It is straightforward to check that (5.11) and (5.12) are valid on the identical class of frames, so the claim is proved.

Proof of (iii). Since $\Diamond\Box p \supset \Box\Diamond p$ and $\blacklozenge\Diamond p \supset \Diamond\blacklozenge p$ define the same class $\mathcal{F}$ of frames, by the Sahlqvist Completeness Theorem, $Kt \oplus \blacklozenge\Diamond p \supset \Diamond\blacklozenge p$ and $K \oplus \Diamond\Box p \supset \Box\Diamond p$ are both sound and complete with respect to $\mathcal{F}$. Thus for any modal formula A, $A \in Kt \oplus \blacklozenge\Diamond p \supset \Diamond\blacklozenge p$ iff $\mathcal{F} \models A$ iff $A \in K \oplus \Diamond\Box p \supset \Box\Diamond p$.

Now, if $A \in K \oplus \Diamond\Box p \supset \Box\Diamond p$ then $A \in Kt \oplus \Diamond\Box p \supset \Box\Diamond p$ and so $\mathbf{I} \vdash A$ is derivable in $DLM + \rho$. Since A is a modal formula, $\mathbf{I} \vdash A$ is derivable in $DLM.K + \rho$. Next, if $\mathbf{I} \vdash A$ is derivable in $DLM.K + \rho$ then $\mathbf{I} \vdash A$ is derivable in $DLM + \rho$, so $A \in Kt \oplus \blacklozenge\Diamond p \supset \Diamond\blacklozenge p$. Since A must be a modal formula, it follows that $A \in K \oplus \Diamond\Box p \supset \Box\Diamond p$. Of course, every rule in $DLM.K + \rho$ obeys (C1)–(C8) so we have proved that $DLM.K + \rho$ properly displays $K \oplus \Diamond\Box p \supset \Box\Diamond p$. Q.E.D.

Next we show that $K \oplus \Diamond\Box p \supset \Box\Diamond p$ cannot be written as a primitive modal axiomatisation over K. Let $Var(A)$ be the set of propositional variables that occur in the formula A. We will require the following lemma.

Lemma 5.13 *Let L be an axiomatic extension of Kt (resp. K). For every primitive tense (modal) formula $A \supset B$, there are a finite number of primitive tense (modal) formulae $A_i^* \supset B_i^*$ satisfying $L \oplus A \supset B = L \oplus_{i\in\Lambda} \{A_i^* \supset B_i^*\}$ such that for each i, either (i) $Var(A_i^*) = Var(B_i^*)$, or (ii) A_i^* is constructed from $\top$ using $\wedge$, $\Diamond$ and $\blacklozenge$ and B_i^* is the propositional variable q.*

Proof. We prove the result for primitive tense formulae. The proof for the modal case is analogous.

Write A and B in disjunctive normal form as the (finite) disjunctions $\bigvee_{i\in I} A_i$ and $\bigvee_{j\in J} B_j$ respectively, where each disjunct is constructed from propositional variables and $\top$ using $\wedge$, $\Diamond$ and $\blacklozenge$. Now $L \oplus (\bigvee_{i\in I} A_i \supset \bigvee_{j\in J} B_j) = L \oplus_{i\in I} \{A_i \supset \bigvee_{j\in J} B_j\}$, where each A_i is constructed from propositional variables and $\top$ using $\wedge$, $\Diamond$ and $\blacklozenge$.

Let $i \in I$. We will show that $L \oplus A_i \supset \bigvee_{j\in J} B_j = L \oplus A_i^* \supset B_i^*$ where $A_i^* \supset B_i^*$ is a primitive tense formula with the required properties. The result follows by repeating the argument for each $i \in I$.

Let A'_i be obtained from A_i by substituting $\top$ for every variable in $Var(A_i) \setminus Var(B)$, and let B'_i be obtained from B by substituting $\bot$ for every variable in $Var(B) \setminus Var(A_i)$.

Since $A'_i \supset B$ is a substitution instance of $A_i \supset B$, it follows that $L \oplus A'_i \supset B \subset L \oplus A_i \supset B$. We have the following results for arbitrary formulae built from propositional variables and $\top$ using $\wedge$, $\Diamond$ and $\blacklozenge$; and a string σ constructed from $\Diamond$ and $\blacklozenge$:

$$(\top \supset \top) \in Kt \text{ and } (p \supset \top) \in Kt \text{ for any propositional variable } p$$

$$D_1 \supset D_2 \in Kt \text{ implies } \sigma D_1 \supset \sigma D_2 \in Kt$$

$$D_1 \supset D_2 \in Kt \text{ and } E_1 \supset E_2 \in Kt \text{ implies } D_1 \wedge E_1 \supset D_2 \wedge E_2 \in Kt$$

Statements 1 and 3 follow from the axioms of Cp (these are present in L too). Statement 2 follows from $(Ax - \Box)$ and $(Ax - \blacksquare)$. These are axioms of Kt and hence are present in L too. Since A_i is constructed from propositional variables and $\top$ using $\wedge$, $\Diamond$ and $\blacklozenge$, it follows that $A_i \supset A'_i$. Together with $A'_i \supset B$ we get $A_i \supset B \in L \oplus A'_i \supset B$. Thus $L \oplus A_i \supset B = L \oplus A'_i \supset B$.

Now let us show that $L \oplus A'_i \supset B = L \oplus A'_i \supset B'_i$. Since $A'_i \supset B'_i$ is a substitution instance of $A'_i \supset B$, it follows that $L \oplus A'_i \supset B'_i \subset L \oplus A'_i \supset B$. We have the following results for arbitrary formulae built from propositional variables and $\top$ using $\wedge$, $\Diamond$ and $\blacklozenge$; and a string σ constructed from $\Diamond$ and $\blacklozenge$:

$$((D \wedge \bot) \supset \bot) \in Kt$$

$$(D \supset \bot) \in Kt \text{ implies } (\sigma D \supset \bot) \in Kt$$

Once again, the above holds for formulae in L too. From the above it follows that for any formula built from propositional variables, $\top$ and $\bot$ using $\wedge$, $\Diamond$ and $\blacklozenge$, the formula is equivalent to $\bot$ whenever it contains an occurrence of $\bot$. Applying these results to B'_i we have that $B'_i \approx \bot \in Kt$ or $B'_i \approx \bigvee_{j \in J_i} B_j \in Kt$ for some non-empty set $J_i \subset J$. It follows that $A'_i \supset B$ is derivable from $A'_i \supset B'_i$, so $A'_i \supset B \in L \oplus A'_i \supset B'_i$. Thus $L \oplus A'_i \supset B = L \oplus A'_i \supset B'_i$.

Now, suppose that $B'_i \approx \bigvee_{j \in J_i} B_j \in Kt$ for some non-empty set $J_i \subset J$. Then $A'_i \supset B'_i$ is already a primitive tense formula such that $Var(A'_i) = Var(B'_i)$. Setting $A^*_i = A'_i$ and $B^*_i = B'_i$ we can satisfy (i) in the statement of the theorem.

Finally, suppose that $B'_i \approx \bot \in Kt$. Then $L \oplus A_i \supset B = L \oplus A'_i \supset \bot$. Set $B^*_i = q$ for some $q \notin Var(A'_i)$. Then we have $L \oplus A'_i \supset \bot = L \oplus A'_i \supset q$. Let A^*_i be the formula obtained from A'_i by substituting every propositional variable in $Var(A'_i)$ with $\top$. Using our work above we obtain $L \oplus A_i \supset B = L \oplus A^*_i \supset B^*_i$, where A_i^* is constructed from $\top$ using $\wedge$, $\Diamond$ and $\blacklozenge$. Clearly $A_i^* \supset B_i^*$ is a primitive tense formula satisfying (ii) in the statement of the theorem. Q.E.D.

The reason for the seemingly unusual case split in the statement of the above lemma is due to the fact that primitive tense formulae are not permitted to contain the symbol $\bot$. For example, consider the primitive tense formula $\Diamond p \supset q$. Substituting the variable p with $\top$ and q with $\bot$ we get $\Diamond\top \supset \bot$. Let L be some extension of Kt. The proof of the above lemma witnesses that $L \oplus \Diamond p \supset q = L \oplus \Diamond\top \supset \bot$. It also witnesses that $L \oplus \Diamond\top \supset \bot = L \oplus \Diamond\top \supset q$. Now $\Diamond\top \supset \bot$ is not a primitive tense formula, but $\Diamond\top \supset q$ *is* a primtive tense formula. Since we wish to remain within the primitive tense fragment, we have formulated the lemma using this case split.

Let us introduce some terminology. We write $X_1 \vdash Y_1 \sim X_2 \vdash Y_2$ to mean that each sequent is derivable from the other using the structural rules in Table 4.2 and Table 4.3 (this is a weaker requirement that display equivalence where only rules from Table 4.2 may be used). A formula occurrence A in the sequent $X \vdash Y$ is $\bullet$*-hugged* if $X \vdash Y \sim \bullet A \vdash Y'$ or $X \vdash Y \sim X' \vdash \bullet A$ for some structure X' or Y'. Similarly, a formula occurrence A in the sequent $X \vdash Y$ is $*\bullet*$*-hugged* if $X \vdash Y \sim *\bullet *A \vdash Y'$ or $X \vdash Y \sim X' \vdash *\bullet *A$ for some structure X' or Y'.

Remark 5.14 *Consider the rules $\Box\vdash$ and $\vdash\Diamond$:*

$$\frac{A \vdash X}{\Box A \vdash \bullet X}\,\Box\vdash \qquad\qquad \frac{X \vdash A}{*\bullet *X \vdash \Diamond A}\,\vdash\Diamond$$

By inspection, in the conclusion sequent of $\Box\vdash$, the formula occurrence $\Box A$ is $\bullet$-hugged, and in the conclusion sequent of $\vdash\Diamond$, the formula occurrence $\Diamond A$ is $*\bullet*$-hugged.

An inspection of the rules in Tables 4.2 and 4.3 reveal that a formula occurrence cannot be both $\bullet$-hugged and $*\bullet*$-hugged. Moreover, for any occurrence of formula A in a sequent, A is either $\bullet$-hugged (resp. $*\bullet*$-hugged) or not $\bullet$-hugged (not $*\bullet*$-hugged).

If the conclusion sequent of a rule instance ρ contains a $\bullet$-hugged (resp. $*\bullet*$-hugged) formula occurrence A and every parametric ancestor of A (see discussion following Theorem 4.48 for the definition) in the premise sequents of ρ is $\bullet$-hugged ($*\bullet*$-hugged) we say that ρ *preserves* $\bullet$*-hugged* ($*\bullet*$*-hugged*) *formulae upwards*. A rule schema ρ preserves preserves $\bullet$-hugged (resp. $*\bullet*$-hugged) formulae upwards if every rule instance of ρ preserves $\bullet$-hugged ($*\bullet*$-hugged) formulae upwards.

By inspection, all the rule schemata in Tables 4.2 and 4.3 preserve $\bullet$-hugged and $* \bullet *$-hugged formulae upwards. In contrast, it should be clear from Remark 5.14 that some of the logical rule schemata (Table 4.1) do *not* preserve $\bullet$-hugged and $* \bullet *$-hugged formulae upwards.

Lemma 5.15 *Let $\{m_s\}_s$ be a finite set of primitive modal formulae. Then there is a finite set $\{\rho_{si}\}_{si}$ of structural rule schemata such that $DLM.K + \{\rho_{si}\}_{si}$ properly displays $K \oplus \{m_s\}_s$. Moreover, the structural rules of the calculus satisfy the following properties:*

(i) each structural rule schema preserves $\bullet$-hugged formulae upwards.

(ii) every schematic structure variable in the conclusion of a structural rule schema must occur in (at least) one of the premises.

(iii) if a premise of a structural rule schema contains a schematic structural variable not present in the conclusion, then the conclusion contains no schematic structural variables.

Proof. By Lemma 5.13, without loss of generality we may assume that each primitive modal formula m_s has the form $\bigvee_{i \in I^s} A^s_i \supset \bigvee_{j \in J^s_i} B^s_{ij}$ where for each i, A^s_i disjunction-free, and either (i) $Var(A^s_i) = Var(\bigvee_j B^s_{ij})$ or (ii) $|J_i| = 1$, B^s_{i1} is some propositional variable q, and A^s_i is built from $\top$ using $\wedge$, $\Diamond$ and $\blacklozenge$.

By Display Theorem I (Theorem 5.2) we can compute the structural rule $\{\rho_{si}\}_i$ corresponding to each m_s so that $DLM + \{\rho_{si}\}_{si}$ properly displays $Kt \oplus \{m_s\}$. Then, by Lemma 5.10 we have that $DLM.K + \{\rho_{si}\}_{si}$ properly displays $K \oplus \{m_s\}$. The structural rules in $DLM_K + \{\rho_{si}\}_{si}$ consist of the rules in Tables 4.2 and 4.3 and the rules $\{\rho_{si}\}_{si}$. By inspection it is easy to verify that all the rules in Tables 4.2 and 4.3 satisfy the claims of this corollary. Thus it remains to prove the claims for the rules $\{\rho_{si}\}_{si}$.

We compute the rule schema ρ_{si} from m_s using the translation σ from Section 5.1:

$$\sigma(\top) = I$$
$$\sigma(p) = X_p$$
$$\sigma(B \wedge A) = \sigma(B) \circ \sigma(A)$$
$$\sigma(\Diamond B) = * \bullet * \sigma(B)$$

where X_p denotes the schematic structure variable corresponding to p. Notice that we have intentionally omitted the line

$$\sigma(\blacklozenge B) = \bullet\sigma(B)$$

This is because we are dealing with primitive modal formulae and hence cannot encounter $\blacklozenge$. Since we are using the algorithm in Section 5.1 we know that each ρ_{si} will have the form

$$\frac{L_1 \vdash Y \qquad \ldots \qquad L_n \vdash Y}{M \vdash Y}$$

where the schematic structures $L_1, \ldots, L_n, M$ are composed from schematic structure variables and $\mathbf{I}$ using only $\circ$ and $* \bullet *$. For example, there can be no rule schema of the following form:

$$\frac{\bullet L \vdash Y}{L \vdash Y}$$

As a consequence, each rule ρ_{si} preserves $\bullet$-hugged formulae upwards, so claim (i) is proved.

Now to prove (ii) and (iii). Suppose that $A_i^s \supset \bigvee_{j \in J_i^s} B_{ij}^s$ $(i \in I^s)$ is a primitive modal formula such that $Var(A_i^s) = Var(\bigvee_j B_{ij}^s)$. Then, in the corresponding rule ρ_{si} under the translation σ, every schematic structure variable in the conclusion $\sigma(A_i^s)$ must occur in (at least) one of the premises $\{\sigma(B_{ij}^s)\}_j$, and every schematic structure variable occurring in a premise will also occur in the conclusion. On the other hand, for a primitive modal formula of the form $A_i^s \supset q$ where A_i^s is constructed from $\top$ using $\wedge$, $\Diamond$ and $\blacklozenge$, then under the translation σ, the conclusion $\sigma(A_i^s)$ of ρ_{si} contains no schematic structure variables. These are the only two cases to check so the result is proved. Q.E.D.

Lemma 5.16 *The logic $K \oplus \Diamond\Box p \supset \Box\Diamond p$ cannot be written as a primitive modal axiom extension over K.*

Proof. Proof by contradiction. Assume that there exists some set $\{m_i\}_{i \in M}$ of primitive modal formulae such that $K \oplus \Diamond\Box p \supset \Box\Diamond p = K \oplus \{m_i\}_{i \in M}$. By the previous lemma, without loss of generality we may assume that each primitive modal formula m_i has the form $A_i \supset \bigwedge_j B_{ij}$ with A_i disjunction-free, and either (i) $Var(A_i) = Var(\bigwedge_j B_{ij})$ or (ii) m_i has the form $A_i \supset q$ where A_i is built from $\top$ using $\wedge$, $\Diamond$ and $\blacklozenge$. Let $\mathcal{F}$ denote the class of frames defined by $\Diamond\Box p \supset \Box\Diamond p$. From the Sahqlvist Completeness Theorem we know that $K \oplus \Diamond\Box p \supset \Box\Diamond p$ is sound and weakly complete with respect to $\mathcal{F}$.

By Lemma 5.15 we can compute the structural rules $\{\rho_i\}_{i\in I}$ corresponding to $\{m_i\}_{i\in M}$ so $DLM+\{\rho_i\}_{i\in I}$ properly displays $Kt\oplus\{m_i\}_{i\in M}$ and $DLM.K+\{\rho_i\}_{i\in I}$ properly displays $K\oplus\{m_i\}_{i\in M}$, and all the structural rules in $DLM.K+\{\rho_i\}_{i\in I}$ satisfy (i)–(iii) in Lemma 5.15.

Since $\Diamond\Box p \supset \Box\Diamond p \in K\oplus\{m_i\}_{i\in M}$, there must be a derivation in $DLM.K+\{\rho_i\}_{i\in I}$ of the sequent $\Diamond\Box p \vdash \Box\Diamond p$. Because

$$\dfrac{\dfrac{\dfrac{\Box p \vdash \Box p}{*\bullet *\Box p \vdash \Diamond\Box p}\vdash\Diamond \qquad \Diamond\Box p \vdash \Box\Diamond p}{*\bullet *\Box p \vdash \Box\Diamond p}\mathit{cut} \qquad \dfrac{\Diamond p \vdash \Diamond p}{\Box\Diamond p \vdash \bullet\Diamond p}\Box\vdash}{*\bullet *\Box p \vdash \bullet\Diamond p}\mathit{cut}$$

from the cut-elimination theorem, there must be a cutfree derivation δ of $*\bullet*\Box p \vdash \bullet\Diamond p$. Since every initial sequent of δ must have the form $p \vdash p$, tracing upwards from the endsequent along every path, we will either encounter $\Box\vdash$ first and then $\vdash\Diamond$, or $\vdash\Diamond$ first and then $\Box\vdash$. Also notice that because the $\Box p$ formula occurrence in $*\bullet*\Box p \vdash \bullet\Diamond p$ is $*\bullet*$-hugged and the $\Diamond p$ formula occurrence is $\bullet$-hugged, neither of the logical rules can occur *immediately* above the sequent $*\bullet*\Box p \vdash \bullet\Diamond p$ (see Remark 5.14). Clearly the structural rules in $\{\rho_i\}_{i\in I}$ must be capable of transforming $\Box p$ into a $\bullet$-hugged formula and $\Diamond p$ into a $*\bullet*$-hugged formula. We will first show that any such transformation will imply derivability in $DLM+\{\rho_i\}_{i\in I}$ of a sequent of a particular form. This corresponds to validity on $\mathcal{F}$ of a formula with a certain syntactical form. We will show that this leads to a contradiction.

Case I. Suppose that the $\Box$ is removed prior to the $\Diamond$ in some path above the endsequent $*\bullet*\Box p \vdash \bullet\Diamond p$ of δ. Then, in the conclusion sequent $\mathcal{S}$ of $\Box\vdash$ on that path,

(i) it must be the case that $\Box p$ is $\bullet$-hugged so $\mathcal{S}$ has the form $\Box p \vdash \bullet X_1$ for some structure X_1.

(ii) The only rules that can occur between the endsequent and $\mathcal{S}$ are the structural rules of the calculus. We already know that the structural rules preserve $\bullet$-hugged formulae upwards. Moreover, because the structural rules satisfy (ii) and (iii) in Lemma 5.15, new formulae cannot appear when passing from the conclusion to a premise of a rule instance. As a result, the sequent $\mathcal{S}$ has the form $\Box p \vdash X_2$ where X_2 is built from $\bullet\Diamond p$ and $\mathbf{I}$ using $*\bullet*$ and $\circ$.

To make $\bullet X_1 = X_2$ the only possibility is that $\mathcal{S}$ is the sequent $\Box p \vdash \bullet\Diamond p$. Thus the sequents $\Box p \vdash \bullet\Diamond p$ and $\Box p \vdash \Box\Diamond p$ are derivable so $\Box p \supset \Box\Diamond p$ must be in $K \oplus \Diamond\Box p \supset \Box\Diamond p$. Now consider the following frame F:

$$\begin{tikzcd} u \arrow[r] & v \arrow[r] & w \arrow[loop above] \end{tikzcd}$$

By inspection, $\Diamond\Box p \supset \Box\Diamond p$ is valid on F and hence $F \in \mathcal{F}$.[5] Therefore $\Box p \supset \Box\Diamond p$ must be valid on F. However $\Box p \supset \Box\Diamond p$ is falsifiable on F at state u by setting $V(p) = \{v\}$. So we have a contradiction.

Case II. The remaining case to consider is when $\Diamond$ is removed prior to $\Box$ in every path above the endsequent of δ. Then, in the conclusion sequent $\mathcal{S}_j$ of every path above the endsequent $*\bullet*\Box p \vdash \bullet\Diamond p$ of δ it must be the case that $\Diamond p$ is $*\bullet*$-hugged in $\mathcal{S}_j$, so $\mathcal{S}_j$ is of the form $*\bullet*X_j \vdash \Diamond p$. Since all the structural rules of the calculus preserve $\bullet$-hugged formulae upwards, the only way to get from a $\bullet$-hugged $\Diamond p$ occurrence $(*\bullet*\Box p \vdash \bullet\Diamond p)$ to a $*\bullet*$-hugged $\Diamond p$ occurrence $(*\bullet*X_j \vdash \Diamond p)$ is if a structure containing $\bullet\Diamond p$ disappears along the path between the endsequent and $\mathcal{S}_j$. However we know that every schematic structural variable occurring in the conclusion of a structural rule in the calculus must occur in (at least) one premise so it is impossible for $\bullet\Diamond p$ to disappear in *every* path above the endsequent and below the $\{\mathcal{S}_j\}$. We have arrived at a contradiction. Q.E.D.

We have shown the existence of a properly displayable modal logic that cannot be written as an axiomatic extension over K by primitive modal formulae. Hence Kracht's claim is contradicted. Admittedly, the display calculus is not an ideal proof-theoretic system to conduct a proof of this nature because of the large diversity of structures that can appear in a derivation. A semantic proof would likely yield a more elegant argument. As we noted before, the statement of Lemma 5.16 is implicitly affirmed in [80, 33] although we are not aware of any proof.

5.2.3 Identifying the error in 'Display Theorem II'

Since we have contradicted Claim 5.11, it follows that there must be some error in Kracht's proof [39] of this claim. Let us identify this error.

[5] We have seen that $\Diamond\Box p \supset \Box\Diamond p$ corresponds to the formula $(\forall u \rhd x)(\forall v \rhd x)(\exists l \rhd u)(Rvl)$. Informally, we can think of this property as saying that 'any two not necessarily distinct states that x sees (u and v say) will themselves see some state y'. So validity of $\Diamond\Box p \supset \Box\Diamond p$ on a frame F can be easily verified by checking this statement for each state in F.

Definition 5.17 ($\mathbf{A^{r'}f\exists^{r'}x}$) *Let* $\mathbf{A^{r'}f\exists^{r'}x}$ *denote the class of modal restrictedly positive formulae (Definition 4.16) with a single free variable* x *of the form* $\forall^{r'}\overline{x}\,\phi(\exists^{r'}\overline{y}\,;x,\overline{x})$ *where* $\phi(x)$ *is constructed from existential forward restricted quantifiers and (positive) atomic formulae of the form* $u=u$, $u=v$ *and* $\mathcal{R}^s uv$ *using* $\wedge$ *and* $\vee$, *and in an atomic formula* $u=v$ *and* $\mathcal{R}^s uv$ *at least one of* u *and* v *is inherently universal (Definition 4.17).*

Clearly $\mathbf{A^{r'}f\exists^{r'}x} \subset \mathbf{A^{r}f\exists^{r}x}$.

Kracht's proof consists of two steps.

Step I. If $\alpha \in \mathbf{A^{r}f\exists^{r}x}$ and $\mathcal{F}_{\forall x\alpha}$ is definable by some set of modal formulae, then there is an $\alpha' \in \mathbf{A^{r'}f\exists^{r'}x}$ such that $\mathcal{F}_{\forall x\alpha} = \mathcal{F}_{\forall x\alpha'}$.

His idea is to get rid of the 'backward restricted quantifiers' $(\forall u \lhd v)$ and $(\exists u \lhd v)$ in α. Let α be the formula $\forall^r\overline{y}\,\phi(\exists^r\overline{z}\,;x,\overline{y})$. Since $\mathcal{F}_{\forall x\alpha}$ is modally definable, $\forall x\alpha$ is preserved under generated modal subframes. It can be shown that this implies that every backward existential restricted quantifier $(\exists u \lhd v)$ in $\exists^r\overline{z}$ can be replaced by some number of forward existential restricted quantifiers, to obtain a formula α' such that $\forall x\alpha = \forall x\alpha'$, where α' has the form

$$\forall^r\overline{y}\,REL(x) \wedge \phi(\exists^{r'}\overline{z}'\,;x,\overline{y})$$

and $REL(x)$ consists of a conjunction of terms Ruv for each $(\exists u \lhd v)$ term in $\exists^r\overline{z}$. To remove the backward universal restricted quantifiers in α', Kracht suggests that we consider the formula $\neg\alpha'$. He claims here that $\forall x\neg\alpha'$ is preserved under generated modal subframes, and hence we can remove the resulting backward existential restricted quantifiers in $\neg\alpha'$ in a similar manner to before. *However, it is not clear that the formula* $\forall x\neg\alpha'$ *is in fact preserved under generated modal subframes, so his proof is incomplete.*[6]

STEP II. Next, Kracht concludes that every formula in $\mathbf{A^{r'}f\exists^{r'}x}$ corresponds to a primitive modal formula. *This is incorrect* — we have seen that the modal formula $\Diamond\Box p \supset \Box\Diamond p$ is a frame correspondent of the formula α:

$$(\forall y \rhd x)(\forall u \rhd x)(\exists v \rhd y)Ruv$$

By inspection, $\alpha \in \mathbf{A^{r'}f\exists^{r'}x}$. Suppose that α has a primitive modal correspondent m (so $\mathcal{F}_m = \mathcal{F}_{\forall x\alpha}$). Since $\Diamond\Box p \supset \Box\Diamond p$ and m are Sahlqvist formulae, using the Sahlqvist completeness theorem, $A \in K \oplus m$ iff

$$\mathcal{F}_m \models A \text{ iff } \mathcal{F}_{\forall x\alpha} \models A \text{ iff } \mathcal{F}_{\Diamond\Box p\supset\Box\Diamond p} \models A$$

[6] M. Kracht completely agrees with our analysis regarding this problem and concedes that it is not clear how to obtain the result: personal correspondence by email dated 13/Dec/2010.

Once again, by the Sahlqvist completeness theorem, $\mathcal{F}_{\Diamond\Box p\supset\Box\Diamond p} \models A$ iff $A \in K \oplus \Diamond\Box p \supset \Box\Diamond p$. Thus $K \oplus m = K \oplus \Diamond\Box p \supset \Box\Diamond p$. However, this is impossible, due to Lemma 5.16.

5.2.4 A syntactic characterisation of $\mathbf{A^{r'}f\exists^{r'}x}$

In this section we will define a subclass of modal Sahlqvist formulae called the *M-formulae* that is more expressive than the class of primitive modal formulae, and show that every $\alpha \in \mathbf{A^{r'}f\exists^{r'}x}$ corresponds to an M-formula, and every M-formula corresponds to a formula from $\mathbf{A^{r'}f\exists^{r'}x}$.

Define an *M-Ant* formula to be a formula of one of the following forms for $n, s_i \geq 0$:

$$p \wedge \Box^{s_1+1}q_1 \wedge \ldots \wedge \Box^{s_n+1}q_n \qquad\qquad \Box^{s_1+1}q_0 \wedge \ldots \wedge \Box^{s_n+1}q_n$$

Define a *basic primitive formula* to be a formula built from $\wedge$ and $\Diamond$ using propositional variables and $\top$.

Then an *M-implication* has the form $A \supset B$ where A is an M-Ant or $\top$, and B is constructed from basic primitive formulae using $\vee$, or $\bot$.

Finally, an *M-formula* is a formula constructed from M-implications $\{A_i \supset B_i\}_{i\in\Lambda}$ by freely applying disjunctions and boxes, that satisfies

$$Var(A_i) \cap Var(A_j) = \emptyset \text{ for } i \neq j$$
$$Var(B_i) \subseteq \bigcup_i Var(A_i)$$

Let $\mathcal{MFORM}$ denote the class of finite axiomatic extensions over K by M-formulae. In notation,

$$\mathcal{MFORM} = \{L \mid L = K \oplus \{M_i\} \text{ for some finite set } \{M_i\} \text{ of M-formulae}\}$$

Lemma 5.18 *Every M-formula is a modal Sahlqvist formula.*

Proof. This follows immediately from a consideration of Definition 4.26. Q.E.D.

Lemma 5.19 *Every $\alpha \in \mathbf{A^{r'}f\exists^{r'}x}$ corresponds to an M-formula, and every M-formula corresponds to a formula from $\mathbf{A^{r'}f\exists^{r'}x}$.*

Proof. It suffices to show that $\alpha \in \mathbf{A^{r'}f\exists^{r'}x}$ in its most general form corresponds to an M-formula in its most general form. Suppose that

$$\alpha = \forall^{r'}x^{(1)} \ldots \forall^{r'}x^{(n)}\, \phi(\exists^{r'}y^{(1)} \ldots \exists^{r'}y^{(m)}\,; x^{(0)}, \ldots, x^{(n)})$$

is an arbitrary element of $\mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$, so the restrictor of any $x^{(i)}$ is some $x^{(j)}$ with $j < i$, and the restrictor of any $y^{(i)}$ is either some $x^{(k)}$ or some $y^{(j)}$ with $j < i$. Also, ϕ is composed from positive atomic formulae of the form $u = u$, $u = v$ and $\mathcal{R}^s uv$ using $\wedge$ and $\vee$, and in an atom $u = v$ and $\mathcal{R}^s uv$ at least one of u and v is inherently universal (see Definition 4.18).

Clearly α is a Kracht formula. Without loss of generality, suppose that ϕ is a disjunction of conjunctions, so α is a type I formula in the terminology of Section 4.2.3. Let $\overline{x}$ and $\overline{y}$ denote the sets $\{x^{(1)}, \ldots, x^{(n)}\}$ and $\{y^{(1)}, \ldots, y^{(m)}\}$ respectively. Following the proof of Theorem 4.31, α is equivalent to

$$\widetilde{\forall} P \widetilde{\forall} Q \forall^{r'} \overline{x} \left[\bigwedge_{i \in S_1} ST_{x^{(i)}}(M^{(i)}) \rightarrow \phi'(\exists^{r'} \overline{y}\,) \right] \tag{5.13}$$

for some set $S_1 \subseteq \{0, \ldots, n\}$ and formulae $\{M^i\}_{i \in S_1}$, where each $M^{(i)}$ has one of the following forms, for $n_i \geq 0, s(i,j) \geq 1$:

$$p^{(i)} \wedge \Box^{s(i,1)} q_1^{(i)} \wedge \ldots \wedge \Box^{s(i,n_i)} q_{n_i}^{(i)} \qquad \Box^{s(i,1)} q_1^{(i)} \wedge \ldots \wedge \Box^{s(i,n_i)} q_{n_i}^{(i)}$$

and ϕ' is obtained from ϕ by replacing (for $x^{(i)} \in \overline{x}$ and $u \in \overline{x} \cup \overline{y}$)

$$\begin{aligned} &u = u \text{ with } ST_u(\top) \\ &u = x^{(i)} \text{ with } ST_u(p_i) \\ &\mathcal{R}^l u x^{(i)} \text{ with } ST_u(\Diamond^l p_i) \end{aligned}$$

and each occurrence (indexed by j) of $\mathcal{R}^{s(i,j)} x^{(i)} u$ ($s_{i,j} \geq 1$) in ϕ with $ST_u(q_j^{(i)})$. Furthermore, inspection of the proof of Theorem 4.31 reveals that

$$Var(M^{(i)}) \cap Var(M^{(j)}) = \emptyset \text{ for } i \neq j$$

and every formula in ϕ' occurs in $\bigcup_{i \in S_1} Var(M^{(i)})$.

Consider $\phi'(\exists^{r'} \overline{y}\,)$. By removing the existential quantifiers in favour of $\Diamond$-occurrences and collecting $ST_{x^{(i)}}$ terms together for each $x^{(i)}$ we obtain a disjunction of formulae of the form

$$ST_{x^{(i)}} \underbrace{\left(\bigvee_j \Diamond \ldots \Diamond (\Diamond \ldots \Diamond pv \wedge \Diamond \ldots \Diamond (\Diamond \ldots \Diamond pv \wedge \ldots)) \right)}_{\psi^{(i)}}$$

for i in some set $S_2 \subseteq \{0, \ldots, n\}$, where pv can be any propositional variable occurring in (5.13) or $\top$. Extend $\psi^{(i)}$ to $\{0, \ldots, n\}$ by setting $\psi^{(i)} = \bot$ for

$i \notin S_2$. Also, extend $M^{(i)}$ from S_1 to $\{0, \ldots, n\}$ by setting $M^{(i)} = \top$ for $i \notin S_1$. Then (5.13) is equivalent to

$$\widetilde{\forall} P \widetilde{\forall} Q \forall^{r'} \overline{x} \left[\bigwedge_{0 \leq i \leq n} ST_{x^{(i)}}(M^{(i)}) \rightarrow \bigvee_{0 \leq i \leq n} ST_{x^{(i)}}(\psi^{(i)}) \right]$$

This is equivalent to

$$\widetilde{\forall} P \widetilde{\forall} Q \forall^{r'} \overline{x} \left[\bigvee_{0 \leq i \leq n} ST_{x^{(i)}}(\neg M^{(i)} \vee \psi^{(i)}) \right]$$

and thus

$$\widetilde{\forall} P \widetilde{\forall} Q \neg \exists^{r'} \overline{x} \left[\bigwedge_{0 \leq i \leq n} ST_{x^{(i)}}(M^{(i)} \wedge \neg \psi^{(i)}) \right]$$

We can now eliminate the restricted quantifiers $\exists^{r'} \overline{x}$ to obtain a formula of the following form, where $(\Pi_1, \ldots, \Pi_n)$ denotes an ordering of the formulae $\{M^{(i)} \wedge \neg\phi^{(i)}\}$ $(1 \leq i \leq n)$, and $t_i \geq 0$.

$$\widetilde{\forall} P \widetilde{\forall} Q \neg ST_{x^{(0)}} \left[(M^{(0)} \wedge \neg\phi^{(0)}) \wedge \left(\bigwedge \underbrace{\Diamond \ldots \Diamond}_{t_1} (\underbrace{\Diamond \ldots \Diamond}_{t_2} \Pi_1 \wedge \underbrace{\Diamond \ldots \Diamond}_{t_3} (\underbrace{\Diamond \ldots \Diamond}_{t_4} \Pi_2 \wedge \ldots)) \right) \right]$$

This is the local second-order frame correspondent of a formula of the form

$$\left[(M^{(0)} \wedge \neg\phi^{(0)}) \wedge \left(\bigwedge \Diamond^{t_1} (\Diamond^{t_2} \Pi_1 \wedge \Diamond^{t_3} (\Diamond^{t_4} \Pi_2 \wedge \ldots)) \right) \right] \supset \bot$$

which is equivalent to

$$\left[(M^{(0)} \wedge \neg\phi^{(0)}) \wedge \left(\bigwedge \neg \Box^{t_1} (\Box^{t_2} \neg \Pi_1 \vee \Box^{t_3} (\Box^{t_4} \neg \Pi_2 \vee \neg \ldots)) \right) \right] \supset \bot$$

Since $\neg\Pi_k$ is a formula of the form $(M^{(i)} \supset \phi^{(i)})$ for some i, this is equivalent to

$$(M^{(0)} \supset \phi^{(0)}) \vee \left(\bigvee \Box^{t_1} [\Box^{t_2} (M^{(i)} \supset \phi^{(i)}) \vee \Box^{t_3} [\Box^{t_4} (M^{(j)} \supset \phi^{(j)}) \vee \neg \ldots]] \right)$$

This is the general form for a modal formula corresponding to a formula in $\mathbf{A^{r'} f \exists^{r'} x}$. Notice that this is precisely the form of an M-formula, as defined at the beginning of this section. Q.E.D.

Theorem 5.20 $\mathcal{MFORM} \subseteq \mathcal{D}_{modal}$.

Proof. Suppose that $K \oplus \{M_i\} \in \mathcal{MFORM}$ for some set $\{M_i\}$ of M-formulae. By Lemma 5.19 each M_i corresponds to a formula from $\mathbf{A^{r'} f \exists^{r'} x}$. So there exists a finite set $\{\alpha_i\}_{i \in J} \subset \mathbf{A^{r'} f \exists^{r'} x}$ such that $\alpha = \bigwedge_i \alpha_i$ corresponds to $\{M_i\}$. Since

$\mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x} \subset \mathbf{A}^{\mathbf{r}}\mathbf{f}\exists^{\mathbf{r}}\mathbf{x}$ it follows that α corresponds to a set $\{t_i\}_{i\in T}$ of primitive tense formulae. By Display Theorem I, there is a set $\{\rho_i\}_{i\in I}$ of structural rules such that $DLM + \{\rho_i\}_{i\in I}$ properly displays $Kt \oplus \{t_i\}_{i\in T}$. Now, because every M-formula is a Sahlqvist formula, by the Sahlqvist completeness theorem, $A \in K \oplus \{M_i\}$ iff $\mathcal{F}_{\{M_i\}} \models A$. This occurs iff $\mathcal{F}_{\{t_i\}_{i\in T}} \models A$ iff $A \in Kt \oplus \{t_i\}$ iff $\mathbf{I} \vdash A$ is derivable in $DLM + \{\rho_i\}_{i\in I}$. Because A is a modal formula, by the cut-elimination result we have $\mathbf{I} \vdash A$ is derivable in $DLM.K + \{\rho_i\}_{i\in I}$. It follows that $DLM.K + \{\rho_i\}_{i\in I}$ properly displays $K \oplus \{M_i\}$. Since the set $\{M_i\}$ was arbitrary, $\mathcal{MFORM} \subseteq \mathcal{D}_{modal}$. Q.E.D.

Example 5.21 *We saw in Section 5.2.2 that $K \oplus \Diamond\Box p \supset \Box\Diamond p$ can be properly displayed over $DLM.K$. Observe that $\Diamond\Box p \supset \Box\Diamond p$ can be written as the following M-formula:*

$$\begin{aligned}\Diamond\Box p \supset \Box\Diamond p &\approx \neg\Diamond\Box p \vee \Box\Diamond p\\ &\approx \Box\neg\Box p \vee \Box\Diamond p\\ &\approx \Box(\Box p \supset \bot) \vee \Box(\top \supset \Diamond p)\end{aligned}$$

Thus $K \oplus \Diamond\Box p \supset \Box\Diamond p \in \mathcal{MFORM}$.

It is straightforward to show that every primitive modal axiomatic extension over K is in $\mathcal{MFORM}$.

Corollary 5.22 *If $\{m_i\}_{i\in M}$ is a set of primitive modal formulae, then $K \oplus \{m_i\}_{i\in M} \in \mathcal{MFORM}$.*

Proof. It is easy to check that each primitive modal formula m_i corresponds to a formula $\alpha_i \in \mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$ (consider the algorithm in Section 4.2.3). Thus $\mathcal{F}_{\{m_i\}_{i\in M}} = \mathcal{F}_{\forall x\wedge\alpha_i}$. Since $\mathcal{F}_{\forall x\wedge\alpha_i}$ is definable by some set $\{M_i\}$ of M-formulae (Lemma 5.19). From the Sahlqvist completeness theorem, we have

$$A \in K \oplus \{M_i\} \text{ iff } \mathcal{F}_{\{M_i\}} \models A \text{ iff } \mathcal{F}_{\{m_i\}_{i\in M}} \models A \text{ iff } A \in K \oplus \{m_i\}_{i\in M}$$

Thus $K \oplus \{m_i\}_{i\in M} = K \oplus \{M_i\} \in \mathcal{MFORM}$. Q.E.D.

In Appendix A we present a direct method for computing the M-formulae corresponding to a given primitive modal formula.

5.2.5 Towards a complete characterisation

In Lemma 5.19 we saw that every M-formula globally corresponds to some formula $\forall x\alpha$ ($\alpha \in \mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$) and $\forall x\alpha$ ($\alpha \in \mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$) globally corresponds to some M-formula. Thus, in order to properly display an axiomatic extension over K by M-formulae $\{M_i\}$, it suffices to compute the first-order correspondents $\{\alpha_i\}_{i\in J}$ and from these compute the primitive tense correspondents. The required structural rules can be then obtained from the primitive tense correspondents. (We are implicitly making use of the Sahlqvist completeness theorem and the fact that every M-formula is a Sahlqvist formula).

Although every axiomatic extension over K by M-formulae is properly displayable, we do not know if there are other properly displayable axiomatic extensions over K. To shed light on this question we take the following approach:

We will look at the form of the structural rules corresponding to M-formulae (call these rules *basic rules*). It then remains to investigate if rules outside this form enable us to properly display more logics. We leave this investigation for future work. However we will present some justifications for the restrictions that the basic rules impose on a general structural rule.

Computing the basic rules

One method for computing the basic rules would be to use the general form of a formula $\alpha \in \mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$ and compute the corresponding primitive tense formula and then the display rule. However, with the objective of obtaining a structural rule of a certain form, we will first transform α into a formula $\alpha' \in \mathbf{A}^{\mathbf{r}}\mathbf{f}\exists^{\mathbf{r}}\mathbf{x}$ satisfying certain syntactic properties, such that for any frame F, $F \models \forall x_0\alpha$ iff $F \models \forall x_0\alpha'$. Then we will compute the display rule corresponding to $\forall x_0\alpha'$ to obtain the basic rule. We will use a running example to illustrate the ideas.

Let $\alpha = \forall^{r'}x_1 \ldots \forall^{r'}x_n\,\phi$ be an arbitrary formula in $\mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$. Using the algorithm in Section 4.2.3 — inserting $\blacklozenge$ as described in Remark 4.30 — we can show (the proof is analogous to Lemma 5.5) that the correspondent of α is the formula

$$f(p_0 \wedge \neg D_0, \ldots, p_n \wedge \neg D_n) \supset \bot \tag{5.14}$$

where

(i) f is a $\Diamond\wedge$-formula,

(ii) the propositional variables $\{p_i\}_{i\in P}$ are distinct

(iii) each D_i is either $\bot$ or constructed from variables in $\{p_i\}$ and $\top$ using $\Diamond$, $\blacklozenge$, $\wedge$ and $\vee$ such that an occurrence of $\blacklozenge$ can only bind other $\blacklozenge$ occurrences and propositional variables. This last condition follows from the fact that the algorithm introduces occurrences of $\blacklozenge$ precisely to replace terms in α of the form $\mathcal{R}^s xu$ (u is inherently universal) with $\blacklozenge^s p_x$.

Condition (iii) means that $\blacklozenge$ only occurs in the context $\blacklozenge \ldots \blacklozenge p_i$ (so $\blacklozenge(p_i \vee p_j)$ is impossible, for example). A formula of this form is called a *blue formula*.

Example 5.23 *Let α be the formula*

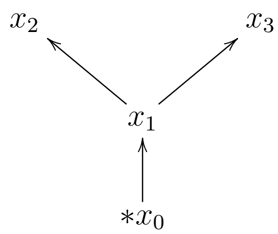
$$(\forall x_1 \rhd x_0)(\forall x_2 \rhd x_1)(\forall x_3 \rhd x_1)[(\exists x_4 \rhd x_3)(x_2 = x_4)\vee (\exists x_4 \rhd x_3)Rx_2x_4 \vee (\exists x_4 \rhd x_3)Rx_4x_2] \quad (5.15)$$

Using the algorithm in Section 4.2.3 we compute the correspondent blue formula:

$$p_0 \wedge \Diamond\,(p_1 \wedge \Diamond(p_2 \wedge \neg\bot) \wedge \Diamond(p_3 \wedge \neg(\Diamond p_2 \vee \Diamond\blacklozenge p_2 \vee \Diamond\Diamond p_2))) \supset \bot \quad (5.16)$$

The formula $\forall x_0 \alpha$ can be interpreted graphically as follows. Let G_α be the directed graph whose nodes consist of every universally quantified variable in $\forall x_0 \alpha$ (ie. the variables $\{x_0, \ldots, x_n\}$), and $v \to u$ iff $(\forall u \rhd v)$ appears in α. The node x_0 is called the *reference point*. Notice that G_α is a tree with root x_0.

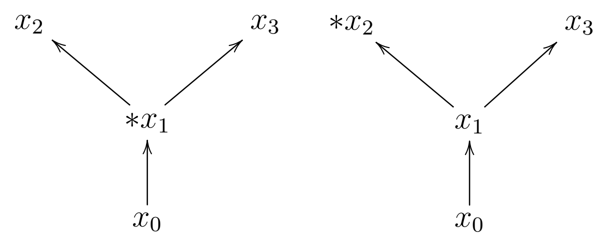
Example 5.24 (cont.) *The graph for the formula $\forall x_0 \alpha$ where α is the formula (5.15):*



Note that the graph contains nodes for universally quantified variables only (so there is no node for x_4). We have marked the reference point with $*$.

Our first observation is that we can shift the reference point to any other node to obtain a frame-equivalent formula $\forall x \alpha'$ where α' is in $\mathbf{A^r f \exists^r x}$ under suitable variable renaming. To see this, note that G_α is a connected graph and hence any node is reachable from any other node using the backward and forward universal restricted quantifiers. Notice that $G_{\alpha'}$ — this graph is obtained analogously to before. In particular $v \to u$ iff either $(\forall u \rhd v)$ or $(\forall v \lhd u)$ appears in α' — is identical to G_α except the reference point need not be the same. In particular, $G_{\alpha'}$ is a tree because G_α is a tree.

Example 5.25 (cont.) *In the example above, the reference point for the graph was x_0. Let us illustrate shifting this reference point with two examples:*



In the graph above left, the reference point has been moved to x_1. The corresponding formula for this graph is

$$\forall x_1(\forall x_2 \rhd x_1)(\forall x_3 \rhd x_1)(\forall x_0 \lhd x_1)[(\exists x_4 \rhd x_3)(x_2 = x_4)\vee \\ (\exists x_4 \rhd x_3)Rx_2x_4 \vee (\exists x_4 \rhd x_3)Rx_4x_2] \quad (5.17)$$

In the graph above right, the reference point has been moved to x_2. The corresponding formula for this graph is

$$\forall x_2(\forall x_1 \lhd x_2)(\forall x_3 \rhd x_1)(\forall x_0 \lhd x_1)[(\exists x_4 \rhd x_3)(x_2 = x_4)\vee \\ (\exists x_4 \rhd x_3)Rx_2x_4 \vee (\exists x_4 \rhd x_3)Rx_4x_2]$$

As expected, aside from the placement of $*$, each of the above graphs is identical to the original graph,

Let u be an arbitrary node in G_α. Our second observation is that any point v on this graph is reachable from u by first taking some number of backward steps and then some number of forward steps. In particular, there is never a need to go forward before going back. To see the consequences of this observation, consider the tense correspondent of $\forall x\alpha'$:

$$f'(p_0 \wedge \neg D_1, \dots, p_n \wedge \neg D_n) \supset \bot \quad (5.18)$$

where the $\{p_i\}_{i\in P}$ and $\{D_i\}_{i\in\mathcal{D}}$ are identical to those appearing in (5.14). It must be the case that f' is a $\blacklozenge\Diamond\wedge$-formula such that there is *no* occurrence of $\blacklozenge$ inside the scope of $\Diamond$ (ie. no 'nesting' of $\blacklozenge$ inside $\Diamond$).

If a formula does *not* contain a subformula of the form $A \wedge B$ where both A and B contain $\blacklozenge$ we say that the formula *resists $\blacklozenge$-conjoining.* Our third observation is that f' resists $\blacklozenge$-conjoining. For, if it did not resist $\blacklozenge$-conjoining, this would imply the existence of two nodes in $G_{\alpha'}$ with no common ancestor Thus

$G_{\alpha'}$ would no longer be a tree — the root of a tree is a common ancestor for any pair of nodes — and this is impossible.

To summarise, we have shown that every blue formula is frame-equivalent to a formula of the form (5.18) where

(i) f' is a $\blacklozenge\Diamond\wedge$-formula

(ii) the propositional variables $\{p_i\}_{i\in P}$ are distinct

(iii) each D_i is either $\bot$ or constructed from variables in $\{p_i\}$ and $\top$ using $\Diamond$, $\blacklozenge$, $\wedge$ and $\vee$ such that an occurrence of $\blacklozenge$ can only bind other $\blacklozenge$ occurrences and propositional variables.

(iv) f' contains no nesting of $\blacklozenge$ inside $\Diamond$ and resists $\blacklozenge$-conjoining.

Call this formula a *green formula.*

In fact, it is also the case that every green formula is frame-equivalent to some blue formula. The reason for this is that due to the nesting and conjoining restrictions, the graph of the first-order formula corresponding to (5.18) is a tree. The result then follows by shifting the reference point to the root of this tree thus eliminating the need for backward universal restricted quantifiers.

Example 5.26 (cont.) *Consider the blue formula (5.17). The frame-equivalent green formula is*

$$((\blacklozenge p_0 \wedge \neg\bot) \wedge (p_1 \wedge \neg\bot) \wedge \Diamond(p_2 \wedge \neg\bot) \wedge \Diamond(p_3 \wedge \neg(\Diamond p_2 \vee \blacklozenge p_2 \vee \Diamond\Diamond p_2))) \supset \bot \tag{5.19}$$

We can represent the above using the $\blacklozenge\Diamond\wedge$-formula

$$f'(X_1, X_2, X_3, X_4) = \blacklozenge X_0 \wedge X_1 \wedge \Diamond X_2 \wedge \Diamond X_3 \tag{5.20}$$

as $f'(p_0 \wedge \neg\bot, p_1 \wedge \neg\bot, p_2 \wedge \neg\bot, p_3 \wedge \neg(\Diamond p_2 \vee \Diamond\blacklozenge p_2 \vee \Diamond\Diamond p_2)$.

Although a green formula is not a primitive tense formula due to the presence of negation symbols, using Lemma 5.7, the green formula (5.18) is frame-equivalent to the following primitive tense formula:

$$f'(p_0, \ldots, p_n) \supset \overset{\vee}{f}(p_0 \wedge D_1, \ldots, p_n \wedge D_n)$$

Example 5.27 (cont.) *From Lemma 5.7, (5.17) is frame-equivalent to*

$$\blacklozenge p_0 \wedge p_1 \wedge \Diamond p_2 \wedge \Diamond p_3 \supset [\blacklozenge(p_0 \wedge \bot) \vee (p_1 \wedge \bot) \vee \Diamond(p_2 \wedge \bot) \vee \\ \Diamond(p_3 \wedge (\Diamond p_2 \vee \Diamond\blacklozenge p_2 \vee \Diamond\Diamond p_2))]$$

For any X_i occurring in f', let the *address* $\mathcal{A}(X_i)$ of X_i in f' be the ordered string constructed from $\Diamond$ and $\blacklozenge$ whose scope contains X_i. It follows that we can push the disjunction symbols outwards in the succedent of the above formula to obtain an equivalent formula of the form

$$f(p_i) \supset \bigvee_{i=0}^{n} \left(\bigvee_{j=1}^{m_i} \mathcal{A}(X_i)(p_i \wedge D_{ij}) \right)$$

where $D_i = \vee_{j=1}^{m_i} D_{ij}$ for each i.

Example 5.28 (cont.) *From (5.20) we compute* $\mathcal{A}(X_0) = \blacklozenge$, $\mathcal{A}(X_1) = \epsilon$ *(empty string),* $\mathcal{A}(X_2) = \Diamond$ *and* $\mathcal{A}(X_3) = \Diamond$. *We obtain the equivalent formula*

$$\begin{aligned}\blacklozenge p_0 \wedge p_1 \wedge \Diamond p_2 \wedge \Diamond p_3 \supset \blacklozenge(p_0 \wedge \bot) \vee (p_1 \wedge \bot) \vee \Diamond(p_2 \wedge \bot)\vee \\ \Diamond(p_3 \wedge \Diamond p_2) \vee \Diamond(p_3 \wedge \Diamond\blacklozenge p_2) \vee \Diamond(p_3 \wedge \Diamond\Diamond p_2)\end{aligned} \tag{5.21}$$

By a monotonicity argument, this formula is frame-equivalent to

$$\Diamond\top \wedge \Diamond p_2 \wedge \Diamond p_3 \supset (\Diamond(p_3 \wedge (\Diamond p_2 \vee \Diamond\blacklozenge p_2 \vee \Diamond\Diamond p_2))) \tag{5.22}$$

Using the function σ defined in Secion 5.1.1 we obtain the corresponding rule for some fresh schematic structure variable Y:

$$\frac{\sigma(\mathcal{A}(p_1)(p_1 \wedge D_{11})) \vdash Y \ldots \sigma(\mathcal{A}(p_n)(p_n \wedge D_{nm_n})) \vdash Y}{\sigma(f(p_i)) \vdash Y}$$

It is easy to verify that this rule is a structural rule obeying (C1)–(C8).

Example 5.29 (cont.) *Finally, let us compute the rule corresponding to (5.22):*

$$\frac{*\bullet*(X_3 \circ *\bullet*X_2) \vdash Y \qquad *\bullet*(X_3 \circ *\bullet*\bullet X_2) \vdash Y \qquad *\bullet*(X_3 \circ (*\bullet*)(*\bullet*) \circ X_2) \vdash Y}{\bullet\mathbf{I} \circ *\bullet*X_2 \circ *\bullet*X_3 \vdash Y}$$

Abstracting from the above, we define a *basic rule* to be a structural rule satisfying the following additional conditions:

(i) the structural connective $*$ does not occur in any sequent.

(ii) any schematic structure variable appears at most once in the conclusion sequent.

(iii) A schematic structure variable appears in the succedent of every sequent in the rule. Moreover this variable does not appear in any antecedent.

(iv) the conclusion sequent antecedent is constructed from the schematic structure variables $\{X_i\}$ and $\mathbf{I}$ using $*\bullet*$, $\bullet$ and $\circ$, and each schematic structure variable occurs exactly once. Let $\mathcal{A}(X_i)$ denote the address of X_i in the conclusion sequent antecedent (we are overloading $\mathcal{A}()$ to apply to either a $\blacklozenge\Diamond\wedge$-formula or a display structure but this should cause no confusion).

(v) the conclusion sequent antecedent does not contain an occurrence of $\bullet$ in the scope of $*\bullet*$, and resists $\bullet$-conjoining (ie there is no substructure of the form $U \circ V$ where both U and V contain occurrences of $\bullet$).

(vi) Each premise is of the form $\mathcal{A}(X_i)(X_i \circ L_i)$ where L_i is constructed from $\{X_i\}$ and $\mathbf{I}$ using $\circ$ and the 'building blocks' $*\bullet*$ and $\bullet$. Moreover, the building block $\bullet$ may only bind other occurrences of $\bullet$ and schematic structure variables (so an L_i of the form $\bullet(X_j \circ X_k)$ is prohibited, for example).

By inspection, every basic rule satisfies (C1)–(C8).

Non-basic structural rules

A structural rule that is not a basic rule is called a *non-basic* rule. Clearly a non-basic rule violates some of the conditions (i)–(vi) for basic rules.

In the remainder of this section we will look at non-basic rules satisfying (C1)–(C8). Of course, it is easy to construct a non-basic rule satisfying (C1)–(C8) that is equivalent to a basic rule. For example, consider the following non-basic rule (below left) and its equivalent basic rule (below right):

$$\frac{*X \vdash Y}{*X \vdash Y} \qquad\qquad \frac{X \vdash Y}{X \vdash Y}$$

The interesting case is when the non-basic rule cannot be rewritten as a basic rule. In particular, we investigate if such rules can be used to display additional logics (ie logics outside the axiomatic extensions of K via M-formulae).

Although we do not have a complete answer to this question, we present the following observations.

Without loss of generality we may assume that any structural rule satisfying (C1)–(C8) satisfies (i) and (ii).

Item (iii) is a sufficient condition to ensure that a rule extension can be expressed as an axiomatic extension. Consider the following rule, where the schematic structure variable Y does not appear in $\{L_i\}_{i\in\mathcal{L}}$ in the following:

$$\frac{L_1 \vdash Y \qquad \ldots \qquad L_n \vdash Y}{L_0 \vdash Y}\,\rho$$

Suppose that some structural rule extension $\mathcal{C}$ of DLM properly displays some extension L of Kt. By the definition of properly displays, $\mathcal{C} + \rho$ properly displays $L + r$ where r is the rule

$$\frac{\tau(L_1) \vdash q \qquad \ldots \qquad \tau(L_n) \vdash q}{\tau(L_0) \vdash q}\, r$$

and the propositional variable q does not appear in $\{\tau(L_i)\}_{i \in \mathcal{L}}$. Let t denote the formula $\tau(L_0) \supset \tau(L_1) \vee \ldots \vee \tau(L_n)$. Let us show that $L + r = L \oplus t$. Clearly the rule r is admissible in $L \oplus t$, so $L + r \subseteq L \oplus t$. Moreover, by substituting $\tau(L_1) \vee \ldots \vee \tau(L_n)$ for the variable q in the rule r we can obtain the formula t. Thus $L \oplus t \subseteq L + r$. We conclude that $L \oplus t = L + r$. Without item (iii) it is not obvious how to compute the formula corresponding a given rule. Indeed it may be the case that any properly displayable axiomatic extension over K can be constructed using only structural rules obeying (iii) but this requires further investigation.

The restriction in (iv) states that each schematic structure variable must appear exactly once. In (ii) we noted that assuming that each schematic structure variable occurs at most once involves no loss of generality. So the restriction is that each X_i *must* appear in the conclusion sequent. This is equivalent to stating that no schematic structure variable can appear in a premise sequent unless it also appears in the conclusion sequent. Suppose that ρ is such a structural rule. Let $A \supset B$ be the corresponding primitive tense formula computed according to the algorithm. It must be the case that B contains a propositional variable not appearing in A. By a monotonicity argument it can be shown that $A \supset B$ is frame-equivalent to a formula $A' \supset B'$ where every propositional variable in B' occurs in A'. Let ρ' be the corresponding structural rule. Clearly ρ' obeys (iv). Moreover, the set of derivable sequents is invariant under the substitution of ρ' for ρ. Thus, without loss of generality, we need only consider structural rule extensions of $DLM.K$ that obey (iv).

Regarding item (v), consider the rule

$$\frac{\bullet(X \circ Z) \vdash Y}{\bullet X \circ \bullet Z \vdash Y}$$

This rule contains $\bullet$-conjoining in the conclusion antecedent (it satisfies all the other conditions of basic rules). The corresponding primitive tense formula is $\blacklozenge p \wedge \blacklozenge q \supset \blacklozenge(p \wedge q)$. This formula globally corresponds to the formula $\alpha = \forall x(\forall u \lhd x)(\forall v \lhd x)(u = v)$ ('every state has at most one predecessor'). In fact, the class of frames defined by this formula is modally undefinable. To see this, consider the frames F_1 (below left) and F_2 (below right).

$$\begin{tikzcd} u_1 \arrow[r] & v_1 \\ u_2 \arrow[r] & v_2 \end{tikzcd} \qquad \begin{tikzcd} u'_1 \arrow[dr] & \\ & v' \\ u'_2 \arrow[ur] & \end{tikzcd}$$

Let f be the function taking $u_i \mapsto u'_i$ ($i \in \{1,2\}$) and $v_i \mapsto v'$. It is straightforward to check that f is a bounded morphism (see [7]) between F_1 and F_2 which means that if the class of frames defined by $\blacklozenge p \wedge \blacklozenge q \supset \blacklozenge(p \wedge q)$ is modally definable, then $F_1 \models \alpha$ implies $F_2 \models \alpha$. However, by inspection $F_1 \models \alpha$ and $F_2 \not\models \alpha$ so our claim follows.

Also consider the rule

$$\frac{Z \vdash Y}{(* \bullet *) \bullet X \circ Z \vdash Y}$$

This rule contains an occurrence of $\bullet$ inside the scope of $* \bullet *$ in the conclusion succedent, violating (v) (it satisfies all the other conditions of basic rules, although strictly speaking, the premise sequent should be written as the display equivalent $Z \circ \mathbf{I} \vdash Y$). The corresponding primitive tense formula is $\Diamond\blacklozenge p \wedge q \supset p \wedge q$ which is frame equivalent to $\Diamond\blacklozenge p \supset p$. This formula globally corresponds to the formula $\forall x(\forall u \rhd x)(\forall v \lhd u)(x = u)$ ('every state has at most one predecessor'). We have already seen that the class of frames with this property is modally undefinable.

Finally let us examine item (vi). First consider the rule

$$\frac{* \bullet * \bullet X \vdash Y}{\bullet * \bullet * X \vdash Y}$$

We have encountered this rule before, it corresponds to the primitive tense formula $\blacklozenge\Diamond p \supset \Diamond\blacklozenge p$ which is equivalent to $\Diamond\Box p \supset \Box\Diamond p$. Notice that this rule is *not* a basic rule. However, it is equivalent to the basic rule

$$\frac{Z \circ (* \bullet *) \bullet X \vdash Y}{Z \circ \bullet(* \bullet *)X \vdash Y}$$

In fact, this 'trick' can be employed to ensure that the real restriction in (vi) is that an occurrence of $\bullet$ in the premise may only bind other occurrences of $\bullet$ and schematic structure variables. Consider the following rule:

$$\frac{\bullet(* \bullet *)X \vdash Y}{X \vdash Y}$$

In the premise antecedent of this rule, the occurrence $\bullet$ binds the structure $*\bullet*X$, violating (vi) (the rule satisfies all the other conditions of basic rules). This rule corresponds to the primitive tense formula $p \supset \blacklozenge\Diamond p$. This formula globally corresponds to the formula $\forall x(\exists u \lhd x)(\exists v \rhd u)(x = v)$ ('every state has a predecessor'). Clearly this property holds for the following frame:

$$\begin{tikzcd} u \arrow[loop left] \arrow[r] & v \end{tikzcd}$$

If this class of frames was modally definable, then the property should hold for the subframe generated by state v (see [7]) — that is, the frame consisting of a single irreflexive state. Clearly the property does not hold for a single irreflexive state so we conclude that this class of frames is *not* modally definable.

These observations suggest (but do not prove) that non-basic rules can either be rewritten as basic rules, or correspond to primitive tense formulae that define a modally undefinable class of frames. We could confirm these observations if we could show that every class of modally definable frames defined by a formula in $\mathbf{A^{r}f\exists^{r}x}$ is also defined by a formula in $\mathbf{A^{r'}f\exists^{r'}x}$ — this is precisely the statement that Kracht [39] claims (see Step I in Section 5.2.3) although he had conceded that his proof is incomplete.[7] Hence we leave this statement as a *conjecture*. Notice that if the conjecture holds, then every set $\{\rho_i\}_{i\in I}$ of structural rules whose primitive tense correspondents define a modally definable class of frames could be rewritten as a set $\{\rho_i'\}$ of basic rules.

Since the M-formulae were computed as the frame-equivalent modal formulae for certain primitive tense formulae it is no surprise that the computation cannot be applied to primitive tense formulae defining modally undefinable classes of frames. Let $\{t_i\}_{i\in T}$ be a set of primitive tense formulae defining a modally undefinable class of frames. The question we ask here is: what can we say about the *modal restriction* of $Kt \oplus \{t_i\}_{i\in T}$? To take a specific case: we have already seen that $\Diamond\blacklozenge p \supset p$ defines a class of frames that is modally undefinable, so how can we describe the modal restriction of $Kt \oplus \Diamond\blacklozenge p \supset p$? If it is the case that the *modal restriction* of $Kt \oplus \{t_i\}_{i\in T}$ is *always* an axiomatic extension of K via M-formulae, that would mean that axiomatic extensions via M-formulae do present the 'full picture' after all, and we would have a complete characterisation. On the other hand, if it is the case that the *modal restriction* of $Kt \oplus \{t_i\}_{i\in T}$ is *always* sound and complete for some modally definable class of frames, then M-formulae present the full picture only if the conjecture in the previous paragraph holds. Investigating these issues is the topic of future work.

[7]Personal correspondence by email dated 13/Dec/2010.

Chapter 6

Displaying superintuitionistic logics

This chapter extends Goré [28] where it is shown that a display calculus for a superintuitionistic logic can be obtained from a display calculus for its modal companion. We exploit the relationship between $\mathbf{A}^{\mathrm{r}'}\mathbf{f}\exists^{\mathrm{r}'}\mathbf{x}$ and M-formulae and the fact that extensions over Kt by M-formulae are properly displayable to show how to construct display calculi for a large class of superintuitionistic logics specified using syntactic and semantic characterisations.

We begin by presenting syntactic and semantic specifications for propositional intuitionistic logic Ip. A (consistent) superintuitionistic logic is a consistent extension of Ip closed under modus ponens and uniform substitution. The Gödel translation induces a map between superintuitionistic logics and modal logics between $S4$ and $S5$. In Section 6.2 we obtain the display calculus $DLS4$ for modal logic $S4$ and then introduce the display calculus DLI. In Section 6.3 we show how the mapping between superintuitionistic logics and modal logics can be used to relate structural rule extensions of $DLS4$ and DLI. In particular, we show that DLI properly displays Ip. We can use these results to properly display an axiomatic extension of Ip whenever the Gödel translation of each new axiom is $S4$-equivalent to some M-formula (we introduced M-formulae in the previous chapter). Finally, in Section 6.5.1 we show how to display superintuitionistic logics characterised by classes of intuitionistic frames definable using formulae from $\mathbf{A}^{\mathrm{r}'}\mathbf{f}\exists^{\mathrm{r}'}\mathbf{x}$.

6.1 Introducing superintuitionistic logics

6.1.1 A Hilbert calculus for intuitionistic logic

Intuitionistic logic Ip can be defined (see [16]) in the language $\mathcal{L}$ using the following Hilbert calculus:

Axioms: All the axioms of classical propositional logic Cp (see Section 4.1) except (A10) $p \vee \neg p$.

Inference rules:

- *Modus ponens*: if $A \in Ip$ and $A \supset B \in Ip$ then $B \in Ip$
- *Uniform substitution* of arbitrary formulae for propositional variables in a formula

As with Cp, the symbols $\neg$ and $\top$ are redundant in the sense that $\neg A$ is equivalent to $A \supset \bot$ for any $A \in \mathbf{For}\mathcal{L}$, and $\top$ is equivalent to $\neg\bot$. However it is known [48] that none of the logical symbols "$\bot$", "$\vee$", "$\wedge$" and "$\supset$" is definable in Ip in terms of the other three symbols. Contrast this situation with Cp.

Notice that if we add the axiom (A10) $p \vee \neg p$ to the above calculus, we can obtain a calculus for Cp. That is, $Cp = Ip \oplus p \vee \neg p$.

6.1.2 Semantics for intuitionistic logic

We can also define intuitionistic logic Ip semantically as follows.

A binary relation R on a set W is called a *partial order* if the following conditions are satisfied for all $x, y, z \in W$:

$$Rxx \qquad (reflexivity)$$

$$Rxy \wedge Ryz \to Rxz \qquad (transitivity)$$

$$Rxy \wedge Ryx \to x = y \qquad (antisymmetry)$$

An *intuitionistic frame* is a pair $F = (W, R)$ where W is a non-empty set and R is a partial order on W. Compare this definition with Definition 4.1. Clearly, an intuitionistic frame is a frame where R is a partial order.

An *intuitionistic valuation* V in a frame (W, R) is a function assigning to each propositional variable $p \in \mathbf{Var}\mathcal{L}$ some (possibly empty) subset $V(p) \subseteq W$ such that for every $x \in V(p)$ and $y \in W$, Rxy implies that $y \in V(p)$. More generally, a set $S \subseteq W$ with the property that for all $x, y \in W$:

$$x \in S \text{ and } Rxy \text{ implies } y \in S$$

is called *upward closed*. Let UpW denote the upward closed subsets of W. Thus an intuitionistic valuation in (W, R) is a map V from $\mathbf{Var}\mathcal{L}$ into UpW.

An *intuitionistic model* (based on F) is a pair (F, V) where F is an intuitionistic frame and V is an intuitionistic valuation in F. Compare this with Definition 4.2. Clearly, an intuitionistic model is a special instance of a model where the valuations are restricted to intuitionistic valuations.

Let $M = ((W, R), V)$ be an intuitionistic model and $w \in W$. Define the *satisfaction* relation $M, w \models_i D$ by induction on the structure of the formula $D \in \mathbf{For}\mathcal{L}$ as follows:

$M, w \models_i \bot$ never

$M, w \models_i \top$ always

$M, w \models_i p$ iff $w \in V(p)$

$M, w \models_i \neg A$ iff for all $v \in W$, if Rwv then not $M, v \models_i A$

$M, w \models_i A \vee B$ iff $M, w \models_i A$ or $M, w \models_i B$

$M, w \models_i A \wedge B$ iff $M, w \models_i A$ and $M, w \models_i B$

$M, w \models_i A \supset B$ iff for all $v \in W$, if Rwv then $M, v \models_i A$ implies $M, v \models_i B$

As we would expect, $M, w \models A \supset \bot$ iff $M, w \models \neg A$. The negation of $M, w \models_i A$ is written $M, w \not\models_i A$.

Formula $A \in \mathbf{For}\mathcal{L}$ is valid at a state w in an intuitionistic frame F (notation: $F, w \models_i A$) if $M, w \models_i A$ for all intuitionistic models M based on F. A formula $A \in \mathbf{For}\mathcal{L}$ is valid in an intuitionistic frame F if it is valid at every state w in F (notation: $F \models_i A$). Observe that if $F \not\models_i A$, then there must be some model M based on F and state w such that $M, w \not\models_i A$. Finally, we say that A is valid on a class $\mathcal{F}$ of intuitionistic frames if $F \in \mathcal{F}$ implies $F \models_i A$.

Example 6.1 *Consider the intuitionistic frame F:*

$$\begin{tikzcd} u \arrow[loop left] \arrow[r] & v \arrow[loop right] \end{tikzcd}$$

Set $V(p) = \{v\}$ and let M be the model (F, V). So $M, u \not\models p$. Also, because $M, v \models p$ it follows that $M, u \not\models \neg p$. Therefore $M, u \not\models p \vee \neg p$ and thus $F \not\models p \vee \neg p$.

Suppose that $\alpha(x)$ is a formula from the basic first-order frame language $\mathcal{L}^f$ (see Definition 4.11) containing the free variable x. Recall that $\mathcal{L}^f$ is simply the first-order language containing the symbols $=$ and R. Let $F = (W, R)$ be an arbitrary intuitionistic frame. Interpreting the binary relation R in the frame F as the symbol R in $\mathcal{L}^f$ (technically speaking, we define a *morphism* from $\mathcal{L}^f$ into the structure $F = (W, R)$), we write $F \models \alpha[w/x]$ to mean that α is satisfied (in the usual sense of first-order classical logic) in the frame F when the free variable x in α is interpreted as the state $w \in W$. If $F \models \alpha[w/x]$ for all $w \in W$, then it follows that $F \models \forall x \alpha(x)$. Furthermore, for a formula α in $\mathcal{L}^f$ containing no free variables, observe that $F \models \neg\alpha$ iff $F \not\models \alpha$. If $\mathcal{F}$ is a class of intuitionistic frames then we write $\mathcal{F} \models \alpha$ to mean that $F \in \mathcal{F}$ implies $F \models \alpha$. Note that we have intentionally dropped the subscript "i", writing $\models$ instead of $\models_i$. This is because satisfiability of a first-order formula (unlike satisfiability of an intuitionistic formula) on an intuitionistic frame is defined identically to the general case.

The following semantic characterisation of Ip is a standard result.

Theorem 6.2 $Ip = \{A \in \mathbf{For}\mathcal{L} \mid F \models_i A \textit{ for every intuitionistic frame } F\}$

Proof. See [16]. Q.E.D.

A frame $F' = (W', R')$ is called a *subframe* of the frame $F = (W, R)$ if $W' \subseteq W$ and R' is the restriction of R to W'. The subframe F' is a *generated subframe* of F if W' is an upward closed subset of W. Also, if W' is the upward closure of some set $X \subseteq W$ (ie. W' is the minimal upward closed set containing X) then we say that W' and F' *are generated by the set* X. If F' is generated by a singleton $\{w\} \subseteq W$ we say that F' is *rooted* and call w the *root*.

The following result is also standard (see [16]).

Lemma 6.3 *For every intuitionistic frame F and every formula A, the following conditions are equivalent:*

(i) $F \models_i A$

(ii) $F' \models_i A$ *for every subframe* F' *of* F

(iii) $F' \models_i A$ *for every rooted subframe* F' *of* F

As an immediate corollary, we have

Corollary 6.4

$$Ip = \{A \in \mathbf{For}\mathcal{L} \mid F \models_i A \textit{ for every rooted intuitionistic frame } F\}$$

We will freely make use of the syntactic and semantic charaterisations of Ip as convenient.

6.1.3 Superintuitionistic logics and the Gödel translation

A logic L in the language $\mathcal{L}$ is called *consistent* if $L \neq \mathbf{For}\mathcal{L}$. It is easy to show that any logic containing the axiom (A9) is consistent iff $\bot \notin L$.

A *superintuitionistic logic* L is a set of formulae in the language $\mathcal{L}$ that is closed under modus ponens and uniform substitution. For every consistent superintuitionistic logic L, it is the case that $Ip \subseteq L \subseteq Cp$ (see [16]). From now on, we will use the term 'superintuitionistic logic' to mean a consistent superintuitionistic logic.

Recall that the logic $S4 = K \oplus \{4, T\}$ is the logic obtained from the axioms

$$(4)\ \Box p \supset p \qquad\qquad (T)\ \Box p \supset \Box\Box p$$

Define the Gödel translation [27, 16] $T : \mathbf{For}\mathcal{L} \mapsto \mathbf{For}\mathcal{ML}$:

$$\begin{aligned}
&T(\bot) = \bot\\
&T(\top) = \Box\top\\
&T(p) = \Box p\\
&T(A \wedge B) = T(A) \wedge T(B)\\
&T(A \vee B) = T(A) \vee T(B)\\
&T(A \supset B) = \Box(T(A) \supset T(B))\\
&T(\neg A) = \Box\neg T(A)
\end{aligned}$$

The Gödel translation induces the following embedding of superintuitionistic logics into modal logics.

Theorem 6.5 *For any set* $\{A_i\}_{i\in\mathcal{A}} \in \mathbf{For}\mathcal{L}$*,*

$$A \in Ip \oplus \{A_i\}_{i\in\mathcal{A}} \Leftrightarrow T(A) \in S4 \oplus \{T(A_i)\}_{i\in\mathcal{A}}$$

Proof. See Dummett and Lemmon [21]. Q.E.D.

A modal logic is called *normal* if it is closed under the Necessitation rule $A/\Box A$. A normal modal logic $M \supseteq S4$ is called a *modal companion* of a superintuitionistic logic L if for every intuitionistic formula A, it is the case that

$$A \in L \text{ iff } T(A) \in M \tag{6.1}$$

If M is a modal companion of L, then we say that L is the *superintuitionistic fragment* of M. For example, by Theorem 6.5, $S4 \oplus \{T(A_i)\}_{i\in\mathcal{A}}$ is a modal companion of $Ip \oplus \{A_i\}_{i\in\mathcal{A}}$, and $Ip \oplus \{A_i\}_{i\in\mathcal{A}}$ is the superintuitionistic fragment of $S4 \oplus \{T(A_i)\}_{i\in\mathcal{A}}$. The modal logic $S5$ is defined as $K \oplus \{T, 5\}$, where (5) is the axiom $\Diamond A \supset \Box\Diamond A$. It is easy to check that the modal companion of the logic $Ip \oplus (p \vee \neg p) = Cp$ is the logic $S5$.

We observe that $S4 \oplus \{T(A_i)\}_{i\in\mathcal{A}}$ is by no means the only modal companion of the logic $Ip \oplus \{A_i\}_{i\in\mathcal{A}}$. In fact, it is known that the set of modal companions of an arbitrary superintuitionistic logic $Ip \oplus \{A_i\}_{i\in\mathcal{A}}$ is infinite, with both a least element ($S4 \oplus \{T(A_i)\}_{i\in\mathcal{A}}$) and a greatest element (in terms of set inclusion). See [17] for a survey of results concerning embeddings of superintuitionistic logics into normal modal logics. In passing we note that if we lift the normality restriction in the definition of modal companion, there are even more modal logics into which superintuitionistic logics are embeddable [17, 81].

6.2 The calculus DLI

Extending Definition 4.55, we say that a display calculus $\mathcal{C}$ *properly displays* the logic L if (i) every rule of $\mathcal{C}$ satisfies the display conditions (C1)–(C8) (see Section 4.3.3), and (ii) for every formula A in the language of L, $A \in L$ iff $\mathbf{I} \vdash A$ is derivable in $\mathcal{C}$. If $\mathcal{C}$ properly displays L, then we say that L is properly displayable, and that $\mathcal{C}$ is a display calculus for L.

We can exploit this relationship between superintuitionistic logics and their modal companions to obtain display calculi for superintuitionistic logics. Let us begin by presenting a display calculus for $S4$. Notice that axiomatic extensions over Kt by (4) and (T) are equivalent, respectively, to axiomatic extensions over Kt using the following primitive modal axioms:

$$(4')\ \Diamond\Diamond p \supset \Diamond p \qquad\qquad (T')\ p \supset \Diamond p$$

Since $(4')$ and (T') are primitive modal axioms, following the procedure in Section 5.1 we can easily compute the structural rules

$$\frac{*\bullet * X \vdash Y}{(*\bullet *)(*\bullet *)X \vdash Y}\ d4 \qquad\qquad \frac{*\bullet * X \vdash Y}{X \vdash Y}\ dT$$

such that the calculus $DLS4 = DLM.K + d4 + dT$ properly displays $S4$.

Next we introduce the display calculus DLI consisting of:

$$\frac{\mathbf{I} \vdash X}{\top \vdash X}\ (\top \vdash) \qquad \frac{X \vdash \mathbf{I}}{X \vdash \bot}\ (\vdash \bot)$$

$$\frac{\bullet X \vdash *A}{X \vdash \neg A}\ (\vdash \neg) \qquad \frac{*A \vdash X}{\neg A \vdash \bullet X}\ (\neg \vdash)$$

$$\frac{X \vdash A \qquad Y \vdash B}{X \circ Y \vdash A \wedge B}\ (\vdash \wedge) \qquad \frac{A \circ B \vdash X}{A \wedge B \vdash X}\ (\wedge \vdash)$$

$$\frac{X \vdash A \circ B}{X \vdash A \vee B}\ (\vdash \vee) \qquad \frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash X \circ Y}\ (\vee \vdash)$$

$$\frac{\bullet X \circ A \vdash B}{X \vdash A \supset B}\ (\vdash \supset) \qquad \frac{X \vdash A \qquad B \vdash Y}{A \supset B \vdash \bullet(*X \circ Y)}\ (\supset \vdash)$$

Table 6.1: Logical rules for the display calculus DLI

(i) the initial sequents $p \vdash p$ for every propositional variable p, and the sequents $\mathbf{I} \vdash \top$ and $\bot \vdash \mathbf{I}$,

(ii) the display (structural) rules (Table 4.2) and the proper structural rules (Table 4.3),

(iii) the logical rules given in Table 6.1,

(iv) the structural rules $d4$ and dT,

(v) the cutrule $\dfrac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\ cut$

(vi) the ($Trivh$) rule — for any propositional variable p, $\dfrac{p \vdash Z}{\bullet p \vdash Z}\ Trivh$

Notice that the $Trivh$ rule schema (introduced in [28]) is neither a logical rule (because it does not introduce a logical connective or constant) nor a structural rule (since it contains a schematic propositional variable). It is straightforward to check that $Trivh$ satisfies all the display conditions.

We are particularly interested in derivations for sequents of the form $\mathbf{I} \vdash T(A)$.

Definition 6.6 (T-derivation) *Let $\{\rho_i\}_{i \in I}$ be any set of structural rules. A T-*derivation (in $DLS4 + \{\rho_i\}_{i \in I}$) *is a cutfree derivation in $DLS4 + \{\rho_i\}_{i \in I}$ whose endsequent has the form $\mathbf{I} \vdash T(A)$ where T is the Gödel translation and $A \in \mathbf{For}\mathcal{L}$.*

When using the above definition, we will drop the reference to the calculus ("in $DLS4 + \{\rho_i\}_{i \in I}$") when this can be inferred from the context.

Lemma 6.7 *Let $\{\rho_i\}_{i \in I}$ be any set of structural rules and suppose that δ is a T-derivation (in $DLS4 + \{\rho_i\}_{i \in I}$). Then the principal formula of each $\Box \vdash$ and $\vdash \Box$ rule occurrence in δ is either a propositional variable or $\top$, or a formula C where $C = A \supset B$ or $\neg A$ for some A, B. Furthermore δ contains no occurrences of the introduction rule for $\Diamond$.*

Proof. Because δ is cutfree and the endsequent has the form $\mathbf{I} \vdash T(A)$, the result follows immediately from the definition of the Gödel translation T. Q.E.D.

Recall that the rules in Table 4.2 are given the name 'display rules' because these are the basic structural rules of the display calculus that enable the display property (see discussion above Theorem 4.44). In a proof diagram, we will use the label 'd/r' to indicate some number of such display rule instances. In a derivation, if the rule instance ρ_2 appears below ρ_1 such that there are no rules between ρ_1 and ρ_2 except possibly display rules, then we say that ρ_1 is *followed immediately* by ρ_2. Also, in the proof of the following result we will make use of the notion of 'parametric ancestor'. See the discussion following Theorem 4.48 for a definition of this term.

Lemma 6.8 *Let $\{\rho_i\}_{i \in I}$ be any set of structural rules. Suppose that δ is a T-derivation in $DLS4 + \{\rho_i\}_{i \in I}$. Then there is a T-derivation δ^N of the identical sequent in $DLS4 + \{\rho_i\}_{i \in I}$ such that*

(i) each occurrence of $\supset\vdash$ and $\vdash\supset$ in δ^N is followed immediately by $\Box \vdash$ and $\vdash \Box$ respectively, and

(ii) each occurrence of $\neg \vdash$ and $\vdash \neg$ in δ^N is followed immediately by $\Box \vdash$ and $\vdash \Box$ respectively, and

(iii) each initial sequent $p \vdash p$ appears in the context

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{\Box p \vdash \bullet p}\,\Box\vdash}{(* \bullet *) * p \vdash * \Box p}\,d/r}{(* \bullet *)(* \bullet *) * p \vdash * \Box p}\,d4}{\Box p \vdash \bullet \bullet p}\,d/r}{\bullet \bullet \Box p \vdash p}\,d/r}{\bullet \Box p \vdash \Box p}\,\vdash\Box$$

Also the initial sequent $\mathbf{I} \vdash \top$ *occurs in the context*

$$\cfrac{\cfrac{\mathbf{I} \vdash \top}{\bullet\mathbf{I} \vdash \top}\ d/r}{\mathbf{I} \vdash \Box\top}\ \vdash \Box$$

Proof. Because δ is a T-derivation, every occurrence of $A \supset B$ and $\neg A$ in the endsequent must appear in the respective contexts $\Box(A \supset B)$ and $\Box\neg A$, and each propositional variable p and $\top$ must appear in the respective contexts $\Box p$ and $\Box\top$. Furthermore, Lemma 6.7 tells us that the principal formula of every $\Box \vdash$ and $\vdash \Box$ rule occurrence must be one of $A \supset B$, $\neg A$, or a propositional variable p or $\top$.

Call an occurrence ρ of the rule $\Box \vdash$ (resp. $\vdash \Box$ ρ) *normal* if it follows *immediately* after either $\supset\vdash$ or $\neg \vdash$ ($\vdash\supset$ or $\vdash \neg$) or occurs in the context shown in (iii). Otherwise such an occurrence is called *abnormal*. It suffices to show that δ can be transformed so that every occurrence of $\Box \vdash$ and $\vdash \Box$ is normal.

Proof by induction on the total number of abnormal rule occurrences in δ. We simply choose an abnormal rule occurrence ρ and convert it into a normal rule occurrence using the transformations below.

Case I. $\rho = \Box \vdash$ with principal formula C.

Suppose $C = A \supset B$. Trace the parametric ancestors of C above ρ in δ to identify the sequents where C is introduced:

$$\begin{array}{c}\cfrac{\{U_i \vdash A \qquad B \vdash V_i\}}{A \supset B \vdash *U \circ V_i}\ \supset\vdash \\ \cdots \\ \cfrac{A \supset B \vdash Z}{\Box(A \supset B) \vdash \bullet Z}\ \rho = \Box \vdash\end{array}$$

Replace with

$$\begin{array}{c}\cfrac{\cfrac{\cfrac{\cfrac{\{U_i \vdash A \qquad B \vdash V_i\}}{U_i \circ A \supset B \vdash V_i}\ \supset\vdash}{A \supset B \vdash *U_i \circ V_i}\ d/r}{\Box(A \supset B) \vdash \bullet(*U_i \circ V_i)}\ \Box\vdash}{\bullet\Box(A \supset B) \vdash *U_i \circ V_i}\ d/r \\ \cdots \\ \cfrac{\bullet\Box(A \supset B) \vdash Z}{\Box(A \supset B) \vdash \bullet Z}\end{array}$$

Observe that in the above derivation, moving down from the sequents $\{\bullet\Box(A \supset B) \vdash *U_i \circ V_i\}$ to the sequent $\bullet\Box(A \supset B) \vdash Z$ is legal. This is because in the original derivation, no parametric ancestor of $A \supset B$ is a principal formula in the portion [original diagram] between the sequents $\{A \supset B \vdash *U_i \circ V_i\}$ and $A \supset B \vdash Z$.

When $C = \neg A$ the transformations are similar.

Case II. $\rho = \vdash \Box$ with principal formula C.

Suppose $C = A \supset B$. Trace the parametric ancestors of C above ρ in δ to identify the sequents where C is introduced:

$$\begin{array}{c} \dfrac{\{U_i \circ A \vdash B\}}{U_i \vdash A \supset B}\vdash\supset \\ \cdots \\ \eta \\ \cdots \\ \dfrac{\bullet Z \vdash A \supset B}{Z \vdash \Box(A \supset B)}\vdash\Box \end{array}$$

It would be tempting to treat this subcase in the same manner we dealt with the corresponding case for $\Box\vdash$ case, that is, by replacing the above derivation with

$$\begin{array}{c} \dfrac{\dfrac{\{U_i \circ A \vdash B\}}{U_i \vdash A \supset B}\vdash\supset}{U_i' \vdash \Box(A \supset B}\vdash\Box \\ \cdots \\ \eta \\ \cdots \\ Z \vdash \Box A \supset B \end{array}$$

where $\bullet U_i' = U_i$. However there is no guarantee that this transformation is legal as it relies on U_i containing a leading $\bullet$. Furthermore, the first structural rule in η (should one exist) may no longer have a legal premise in the transformed derivation because an occurrence of $\bullet$ has been 'stolen' by the $\vdash\Box$ rule. Solution: instead of taking the $\vdash\Box$ rule upwards, bring the $\vdash\supset$ rule down as follows:

$$\begin{array}{c} \dfrac{\{U_i \circ A \vdash B\}}{U_i \vdash *A \circ B} \\ \cdots \\ \eta \\ \cdots \\ \dfrac{\dfrac{\dfrac{\bullet Z \vdash *A \circ B}{\bullet Z \circ A \vdash B}d/r}{\bullet Z \vdash A \supset B}\vdash\supset}{Z \vdash \Box(A \supset B}\vdash\Box \end{array}$$

When $C = \neg A$ the transformation is similar.

It remains to deal with the case when the principal formula of $\Box\vdash$ or $\vdash\Box$ is a propositional variable or $\top$. Because every propositional variable appears boxed in the endsequent, every initial sequent $p \vdash p$ in δ appears in one of the following contexts.

$$
\begin{array}{c}
p \vdash p \\ \vdots \\ \eta_1 \\ \vdots \\
\dfrac{Y[p] \vdash \bullet p}{Y[p] \vdash \Box p}\ \vdash \Box \\
\vdots \\ \eta_2 \\ \vdots \\
\dfrac{p \vdash Z[\Box p]}{\Box p \vdash \bullet Z[\Box p]}\ \Box \vdash
\end{array}
\qquad\qquad
\begin{array}{c}
p \vdash p \\ \vdots \\ \eta_3 \\ \vdots \\
\dfrac{p \vdash Y[p]}{\Box p \vdash \bullet Y[p]}\ \Box \vdash \\
\vdots \\ \eta_4 \\ \vdots \\
\dfrac{Z[\Box p] \vdash \bullet p}{Z[\Box p] \vdash \Box p}\ \vdash \Box
\end{array}
$$

The transformations for left above and right above are respectively,

$$
\begin{array}{c}
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{\Box p \vdash \bullet p}\ \Box\vdash}{(* \bullet *) * p \vdash * \Box p}\ d/r}{(* \bullet *)(* \bullet *) * p \vdash * \Box p}\ d4}{\bullet \bullet \Box p \vdash p}\ d/r}{\bullet \Box p \vdash \Box p}\ \vdash \Box \\
\vdots \\ \eta_1 \\ \vdots \\
\dfrac{\dfrac{\dfrac{Y[\bullet \Box p] \vdash \bullet \Box p}{(* \bullet *) * \Box p \vdash * Y[\bullet \Box p]}\ d/r}{* \Box p \vdash * Y[\bullet \Box p]}\ dT}{Y[\bullet \Box p] \vdash \Box p}\ d/r \\
\vdots \\ \eta_2 \\ \vdots \\
\dfrac{\bullet \Box p \vdash Z[\Box p]}{\Box p \vdash \bullet Z[\Box p]}\ d/r
\end{array}
\qquad
\begin{array}{c}
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{\Box p \vdash \bullet p}\ \Box\vdash}{(* \bullet *) * p \vdash * \Box p}\ d/r}{(* \bullet *)(* \bullet *) * p \vdash * \Box p}\ d4}{\bullet \bullet \Box p \vdash p}\ d/r}{\bullet \Box p \vdash \Box p}\ \vdash \Box \\
\vdots \\ \eta_3 \\ \vdots \\
\dfrac{\bullet \Box p \vdash Y[\Box p]}{\Box p \vdash \bullet Y[\Box p]}\ d/r \\
\vdots \\ \eta_4 \\ \vdots \\
\dfrac{Z[\Box p] \vdash \bullet \Box p}{Z[\Box p] \vdash \Box p}\ d/r, dT
\end{array}
$$

Similar arguments can be used to show that $\mathbf{I} \vdash \top$ appears as in the statement of the lemma.

It is easy to see that each of the transformations preserves the endsequent of δ. Also, each iteration of the transformation does not affect existing normal rules in the derivation or introduce any new abnormal rules. Thus the result follows from the induction hypothesis. Q.E.D.

6.3 Structural rule extensions of DLI and $DLS4$

Let $DLS4_{aux}$ be the calculus obtained from $DLS4$ by replacing the initial sequent schema $p \vdash p$ with $\bullet \Box p \vdash \Box p$, replacing $\mathbf{I} \vdash \top$ with $\mathbf{I} \vdash \Box \top$, and replacing the logical rule schemata in Table 4.1 with the rules in Table 6.2.

Define a *proper formula occurrence* A to be a formula occurrence in a sequent or derivation such that there is no formula occurrence properly containing the

$$\frac{\mathbf{I} \vdash X}{\top \vdash X}\ (\top \vdash) \qquad \frac{X \vdash \mathbf{I}}{X \vdash \bot}\ (\vdash \bot)$$

$$\frac{\bullet X \vdash *A}{X \vdash \Box\neg A}\ (\vdash \neg\Box) \qquad \frac{*A \vdash X}{\Box\neg A \vdash \bullet X}\ (\neg\Box \vdash)$$

$$\frac{X \vdash A \qquad Y \vdash B}{X \circ Y \vdash A \wedge B}\ (\vdash \wedge) \qquad \frac{A \circ B \vdash X}{A \wedge B \vdash X}\ (\wedge \vdash)$$

$$\frac{X \vdash A \circ B}{X \vdash A \vee B}\ (\vdash \vee) \qquad \frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash X \circ Y}\ (\vee \vdash)$$

$$\frac{\bullet X \circ A \vdash B}{X \vdash \Box(A \supset B)}\ (\vdash\supset\Box) \qquad \frac{X \vdash A \qquad B \vdash Y}{\Box(A \supset B) \vdash \bullet(*X \circ Y)}\ (\supset\Box\vdash)$$

Table 6.2: Logical rules for the display calculus $DLS4_{aux}$

occurrence A. Consider the sequent $\bullet\Box(A \supset B) \vdash *\bullet*(\Box(A \supset B) \wedge C)$. Then the proper formula occurrences in this sequent are $\Box(A \supset B)$ (from the antecedent) and $(\Box(A \supset B) \wedge C)$ (from the succedent). Notice that although $\Box(A \supset B)$ occurs as a subformula of the formula $(\Box(A \supset B) \wedge C)$, this does not undermine the status of the formula occurrence $\Box(A \supset B)$ as a proper formula occurrence in the antecedent. Informally, the proper formula occurrences are the 'largest' formulae occurring in a sequent or derivation.

Lemma 6.9 *Let $\{\rho_i\}_{i\in I}$ be any set of structural rules.*

(i) $\mathbf{I} \vdash T(A)$ *is derivable in* $DLS4_{aux} + \{\rho_i\}_{i\in I}$ *iff* $\mathbf{I} \vdash T(A)$ *is derivable in* $DLS4 + \{\rho_i\}_{i\in I}$.

(ii) For any derivation δ in $DLS4_{aux} + \{\rho_i\}_{i\in I}$, any proper formula occurrence in δ must be of the form $T(A)$ for some $A \in$ **For**$\mathcal{L}$.

Proof. Proof of (i). The left-to-right direction follows from the observation that every rule in $DLS4_{aux} + \{\rho_i\}_{i\in I}$ is some combination of rules in $DLS4 + \{\rho_i\}_{i\in I}$. In fact, we only need to check the logical rules of $DLS4_{aux} + \{\rho_i\}_{i\in I}$. For example, the $\vdash\supset\Box$ rule is obtained in $DLS4 + \{\rho_i\}_{i\in I}$ by first applying $\vdash\supset$ and then $\vdash\Box$.

For the right-to-left direction, suppose that $\mathbf{I} \vdash T(A)$ is derivable in $DLS4 + \{\rho_i\}_{i\in I}$. We can then obtain a derivation δ^N of $\mathbf{I} \vdash T(A)$ satisfying the conditions in Lemma 6.7 and Lemma 6.8. The translation from δ^N to a derivation δ' of $\mathbf{I} \vdash T(A)$ in $DLS4_{aux} + \{\rho_i\}_{i\in I}$ is straightforward. Replace each initial fragment in δ^N of the form

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{\Box p \vdash \bullet p}\,\Box\vdash}{(\ast \bullet \ast) \ast p \vdash \ast \Box p}\,d/r}{(\ast \bullet \ast)(\ast \bullet \ast) \ast p \vdash \ast \Box p}\,d4}{\bullet \bullet \Box p \vdash p}\,d/r}{\bullet \Box p \vdash \Box p}\,\vdash\Box$$

with the initial sequent $\bullet\Box p \vdash \Box p$. Replace each initial fragment

$$\dfrac{\dfrac{\mathbf{I} \vdash \top}{\bullet\mathbf{I} \vdash \top}\,d/r}{\mathbf{I} \vdash \Box\top}\,\vdash\Box$$

with the initial sequent $\mathbf{I} \vdash \Box\top$. Similarly, each rule pair is replaced with the combination rule in the obvious way. For example, the rule pair $\supset\vdash, \Box\vdash$ is replaced with the rule $\supset\Box\vdash$.

Proof of (ii). Inspection of the initial sequents and logical rules of $DLS4_{aux} + \{\rho_i\}_{i\in I}$ reveal that the only proper formula occurrences that may occur in a derivation have the form $T(A)$ for some $A \in \mathbf{For}\mathcal{L}$. Q.E.D.

Lemma 6.10 *Let $\{\rho_i\}_{i\in I}$ be any set of structural rules. For any $A \in \mathbf{For}\mathcal{L}$, if $\mathbf{I} \vdash A$ is derivable in $DLI + \{\rho_i\}_{i\in I}$, then $\mathbf{I} \vdash T(A)$ is derivable in $DLS4_{aux} + \{\rho_i\}_{i\in I}$.*

Proof. Without loss of generality, we may assume the property that in a derivation in $DLI + \{\rho_i\}_{i\in I}$, the initial sequent $p \vdash p$ is immediately followed by the $Trivh$ rule. In other words, the derivation commences

$$\dfrac{p \vdash p}{\bullet p \vdash p}\,Trivh$$

To see this, observe that the principal formula of $Trivh$ (by definition) must be a propositional variable. In particular, this means that for any propositional variable p that is principal by a $Trivh$ rule ρ, there can be no logical rules above ρ in the derivation that make p principal. Thus we can push the $Trivh$ rule upwards so that it appears immediately after the initial sequent. Moreover, should we require a sequent of the form $p \vdash p$, then we can write:

$$\dfrac{\dfrac{p \vdash p}{\bullet p \vdash p}\,Trivh}{p \vdash p}\,d/r, dT$$

We will prove that if δ is a derivation of $X \vdash Y$ in $DLI + \{\rho_i\}_{i\in I}$, then the sequent $X^T \vdash Y^T$ obtained from $X \vdash Y$ by substituting every proper formula occurrence A in X and Y with $T(A)$ is derivable in $DLS4_{aux} + \{\rho_i\}_{i\in I}$. The statement of the Lemma clearly follows from this result.

Without loss of generality, take δ to be cutfree. Induction on the height of δ. If the height is one, then δ must be either $\mathbf{I} \vdash \top$ or $\bot \vdash \mathbf{I}$. The required derivations are, respectively, $\mathbf{I} \vdash \Box\top$ and $\bot \vdash \mathbf{I}$. If δ has the form

$$\dfrac{p \vdash p}{\bullet p \vdash p}\ Trivh$$

then replace with the initial sequent $\bullet\Box p \vdash \Box p$.

Otherwise, consider the last rule ρ in δ (we will illustrate a few of the cases that arise, the remaining cases are similar).

If $\rho = \vee\vdash$ then δ has the form

$$\dfrac{\begin{matrix}\delta' \\ A \vdash X\end{matrix} \qquad \begin{matrix}\delta'' \\ B \vdash Y\end{matrix}}{A \vee B \vdash X \circ Y}$$

By the induction hypothesis the sequents $T(A) \vdash X^T$ and $T(B) \vdash Y^T$ are derivable in $DLS4_{aux} + \{\rho_i\}_{i\in I}$. Then we can write

$$\dfrac{T(A) \vdash X^T \qquad T(B) \vdash Y^T}{T(A) \vee T(B) \vdash X^T \circ Y^T}\ \vee\vdash$$

If $\rho = \vdash\supset$, then δ has the form

$$\dfrac{\begin{matrix}\delta' \\ \bullet X \circ A \vdash B\end{matrix}}{X \vdash A \supset B}$$

By the induction hypothesis, the sequent $\bullet X^T \circ T(A) \vdash T(B)$ is derivable in $DLS4_{aux} + \{\rho_i\}_{i\in I}$. Then we can write

$$\dfrac{\bullet X^T \circ T(A) \vdash T(B)}{X^T \vdash \Box(T(A) \supset T(B))}\ \vdash\supset\Box$$

If $\rho = \supset\vdash$, then δ has the form

$$\dfrac{\begin{matrix}\delta' \\ X \vdash A\end{matrix} \qquad \begin{matrix}\delta'' \\ B \vdash Y\end{matrix}}{A \supset B \vdash \bullet(*X \circ Y)}$$

By the induction hypothesis, the sequents $X^T \vdash T(A)$ and $T(B) \vdash Y^T$ are derivable in $DLS4_{aux} + \{\rho_i\}_{i\in I}$. Then we can write

$$\dfrac{X^T \vdash T(A) \qquad T(B) \vdash Y^T}{\Box(T(A) \supset T(B)) \vdash \bullet(*X^T \circ Y^T)}\ \supset\Box\vdash$$

If the last rule is a structural rule, we can simply apply the induction hypothesis to the premise sequent(s) and then apply ρ. Q.E.D.

Lemma 6.11 *Let $\{\rho_i\}_{i\in I}$ be any set of structural rules. For any $A \in \mathbf{For}\mathcal{L}$, if $\mathbf{I} \vdash T(A)$ is derivable in $DLS4_{aux} + \{\rho_i\}_{i\in I}$, then $\mathbf{I} \vdash A$ is derivable in $DLI + \{\rho_i\}_{i\in I}$.*

Proof. We will prove that if δ is a derivation of $X \vdash Y$ in $DLS4_{aux} + \{\rho_i\}_{i\in I}$, then the sequent $X^{T-} \vdash Y^{T-}$ obtained from $X \vdash Y$ by substituting every proper formula occurrence $T(B)$ in $X \vdash Y$ with B is derivable in $DLI + \{\rho_i\}_{i\in I}$. Notice that by Lemma 6.9(ii), it is the case that every proper formula occurrence in $X \vdash Y$ is a formula of the form $T(B)$ for some $B \in \mathbf{For}\mathcal{L}$. The statement of the Lemma clearly follows from this result.

Without loss of generality, take δ to be cutfree. Induction on the height of δ. If δ is $\bullet\Box p \vdash \Box p$, then write

$$\frac{p \vdash p}{\bullet p \vdash p}\,Trivh$$

If δ is $\mathbf{I} \vdash \Box\top$ or $\bot \vdash \mathbf{I}$, then write, respectively, $\mathbf{I} \vdash \top$ or $\bot \vdash \mathbf{I}$.

Otherwise, consider the last rule ρ in δ. Once again, we illustrate with a few of the cases that arise.

If $\rho = \vdash \wedge$, then δ has the form

$$\frac{\begin{array}{c}\delta' \\ X \vdash T(A)\end{array} \qquad \begin{array}{c}\delta'' \\ Y \vdash T(B)\end{array}}{X \circ Y \vdash T(A) \vdash (B)}$$

By the induction hypothesis, the sequents $X^{T-} \vdash A$ and $Y^{T-} \vdash B$ are derivable in $DLI + \{\rho_i\}_{i\in I}$. Then we can write

$$\frac{X^{T-} \vdash A \qquad Y^{T-} \vdash B}{X^{T-} \circ Y^{T-} \vdash A \wedge B}\vdash\wedge$$

If $\rho = \vdash\supset\Box$, then δ has the form

$$\frac{\begin{array}{c}\delta' \\ \bullet X \circ T(A) \vdash T(B)\end{array}}{X \vdash \Box(T(A) \supset T(B))}$$

By the induction hypothesis, the sequent $\bullet X^{T-} \circ A \vdash B$ is derivable in $DLI + \{\rho_i\}_{i\in I}$. Then we can write

$$\frac{\bullet X^{T-} \circ A \vdash B}{X^{T-} \vdash A \supset B}\vdash\supset$$

If $\rho = \supset\Box\vdash$, then δ has the form

$$\dfrac{\begin{array}{cc}\delta' & \delta'' \\ X \vdash T(A) & T(B) \vdash Y\end{array}}{\Box(T(A) \supset T(B)) \vdash \bullet(*X \circ Y)}$$

By the induction hypothesis, the sequents $X^{T-} \vdash A$ and $B \vdash Y^{T-}$ are derivable in $DLI + \{\rho_i\}_{i \in I}$. Then we can write

$$\dfrac{X^{T-} \vdash A \qquad B \vdash Y^{T-}}{A \supset B \vdash \bullet(*X^{T-} \circ Y^{T-})} \supset\Box\vdash$$

If the last rule is a structural rule, we can simply apply the induction hypothesis to the premise sequent(s) and then apply ρ. Q.E.D.

Corollary 6.12 *Let $\{\rho_i\}_{i \in I}$ be any set of structural rules. Then, for any $A \in$ **For**$\mathcal{L}$,*

$$\mathbf{I} \vdash_{DLI+\{\rho_i\}_{i \in I}} A \textit{ iff } \mathbf{I} \vdash_{DLS4+\{\rho_i\}_{i \in I}} T(A)$$

Proof. Suppose that $\mathbf{I} \vdash A$ is derivable in $DLI + \{\rho_i\}_{i \in I}$. From Lemma 6.10 we know that $\mathbf{I} \vdash T(A)$ is derivable in $DLS4_{aux} + \{\rho_i\}_{i \in I}$ and then from Lemma 6.9(i), $\mathbf{I} \vdash T(A)$ is derivable in $DLS4 + \{\rho_i\}_{i \in I}$.

Next, suppose that $\mathbf{I} \vdash T(A)$ is derivable in $DLS4 + \{\rho_i\}_{i \in I}$. From Lemma 6.9(i) we know that $\mathbf{I} \vdash T(A)$ is derivable in $DLS4_{aux} + \{\rho_i\}_{i \in I}$. Now the result follows from Lemma 6.11. Q.E.D.

The next lemma provides a method of generating display calculi over DLI to display a large class of superintuitionistic logics.

Lemma 6.13 *Let $\{\rho_i\}_{i \in I}$ be any set of structural display rules and suppose that $DLS4 + \{\rho_i\}_{i \in i}$ properly displays the logic M. If M is a modal companion of some superintuitionistic logic L, then $DLI + \{\rho_i\}_{i \in I}$ properly displays L.*

Proof. It suffices to show that for any $A \in$ **For**$\mathcal{L}$, $A \in L$ iff $\mathbf{I} \vdash A$ is derivable in $DLI + \{\rho_i\}_{i \in I}$.

First suppose that $\mathbf{I} \vdash A$ is derivable in $DLI + \{\rho_i\}_{i \in I}$. By Corollary 6.12 we know that $\mathbf{I} \vdash T(A)$ is derivable in $DLS4 + \{\rho_i\}_{i \in I}$ and thus $T(A) \in M$. Since M is a modal companion of L, we have $A \in L$.

Next suppose that $A \in L$. Since M is a modal companion of L we have $T(A) \in M$. Thus $\mathbf{I} \vdash T(A)$ is derivable in $DLS4 + \{\rho_i\}_{i \in I}$. From Corollary 6.12 we know that $\mathbf{I} \vdash A$ is derivable in $DLI + \{\rho_i\}_{i \in I}$ so we are done. Q.E.D.

Since $S4$ is a modal companion of Ip and $DLS4$ properly displays $S4$, the above Lemma yields

Corollary 6.14 *The display calculus DLI properly displays Ip.*

6.4 Displaying GD; recovering Cp

Gödel-Dummett Logic GD (also called LC) is obtained by the addition of the axiom gd: $(p \supset q) \vee (q \supset p)$ to Ip. From Theorem 6.5 we know that $S4 \oplus \Box(\Box p \supset \Box q) \vee \Box(\Box q \supset \Box p)$ is a modal companion. In the presence of the axioms (4) and (T) it is known (see [31]) that the following axioms are equivalent:

$$\Box(\Box p \supset \Box q) \vee \Box(\Box q \supset \Box p) \qquad\qquad \Box(\Box p \supset q) \vee \Box(\Box q \supset p)$$

The formula above right is an M-formula, so we know that $S4 \oplus \Box(\Box p \supset q) \vee \Box(\Box q \supset p)$ is properly displayable over $DLM.K$. To compute the corresponding display rule, we must first express the M-formula as a formula in $\mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$ and then compute the frame equivalent primitive tense formula using the tools of Sections 4.2.2 and 4.2.3. Then, using the procedure of Section 5.1 we obtain the display rule:

$$\frac{X \vdash Y \qquad \bullet X \vdash Y \qquad * \bullet * X \vdash Y}{\bullet * \bullet * X \vdash Y}\; dgd$$

such that $DLS4 + dgd$ properly displays $S4 \oplus T((p \supset q) \vee (q \supset p))$. Then from Lemma 6.13 we have that $DLI + dgd$ properly displays the superintuitionistic logic GD.

We have already noted that classical propositional logic Cp can be obtained by the addition of the axiom $p \vee \neg p$ to Ip. If we compute the Gödel translation of $p \vee \neg p$ we get

$$\begin{aligned} T(p \vee \neg p) &= \Box p \vee \Box\neg\Box p \\ &\approx (\top \supset \Box p) \vee \Box(\Box p \supset \bot) \end{aligned}$$

This is an M-formula so we can compute the corresponding structural rule as before. Alternatively, observe that this formula is equivalent in K to the (5) axiom $\Diamond p \supset \Box\Diamond p$. In the presence of axiom (T), the (5) axiom is equivalent to the primitive tense formula $p \wedge \Diamond q \supset \Diamond(q \wedge \Diamond p)$ (see [31]). It follows that

$$S4 \oplus p \wedge \Diamond q \supset \Diamond(q \wedge \Diamond p) = S4 \oplus \Diamond p \supset \Box\Diamond p = S5$$

From $p \wedge \Diamond q \supset \Diamond(q \wedge \Diamond p)$ we can compute the display rule

$$\frac{* \bullet *(M \circ * \bullet * L) \vdash Y}{L \circ * \bullet * M \vdash Y}\; dS5$$

such that $DLS4+dS5$ properly displays $S5$. So $DLI+dS5$ properly displays Cp.

Here is a derivation of the sequent $\mathbf{I} \vdash p \vee \neg p$ in $DLI + dS5$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{\bullet p \vdash p}\,Trivh}{p \circ * \bullet p \vdash \mathbf{I}}}{* \bullet *(p \circ * \bullet * * p) \vdash \mathbf{I}}\,d/r}{*p \circ * \bullet *p \vdash \mathbf{I}}\,dS5}{\bullet * p \vdash *p}}{*p \vdash \neg p}\,\neg\vdash}{\mathbf{I} \vdash p \circ \neg p}}{\mathbf{I} \vdash p \vee \neg p}\,\vdash\vee$$

6.5 Displaying superintuitionistic logics characterised by semantic conditions

In the previous section, our mode of operation was as follows: given the superintuitionistic logic $Ip \oplus \{A_i\}_{i\in\mathcal{A}}$, write $\{T(A_i\}_{i\in\mathcal{A}}$ as a set of M-formulae. Since axiomatisations by M-formulae are properly displayable over $DLS4$, we obtain a calculus $DLS4 + \{\rho_i\}_{i\in I}$ that properly displays $S4 \oplus \{T(A_i)\}_{i\in\mathcal{A}}$. From Lemma 6.13 it follows that $DLI + \{\rho_i\}_{i\in I}$ properly displays $Ip \oplus \{A_i\}_{i\in\mathcal{A}}$. However, it is not always clear how to (or possible to) transform the Gödel translation of a given axiom into M-formulae as we did in the previous section. When the superintuitionistic logic has a semantic characterisation, we can sometimes make use of that characterisation to display the logic.

Let $F = (W, R)$ be a transitive frame. Define the equivalence relation $\sim$ on W by taking, for all $x, y \in W$:

$$x \sim y \text{ iff either } x = y \text{ or } (Rxy \text{ and } Ryx)$$

The equivalence classes with respect to $\sim$ are called *clusters*. The cluster containing state x will be denoted $C(x)$.

Definition 6.15 (skeleton of a frame) *The* skeleton ρF *of a transitive frame* F *is the quotient frame with respect to* $\sim$. *That is,* $\rho F = (\rho W, \rho R)$ *where*

$$\rho W = \{C(x) \,|\, x \in W\} \qquad\qquad \rho R C(x) C(y) \textit{ iff } Rxy$$

Observe that ρF is antisymmetric for any transitive frame F. Furthermore, if $F = (W, R)$ is reflexive and transitive, then ρR is a partial order on ρF (ie. ρF is reflexive, transitive and antisymmetric). In this chapter, we are concerned mainly with $S4$-frames (ie. reflexive and transitive frames).

Lemma 6.16 *If $F = (W, R)$ is an antisymmetric $S4$-frame then $\rho F = F$.*

Proof. First notice that for states $x, y \in W$: $x \sim y$ iff $x = y$. The only case to check is when $x \sim y$ and Rxy and Ryx. Since F is antisymmetric, it follows that $x = y$ as required. Thus it follows that $\rho W = W$.

The next thing to check is that for $x, y \in W$: ρRxy iff Rxy. This follows immediately from the definition, so $\rho R = R$.

Since $\rho W = W$ and $\rho R = R$ we conclude that $\rho F = F$. Q.E.D.

Lemma 6.17 *For every reflexive and transitive frame F, and every intuitionistic formula A,*

$$\rho F \models_i A \text{ iff } F \models T(A)$$

Proof. Induction on the size of A. See [16] for details. Q.E.D.

In the following, $(\forall x \rhd w)B$ and $(\exists x \rhd w)B$ respectively stand for either

$$\forall x(Rwx \to B) \qquad\qquad \exists x(Rwx \wedge B)$$

or

$$\forall x(\rho Rwx \to B) \qquad\qquad \exists x(\rho Rwx \wedge B).$$

The context will make it clear if we mean the relation R or ρR.

Recall that we write $F \models \alpha$ for a $\mathcal{L}^f$ formula α to mean that the formula is satisfied on the frame F in the usual sense of first-order logic.

Lemma 6.18 *Let $\alpha(x)$ be an arbitrary formula in $\mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$. If $\forall x\alpha$ is satisfied on an $S4$-frame F, then $\forall x\alpha$ is satisfied on ρF.*

Proof. In order to simplify the notation, we will suppose that α is the modal Kracht formula $(\forall w \rhd x)\phi((\exists y \rhd w); x, w)$ (the generalisation to arbitrary $\alpha \in \mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$ is straightforward). Let F be an $S4$ frame such that $F \models (\forall x)(\forall w \rhd x)\phi((\exists y \rhd w); x, w)$. By definition of $\mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$, ϕ can always be written as a disjunction of conjunctions of atoms comprising of existential restricted quantifiers and formulae Ruv and $u = v$.

We aim to show that $\rho F \models (\forall x)(\forall w \rhd x)\phi((\exists y \rhd w); x, w)$. Argue by contradiction by assuming that (†): $(\forall x)(\forall w \rhd x)\phi((\exists y \rhd w); x, w)$ is not satisfied on ρF. Then $\rho F \models (\exists x)(\exists w \rhd x)\neg\phi((\exists y \rhd w); x, w)$. Thus there exist $C(l), C(m)$

satisfying $\rho RC(l)C(m)$ such that (in the following, variables x and w have been uniformly substituted with $C(l)$ and $C(m)$ respectively):

$$\rho F \models \neg\phi((\exists y \rhd C(m)); C(l), C(m)) \tag{6.2}$$

Consider the statement $F \models \neg\phi((\exists y \rhd m); l, m)$. Let (‡) denote the assumption that this statement does not hold. Then there is an instantiation $\phi'(l, m, z_i)$ with $\{z_i\}$ for y — for some of the existential restricted quantifiers occurrences in $\phi((\exists y \rhd m); l, m)$ — so that Rmz_i and $F \models \phi'(l, m, z_i)$. Now observe that

$F \models Rab$ iff $\rho F \models \rho RC(a)C(b)$

$F \models a = b$ implies $\rho F \models C(a) = C(b)$ (reverse direction need not hold)

Using the above, from $F \models \phi'(l, m, z_i)$ by induction on the size of ϕ we can show that $\rho F \models \phi'(C(l), C(m), C(z_i))$. Since Rmz_i, we have $\rho RC(m)C(z)$ and thus

$$\rho F \models \phi((\exists y \rhd C(m)); C(l), C(m))$$

This contradicts (6.2).

We have made two assumptions (†) or (‡). The contradiction we have obtained tells us that one of these assumptions is incorrect. If (†) is incorrect we are done. Instead, if (‡) is incorrect then it is indeed the case that $F \models \neg\phi((\exists y \rhd m); l, m)$. Since $\rho RC(l)C(m)$ we have Rlm, and thus $F \models (\exists l)(\exists m \rhd l)\neg\phi((\exists y \rhd m); l, m)$. However this contradicts the original premise we were given, that $F \models (\forall x)(\forall w \rhd x)\phi((\exists y \rhd w); x, w)$. The only possibility is that (†) must be incorrect. Q.E.D.

Lemma 6.19 *Let α be a conjunction $\sigma_1 \wedge \ldots \wedge \sigma_n$ of formulae from $\mathbf{A^{r'}f\exists^{r'}x}$. If $\forall x\alpha$ is satisfied on a frame F, then $\forall x\alpha$ is satisfied on ρF.*

Proof. Suppose that $F \models \forall x \bigwedge_i \sigma_i(x)$. Then for every state u in F, $F \models \sigma_i(u)$ and thus $F \models \forall x\sigma_i$ $(1 \leq i \leq n)$. From the previous lemma we know that $\rho F \models \forall x\sigma_i$ for each i. Then for every state u and index i we have $\rho F \models \sigma_i(u)$ and thus $\rho F \models \forall x \bigwedge_i \sigma_i(x)$. Q.E.D.

Lemma 6.20 *Suppose that a class $\mathcal{F}_i$ of intuitionistic frames and a class $\mathcal{F}^{S4}$ of S4-frames are both defined by $\forall x\alpha$, where α is a conjunction of formulae $\{\alpha_i\}_{i\in J} \subset \mathbf{A^{r'}f\exists^{r'}x}$. Then for any intuitionistic formula A*

$$\mathcal{F}_i \models_i A \textit{ iff } \mathcal{F}^{S4} \models T(A)$$

Proof. First suppose that $\mathcal{F}_i \not\models_i A$. Then there must be some intuitionistic frame $F \in \mathcal{F}_i$ such that $F \not\models_i A$. Since F is a reflexive, transitive and antisymmetric frame such that $F \models \forall x\alpha$, it follows that $F \in \mathcal{F}^{S4}$ and $\rho F = F$ (Lemma 6.16). Therefore $\rho F \not\models_i A$ and thus from Lemma 6.17 it follows that $F \not\models T(A)$ and thus $\mathcal{F}^{S4} \not\models T(A)$.

Now suppose that $\mathcal{F} \not\models T(A)$. Then there must be some $S4$-frame $F \in \mathcal{F}^{S4}$ such that $F \not\models T(A)$. By Lemma 6.17 it follows that $\rho F \not\models_i A$. Since $F \models \forall x\alpha$, from Lemma 6.19 it follows that $\rho F \models \forall x\alpha$. Since $\forall x\alpha$ defines $\mathcal{F}_i$, it follows that $\rho F \in \mathcal{F}_i$. Since $\rho F \not\models_i A$ we get $\mathcal{F}_i \not\models_i A$. Q.E.D.

Suppose that the superintuitionistic logic L is sound and weakly complete for the class $\mathcal{F}_i$ of intuitionistic frames defined by $\forall x\alpha$ where α is a conjunction of formulae from $\mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$. For any intuitionistic formula A, $A \in L$ iff $\mathcal{F}_i \models_i A$. If $\mathcal{F}^{S4}$ is the class of $S4$-frames defined by $\forall x\alpha$, from Lemma 6.20 it follows that $\mathcal{F}_i \models_i A$ iff $\mathcal{F}^{S4} \models T(A)$. In Section 5.1 we saw how to compute the set $\{M_i\}_{i\in\Lambda}$ of M-formulae corresponding to α. By the Sahlqvist completeness theorem, $S4 \oplus \{M_i\}_{i\in\Lambda}$ is sound and weakly complete for $\mathcal{F}^{S4}$, and thus $\mathcal{F}^{S4} \models T(A)$ iff $T(A) \in S4 \oplus \{M_i\}_{i\in\Lambda}$. We have proved the following theorem.

Theorem 6.21 *Suppose that the superintuitionistic logic L is sound and weakly complete for the class of intuitionistic frames defined by $\forall x\alpha$ where α is a conjunction of formulae from $\mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$. Let $\{M_i\}_{i\in\Lambda}$ be the M-formulae corresponding to α. Then, for any intuitionistic formula A,*

$$A \in L \textit{ iff } T(A) \in S4 \oplus \{M_i\}_{i\in\Lambda}$$

It follows that $S4 \oplus \{M_i\}_{i\in\Lambda}$ is a modal companion of L.

Corollary 6.22 *If the superintuitionistic logic L is sound and weakly complete for some class of intuitionistic frames defined by $\forall x\alpha$ where α is a conjunction of formulae from $\mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$, then L is properly displayable.*

Proof. Theorem 6.21 tells us that we can compute the axiomatic extension L' of $S4$ by M-formulae such that L' is a modal companion of L. Since any axiomatic extension of $S4$ by M-formulae is properly displayable (Theorem 5.20), the result follows from Lemma 6.13. Q.E.D.

Name	Axiom
kc,wem	$\neg p \vee \neg\neg p$
lc,gd,da	$(p \supset q) \vee (q \supset p)$
kp	$(\neg p \supset q \vee r) \supset (\neg p \supset q) \vee (\neg p \supset r)$
bd_n	$bd_1 = p_1 \vee \neg p_1$; $bd_{n+1} = p_{n+1} \vee (p_{n+1} \supset bd_n)$
btw_n	$\wedge_{0 \leq i \leq j \leq n} \neg(\neg p_i \wedge \neg p_j) \supset \vee_{i=0}^{n}(\neg p_i \supset \vee_{i \neq j} \neg p_j)$

Table 6.3: The syntactic form of some common superintuitionistic axioms as presented in [16]. Notice that the axioms bd_n and btw_n are parametrised by the index n.

6.5.1 Applications

In this section, we will apply Corollary 6.22 to display suitable superintuitionistic logics. Our plan of action is as follows.

Suppose that we are given the superintuitionistic logic $Ip \oplus \{A_i\}_{i \in \mathcal{A}}$ and suppose that for each i, β_i is the global frame correspondent of A_i. In other words, for each i, $F \models_i A_i$ iff $F \models \beta_i$. There are two things we need to be able to do before deploying Corollary 6.22.

Firstly, we need to be able to write each β_i as a formula $\forall x \alpha_i$ for $\alpha_i \in \mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$. If this is possible, then $\mathcal{F}_{\wedge_i \forall x \alpha_i}$ defines $\mathcal{F}_{\{A_i\}_{i \in \mathcal{A}}}$. It is easy to check that for any frame F, $F \models \forall x \wedge_i \alpha_i$ iff $F \models \wedge_i \forall x \alpha_i$. So $\forall x \wedge_i \alpha_i$ properly defines $\mathcal{F}_{\{A_i\}_{i \in \mathcal{A}}}$.

Thus if $Ip \oplus \{A_i\}_{i \in \mathcal{A}}$ is sound and weakly complete for $\mathcal{F}_{\{A_i\}_{i \in \mathcal{A}}}$, from Corollary 6.22 we can obtain a calculus for $Ip \oplus \{A_i\}_{i \in \mathcal{A}}$. So the second thing to check is that $Ip \oplus \{A_i\}_{i \in \mathcal{A}}$ is sound and weakly complete for $\mathcal{F}_{\{A_i\}_{i \in \mathcal{A}}}$.

Certainly, if $\{A_i\}_{i \in \mathcal{A}}$ defines the class $\mathcal{F}$ of intuitionistic frames, it is easy to check that $B \in Ip \oplus \{A_i\}_{i \in \mathcal{A}}$ implies $\mathcal{F} \models_i B$ so we have soundness.

Of course, it need not be the case that $Ip \oplus \{A_i\}_{i \in \mathcal{A}}$ is weakly complete for $\mathcal{F}_{\{A_i\}_{i \in \mathcal{A}}}$. However there is a class of intuitionistic formulae for which we can obtain completeness making use of the notion of canonicity (we used a similar approach for modal logics in Section (4.2.4)).

We say that the intuitionistic formula B is *canonical* if for any superintuitionistic logic L, $B \in L$ implies that B is valid on the intuitionistic canonical frame F_c (see [15] for a definition) for L. It is known that $F_c \models_i B$ iff $B \in L$ for any intuitionistic formula B. It follows that if $\{A_i\}_{i \in \mathcal{A}}$ is a set of canonical formulae defining the class $\mathcal{F}$ of intuitionistic frames, then $\mathcal{F} \models_i B$ implies that $B \in Ip \oplus \{A_i\}_{i \in \mathcal{A}}$. In other words, $Ip \oplus \{A_i\}_{i \in \mathcal{A}}$ is weakly complete with respect to $\mathcal{F}$.

Name	First-order characterisation on intuitionistic frames
kc,*wem*	$\forall xyz(Rxy \wedge Rxz \rightarrow \exists u(Ryu \wedge Rzu))$ $[\forall x(\forall y \rhd x)(\forall z \rhd x)(\exists u \rhd y)Rzu]$
lc,*gd*,*da*	$\forall xy(Rxy \vee Ryx)$ $[\forall w(\forall x \rhd w)(\forall y \rhd w)(Rxy \vee Ryx)]$*
kp	$\forall xyz(Rxy \wedge Rxz \wedge \neg Ryz \wedge \neg Rzy \rightarrow \exists u(Rxu \wedge Ruy \wedge Ruz \wedge$ $\forall v(Ruv \rightarrow \exists w(Rvw \wedge (Ryw \vee Rzw)))))$ [no obvious $\mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$ formula]
bd_n	$\forall x_0 \ldots x_n(\wedge_{i=0}^{n-1} Rx_ix_{i+1} \rightarrow \vee_{i\neq j} x_i = x_j)$ $[\forall x_0(\forall x_1 \rhd x_0)\ldots(\forall x_n \rhd x_{n-1})(\vee_{i\neq j} x_i = x_j)]$
btw_n	$\forall xx_0 \ldots x_n(\wedge_{i=1}^{n} Rxx_i \rightarrow \exists y \bigvee_{i\neq j}(Rx_iy \wedge Rx_jy))$ $[\forall w(\forall x \rhd w)(\forall x_0 \rhd x)\ldots(\forall x_n \rhd x)(\exists y \rhd w)\bigvee_{1\leq i,j\leq n; i\neq j}(Rx_iy \wedge Rx_jy)]$

Table 6.4: First-order characterisation of some common superintuitionistic axioms (see [16]). In the cases where it is possible to directly deduce the equivalent formula of the form $\forall\alpha$ ($\alpha \in \mathbf{A}^{\mathbf{r}}\mathbf{f}\exists^{\mathbf{r}}\mathbf{x}$), this formula is enclosed in [ ]. The symbol $*$ denotes that the characterisation is for a rooted intuitionistic frame.

In Table 6.3 we present a list of standard superintuitionistic axioms. All the formulae in this table are known to be canonical (see [16]). In Table 6.4 we present the first-order formulae characterising these axioms on intuitionistic frames (in the case of *lc* we use rooted intuitionistic frames), as well an an equivalent formulation as formulae of the form $\forall\alpha$ for $\alpha \in \mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$ (with the exception of *kp*). We can utilise Corollary 6.22 to construct display calculi for axiomatic extensions over Ip by the formulae in Table 6.3 (except *kp*). As an aside, we observe that Ghilardi and Meloni [26] present a simple algorithm that is capable of identifying a large class of canonical intuitionistic formulae, including all the formulae in Table 6.3 with the exception of btw_n.

Consider the seven interpolable superintuitionistic logics (see Maksimova [45]). Negri [53] has presented *labelled sequent calculi* for these logics. Here we show how to construct display calculi for these logics following the procedure we described above.

Ip Already done (DLI).

Jan Jankov-De Morgan logic is axiomatised $Ip \oplus kc$. Write f_{kc} to denote the $\forall x\mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$ formula corresponding to *kp*. Because *kc* is canonical we know that $Ip \oplus kc$ is sound and weakly complete with respect to the class of intuitionistic frames satisfying f_{kc}. We have already encountered f_{kc} in Lemma 5.12 where we saw that it globally corresponds to $\blacklozenge\Diamond p \supset \Diamond\blacklozenge p$

and $\Diamond\Box p \supset \Box\Diamond p$. From $\blacklozenge\Diamond p \supset \Diamond\blacklozenge p$ we can compute the corresponding display rule (Section 5.1):

$$\frac{(*\bullet*)(*\bullet*)\bullet X \vdash Y}{(*\bullet*)X \vdash Y}\, dJan$$

Thus, $DLS4 + dJan$ properly displays $S4 \oplus \Diamond\Box p \supset \Box\Diamond p$.

From Theorem 6.21, the above logic is a modal companion of $Ip \oplus kc$. Therefore by Lemma 6.13, $DLI + dJan$ properly displays $Ip \oplus kc$.

GD Gödel-Dummett Logic is axiomatised as $Ip \oplus (p \supset q) \vee (q \supset p)$. We have already presented a calculus for this logic (Section 6.4).

BD₂ This logic can be axiomatised as $Ip \oplus bd_2$. Notice that bd_2 is the formula $p \vee (p \supset (q \vee \neg q))$. From Table 6.4 we see that this formula is characterised by the class of intuitionistic frames satisfying

$$\forall x(\forall y \rhd x)(\forall z \rhd y)(x = y \vee y = z \vee x = z)$$

Since intuitionistic frames are antisymmetric, we can simplify this to

$$\forall x(\forall y \rhd x)(\forall z \rhd y)(Ryx \vee Rzy \vee Rzx))$$

As an aside, this is an illustration of Rodenburg's [61] observation that the '=' symbol has no part to play in intuitionistic correspondence theory. In fact, by the reflexivity of intuitionistic frames, the above is frame-equivalent to

$$\forall x(\forall y \rhd x)(\forall z \rhd y)(Ryx \vee Rzy) \tag{6.3}$$

Because bd_2 is canonical, $Ip \oplus Bd_2$ is sound and complete for the class of frames satisfying (6.3). Following the procedure in Section 4.2.3, (6.3) locally corresponds to

$$\begin{aligned}
&\tilde{\forall} PQ(\forall y \rhd x)(\forall z \rhd y)(ST_x(p) \wedge ST_y(q) \rightarrow ST_y(\Diamond p) \vee ST_z(\Diamond q))\\
&\Leftrightarrow \tilde{\forall} PQ\neg(\exists y \rhd x)(\exists z \rhd y)(ST_x(p) \wedge ST_y(q \wedge \neg\Diamond p) \wedge ST_z(\neg\Diamond q))\\
&\Leftrightarrow \tilde{\forall} PQ\neg(ST_x(p \wedge \Diamond(q \wedge \neg\Diamond p \wedge \Diamond\neg\Diamond q)))
\end{aligned}$$

This corresponds to the formula $p \wedge \Diamond(q \wedge \neg\Diamond p \wedge \Diamond\neg\Diamond q) \supset \bot$ which is frame equivalent to

$$(p \wedge \neg\bot) \wedge \Diamond((q \wedge \neg\Diamond p) \wedge \Diamond(r \wedge \neg\Diamond q)) \supset \bot$$

Since we want to compute the corresponding display rule, we need write this formula as a frame-equivalent primitive tense formula. From Lemma 5.7 the above formula is frame-equivalent to

$$p \wedge \Diamond(q \wedge \Diamond r) \supset (p \wedge \bot) \vee \Diamond((q \wedge \Diamond p) \vee \Diamond(r \wedge \Diamond q))$$

This simplifies to $p \wedge \Diamond(q \wedge \Diamond r) \supset \Diamond(q \wedge \Diamond p) \vee \Diamond\Diamond(r \wedge \Diamond q)$. From Section 5.1 we compute the display rule

$$\frac{* \bullet *(M \circ * \bullet * L) \vdash Z \qquad (* \bullet *)(* \bullet *)(N \circ * \bullet * M) \vdash Z}{L \circ * \bullet *(M \circ * \bullet * N) \vdash Z}\ dbd_2$$

So $DLI + dbd_2$ properly displays $Ip \oplus bd_2$.

GS The greatest semi-constructive logic can be axiomatised as

$$Ip \oplus \{gs, bd_2\}$$

where gs is the formula $(p \supset q) \vee (q \supset p) \vee ((p \supset \neg q) \wedge (\neg q \supset p))$. We already noted that the formula bd_2 is canonical. From Ghilardi and Meloni [26] we may verify that $(p \supset q) \vee (q \supset p) \vee ((p \supset \neg q)$ is also canonical. Thus the logic GS is sound and weakly complete for the class of intuitionistic frames satisfying (6.3) and the following formula [53]:

$$\forall xyz \exists v((Rxv \wedge Ryv) \vee (Ryv \wedge Rzv) \vee (Rxv \wedge Rzv)) \qquad (6.4)$$

It is not obvious that the above formula can be written in the form $\forall x\alpha$ for $\alpha \in \mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$. However, notice that the logic GS is sound and weakly complete for the class of rooted intuitionistic frames with root w (say) satisfying (6.3) and (6.4).

To see this, let $\mathcal{F}_{GS}$ (resp. $\mathcal{F}^r_{GS}$) denote the class of intuitionistic frames (intuitionistic frames with root w) satisfying (6.3) and (6.4) (or equivalently, those frames defined by $\{gs, bd_2\}$). Let us first show that for any $B \in \mathbf{For}\mathcal{L}$, $\mathcal{F}^r_{GS} \models_i B$ iff $\mathcal{F}_{GS} \models_i B$. The right-to-left direction is obvious. For the left-to-right direction, suppose that $\mathcal{F}^r_{GS} \models_i B$ and $\mathcal{F}_{GS} \not\models_i B$. Then there is some model M based on some $F \in \mathcal{F}_{GS}$ and state u such that $M, u \not\models_i B$. Let $u\!\uparrow$ be the subframe of F generated by $\{u\}$. From Lemma 6.3 we know that $u\!\uparrow \models_i \{gs, bd_2\}$ and $u\!\uparrow \not\models_i B$. It follows that $u\!\uparrow$ is a frame in $\mathcal{F}^r_{GS}$ (upto suitable renaming of state variables) on which formula B is not valid. This contradicts our assumption that $\mathcal{F}^r_{GS} \models_i B$ so we are done.

Now notice that on any intuitionistic frame with root w, formula (6.4) is equivalent to

$$(\forall x \rhd w)(\forall y \rhd w)(\forall z \rhd w)(\exists v \rhd w)(Rxv \wedge Ryv) \vee (Ryv \wedge Rzv) \vee (Rxv \wedge Rzv)$$

We can compute the corresponding primitive tense formula as

$$\Diamond p \wedge \Diamond q \wedge \Diamond r \supset \Diamond(\blacklozenge p \wedge \blacklozenge q) \vee \Diamond(\blacklozenge q \wedge \blacklozenge r) \vee \Diamond(\blacklozenge p \wedge \blacklozenge r)$$

The corresponding display rule is

$$\frac{* \bullet *(\bullet L \circ \bullet M) \vdash Y \qquad * \bullet *(\bullet M \circ \bullet N) \vdash Y \qquad * \bullet *(\bullet L \circ \bullet N) \vdash Y}{* \bullet * L \circ * \bullet * M \circ * \bullet * N \vdash Y} \; dgs$$

Thus $DLI + \{dbd_2, dgs\}$ properly displays the logic GS.

SM Smetanich logic [53] can be axiomatised as $Ip \oplus \{gd, bd_2\}$. Using our results above we see that $DLI + \{dgd, dbd_2\}$ properly displays SM.

Cp Already done ($DLI + dS5$).

Negri [53] shows how to construct a labelled sequent calculus for any superintuitionistic logic characterised by a certain syntactically specified class of first-order formulae called *geometric implications*. In contrast, to compute a display calculus, we need to express the first-order formulae in terms of $\mathbf{A^r f \exists^r x}$ formulae. It can be checked easily that every $\mathbf{A^{r'} f \exists^{r'} x}$ formula can be written as a geometric implication. However it is unclear if the other direction holds. In particular, we would like to investigate the following: suppose that the geometric implication g defines a class $\mathcal{F}$ of intuitionistic frames. Then, can we always find some conjunction α of formulae from $\alpha \in \mathbf{A^{r'} f \exists^{r'} x}$ such that $\forall x \alpha$ defines $\mathcal{F}$? Or else, under what conditions does this hold? This is the topic of future research.

Nonetheless, we have seen that for every geometric implication appearing in [53] and characterising some class of intuitionistic frames, it is straightforward enough to deduce the equivalent $\mathbf{A^{r'} f \exists^{r'} x}$ formula. Finally, we observe that it is unclear how to write the first-order correspondent of the formula kp as a $\mathbf{A^{r'} f \exists^{r'} x}$ formula, and hence we are unable to provide a display calculus for Kreisel-Putnam logic $Ip \oplus kp$. Since it is also unclear how to write that first-order formula as a geometric implication, it is unclear if there is a labelled sequent calculus for this logic either.

Part III

Importing results from labelled sequent calculi

Chapter 7

Labelled tree sequent calculi

Tree-hypersequents [57] and nested sequents [13, 37, 11] are generalisations of the traditional sequent calculus obtained via the addition of new symbols into the sequent. It is well-known [12, 57] that these sequents are notational variants of each other. Here we identify a subclass of the labelled sequents of Negri [52] called *labelled tree sequents* that is yet another notational variant of these sequents. The relationship between these sequents can be extended in the obvious way to calculi built from these sequents.

Poggiolesi [58] has presented a cutfree tree-hypersequent calculus $CSGL$ for provability logic GL and posed a question regarding the relationship of this calculus with Negri's [52] cutfree labelled sequent calculus $G3GL$ for GL — remember that the labelled sequent framework is more general that the labelled tree sequent/tree-hypersequent framework. Here we answer this question in full by presenting transformations between derivations in each system when the derivation endsequent has the form $\Rightarrow A$ (in $CSGL$) or $\Rightarrow x : A$ (in $G3GL$). Poggiolesi expends considerable effort in proving soundness and completeness for $CSGL$, and has to consider numerous cases in the proof of cut-admissibility. Using the existing results for $G3GL$ we can directly obtain these results for the $CSGL$ calculus. A key aspect of this work is the method of importing results from the labelled sequent calculus into a labelled tree sequent calculus using a translation between derivations in these calculi.

Hein [35] has suggested a method of constructing labelled tree sequen calculi for logics over K axiomatised by formule from a proper subclass of the Lemmon-Scott axioms [43]. However the cut-elimination result for these calculi is conjectured but not proved. Here we modify Hein's scheme and obtain cut-elimination for the resulting calculi for some concrete modal logics. Although we do not yet have a general proof of cut-elimination, the work here indicates how this problem can be phrased in terms of the extension of the method of importing results from suitable labelled sequent calculi into labelled tree sequent calculi.

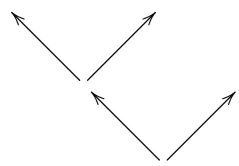
7.1 Introduction

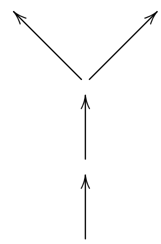
7.1.1 Tree-hypersequents, nested sequents and labelled tree sequents

Gentzen [25] introduced the *sequent calculus* as a tool for studying proof systems for classical and intuitionistic logics. Gentzen sequent calculi are built from *traditional sequents* of the form $X \Rightarrow Y$ where X and Y are formula multisets. The main result is the cut-elimination theorem, which shows how to eliminate the cut-rule from these calculi. The resulting sequent calculi are said to be cutfree. A significant drawback of the Gentzen sequent calculus is the difficulty of adapting the calculus to new logics. In particular, these calculi often fail to be *modular* — informally this means that there is a weak correspondence between the rules of the calculus and the logical axioms, so much effort is required to generate new calculi from an existing calculus, even when the corresponding logics have a close connection. For example, although there is a cutfree Gentzen sequent calculus for $S4$, there is no known cutfree Gentzen calculus for $S5$ despite the fact that the logic $S5$ can be directly obtained from a Hilbert calculus [16] for $S4$ by the addition of a single axiom corresponding to symmetry.

This has lead to various generalisations of the Gentzen sequent calculus in an attempt to present logics using proof systems with nice properties. *Hypersequent calculi* [3] generalise Gentzen sequent calculi by using a /-separated list of traditional sequents (a *hypersequent*) rather than just a single one. Usually, the order of the sequents is not important so a multiset can be used instead of a list. In this case the hypersequent $X \Rightarrow Y/U \Rightarrow V$ is the same as $U \Rightarrow V/X \Rightarrow Y$, for example.

Tree-hypersequents generalise hypersequents through the addition of the symbols ; and ( ) to the syntax, and by attaching importance to the *order* of the traditional sequents. Furthermore, the placement of the / and ; symbols play a crucial role in the semantic meaning of a tree-hypersequent, enabling each tree-hypersequent to be associated with a tree-like frame (see Definition 4.1). For example, the tree-hypersequents $-/(-/(-;-));-$ and $-/-/-/(-;-)$, where the dashes stand for sequents, correspond to the (tree) frame figures below left and below right respectively:





Nested sequents (also called deep sequents) have been invented several times independently (for example, see [13, 37, 11]). Nested sequents generalise the traditional sequent by permitting a nesting of formulae multisets. The nesting is denoted through the addition of the symbols [ ] to the syntax. Indeed, the nesting of formulae multisets using [ ] conveys the same semantic information as the ordering of the symbols /, ; and ( ) in a tree-hypersequent. For example, the tree figures above left and above right can be written, respectively, as nested sequents of the following form:

$$-, [-, [-], [-]], [-] \qquad\qquad -, [-, [-, [-], [-]]]$$

It is easy to see that nested sequents and tree-hypersequents are notational variants of the same object. Following standard notation, we will use a one-sided sequent to capture the information at each node rather than the two-sided traditional sequent we used for tree-hypersequents. For this reason, each dash in the above nested sequents corresponds to a formula multiset.

Labelled sequents [24, 50] generalise the traditional sequent by the prefixing of indices or labels to formulae occurring in the sequent. As Restall [59] observes, a labelled sequent can be viewed as a directed graph with sequents at each node Negri [52] has presented a method for generating cutfree labelled sequent calculi for a large family of modal logics. These labelled sequent calculi incorporate the frame accessibility relation into the syntax of the calculi. These calculi are modular, since a new logic can be presented by the inclusion of a rule corresponding to the properties of its accessibility relation. In the case of $S5$ for example, labelled sequent rules for reflexivity, transitivity and symmetry are added to the base calculus.

A *labelled tree sequent* is a special instance of a labelled sequent where the underlying graph structure is restricted to a tree. Under this restriction, the connection between these sequents and tree-hypersequents and nested sequents becomes apparent.

Since this chapter concerns mappings between various types of sequent calculi, for the sake of definiteness, in the following section we present a formal definition for each type of calculus we will encounter.

7.1.2 Basic definitions

The *basic modal language* $\mathcal{ML}$ is defined using a countably infinite set of propositional variables p_i, the propositional connectives $\neg$, $\vee$, $\wedge$ and $\supset$, the unary modal operators $\Box$ and $\Diamond$, and the punctuation marks ( ).

A *modal formula* is a formula belonging to the set defined by the following grammar

$$A ::= p \,|\, \neg A \,|\, (A \vee B) \,|\, (A \wedge B) \,|\, (A \supset B) \,|\, \Box A \,|\, \Diamond A$$

where p ranges over the set of propositional variables.

In this chapter we work exclusively with classical modal logics. In this context we have the freedom of working with certain proper subsets of $\{\neg, \vee, \wedge, \supset, \Box, \Diamond\}$, as the missing language elements can be defined in terms of the remaining ones. For example, in Section 7.3 we restrict ourselves to the subset $\{\neg, \wedge, \Box\}$. Then a modal formula is implicitly taken as being defined by the simplified grammar

$$A ::= p \,|\, \neg A \,|\, (A \wedge B) \,|\, \Box A$$

A *traditional sequent* (denoted $X \Rightarrow Y$) is an ordered pair (X, Y) where X (the 'antecedent') and Y (the 'succedent') are finite multisets of formulae.

The syntactic equality of two structures (such as sequents or formulae) is denoted using $\equiv$. The negation is denoted $\not\equiv$. So, for example $\neg\neg p \not\equiv p$ although the formulae $\neg\neg p$ and p are logically equivalent in extensions of classical logic.

Definition 7.1 (Gentzen sequent calculus) *The* Gentzen sequent calculus *consists of some set of traditional sequents (the* initial sequents*) and some set of* inference rules *of the form*

$$\frac{\mathcal{S}_1 \ldots \mathcal{S}_n}{\mathcal{S}}$$

where the traditional sequents $\mathcal{S}_1, \ldots, \mathcal{S}_n$ are called the premises *of the rule, and $\mathcal{S}$ is called the* conclusion *sequent.*

The two common flavours of the cut-rule are the additive cut, where the contexts Γ, Δ are identical in each premise,

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ cut$$

and the multiplicative cut

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ cut$$

Tree-hypersequent calculi

A tree-hypersequent is built from traditional sequents using the symbols /, ; and ( ).

Definition 7.2 *A* tree-hypersequent *is defined inductively as follows:*

(i) if $\mathcal{S}$ is a traditional sequent, then $\mathcal{S}$ is a tree-hypersequent,

(ii) if $\mathcal{S}$ is a traditional sequent and $G_1, G_2, \ldots, G_n$ are tree-hypersequents, then $\mathcal{S}/G_1; G_2; \ldots; G_n$ is a tree-hypersequent.

For the sake of clarify, we will often use parentheses, for example, writing the tree-hypersequent $\mathcal{S}/G_1; \ldots; G_n$ as $\mathcal{S}/(G_1; \ldots; G_n)$, although, strictly speaking, these symbols are not a part of the formal language.

We will use the following syntactic conventions (possibly with subscripts):

$A, B, \ldots$: modal formulae,

$X, Y, U, V, \Gamma, \Delta$: finite multisets of formulae

$\mathcal{S}, \mathcal{T}$: traditional sequents

$G, H, \ldots$: tree-hypersequents.

$\underline{X}, \underline{Y}, \ldots$: finite multisets of tree-hypersequents.

Following are some examples of tree-hypersequents:

$$X \Rightarrow Y \qquad X \Rightarrow Y/(U \Rightarrow V; \Gamma \Rightarrow \Delta) \qquad \mathcal{S}/((\mathcal{T}/\underline{X}); \underline{Y})$$

Let $\emptyset$ denote the empty tree-hypersequent. We write $G\{H\}$ to mean that the tree-hypersequent G contains a specific occurrence of the tree-hypersequent H. Then $G\{\emptyset\}$ denotes the tree-hypersequent obtained from $G\{H\}$ by substituting that specific occurrence of H with the empty tree-hypersequent. The notation here differs from Poggiolesi [57, 58] where square brackets [ ] are used instead of { }. The square brackets are reserved here for the language of nested sequents.

Given two tree-hypersequents $G_1\{\mathcal{S}_1\}$ and $G_2\{\mathcal{S}_2\}$, the relation of *equivalent position* between the traditional sequents $\mathcal{S}_1$ and $\mathcal{S}_2$ (denoted $G_1\{\mathcal{S}_1\} \sim G_2\{\mathcal{S}_2\}$) is defined inductively as follows:

(i) $\mathcal{S}_1 \sim \mathcal{S}_2$

(ii) $\mathcal{S}_1/\underline{X} \sim \mathcal{S}_2/\underline{X}'$

(iii) If $H_1\{\mathcal{S}_1\} \sim H_2\{\mathcal{S}_2\}$ then $\mathcal{T}/(H_1\{\mathcal{S}_1\};\underline{X}) \sim \mathcal{T}'/(H_2\{\mathcal{S}_2\};\underline{X}')$ where $\mathcal{T}$ and $\mathcal{T}'$ are traditional sequents

The intended interpretation $\mathcal{I}$ of a tree-hypersequent as a modal formula is defined as follows

$$(X \Rightarrow Y)^{\mathcal{I}} = \wedge X \supset \vee Y$$
$$(\mathcal{S}/(G_1;\dots G_n))^{\mathcal{I}} = \mathcal{S}^{\mathcal{I}} \vee \Box G_1^{\mathcal{I}} \vee \dots \vee \Box G_n^{\mathcal{I}}$$

Definition 7.3 (tree-hypersequent calculus) *Obtained from Definition 7.1 with the phrase 'traditional sequent' replaced with 'tree-hypersequent'.*

For brevity, from here onwards we will write THS to stand for the words 'tree-hypersequent(s)'.

For THS $G\{X \Rightarrow Y, A\} \sim G'\{A, U \Rightarrow V\}$ (so the sequents $X \Rightarrow Y, A$ and $A, X \Rightarrow Y$ are in an equivalent position), define the cut-rule:

$$\frac{G\{X \Rightarrow Y, A\} \qquad G'\{A, U \Rightarrow V\}}{G \star G'\{X, U \Rightarrow Y, V\}}\,cut$$

where $X \Rightarrow Y \otimes U \Rightarrow V$ is defined as $X, U \Rightarrow Y, V$, and the operator $\star$ is defined inductively for THS $H\{\mathcal{S}\} \sim H'\{\mathcal{S}'\}$ as follows

(i) $\mathcal{S} \star \mathcal{S}' = \mathcal{S} \otimes \mathcal{S}'$

(ii) $(\mathcal{S}/\underline{X}) \star (\mathcal{S}'/\underline{Y}) = (\mathcal{S} \otimes \mathcal{S}'/\underline{X};\underline{Y})$

(iii) $(\mathcal{T}/H\{\mathcal{S}\};\underline{X}) \otimes (\mathcal{T}'/H'\{\mathcal{S}'\};\underline{Y}) = \mathcal{T} \otimes \mathcal{T}'/(H\{\mathcal{S}\} \star H'\{\mathcal{S}'\});\underline{X};\underline{Y}$

Remember that $\mathcal{S}, \mathcal{S}', \mathcal{T}, \mathcal{T}'$ denote traditional sequents. The $\star$ operation can be viewed as a merge operation on trees, and it ensures that the conclusion sequent of the cut-rule is indeed a THS.

Nested sequent calculi

A nested sequent is a finite multiset of modal formulae and boxed sequents. A boxed sequent is a term of the form $[\Gamma]$ where Γ is a nested sequent. Thus a nested sequent has the form

$$A_1, \dots, A_m, [\Gamma_1], \dots, [\Gamma_n]$$

where the $\{A_i\}$ are modal formulae and the $\{\Gamma_i\}$ are nested sequents. We write $\Gamma\{\Delta\}$ to denote the nested sequent containing a specific occurrence of Δ. Similarly, the notation $\Gamma\{\}$ denotes a nested sequent with a single *hole* which does not occur inside formulae. The nested sequent $\Gamma\{\Delta\}$ is obtained from $\Gamma\{\}$ by replacing the hole with Δ.

Definition 7.4 (nested sequent calculus) *Obtained from Definition 7.1 with the phrase 'traditional sequent' replaced with 'nested sequent'.*

The cut-rule for nested sequent calculi is defined as follows:

$$\frac{\Gamma\{A\} \qquad \Gamma\{\neg A\}}{\Gamma\{\emptyset\}}\, cut$$

Labelled sequents and labelled tree sequent calculi

Fitting [24] has described the incorporation of frame semantics into tableau proof systems for the purpose of obtaining tableau systems for certain logics. Approaches to internalise the frame semantics into the Gentzen sequent calculus via the labelling of formulae appear in Mints [50], Vigano [76] and Kushida and Okada [41]. In this Chapter we use the labelled systems for modal logic presented in Negri [52].

Assume that we have at our disposal an infinite set $\mathbb{SV}$ of variables ('state variables') disjoint from the set of propositional variables. We will use the letters $x, y, z \ldots$ to denote state variables. A *labelled formula* has the form $x : A$ where x is a state variable and A is a modal formula. If $X = \{A_1, \ldots A_n\}$ is a multiset of formulae, then $x : X$ denotes the multiset $\{x : A_1, \ldots, x : A_n\}$ of labelled formulae. Notice that if the formula multiset X is empty, then $x : X$ is the empty labelled formula multiset. A *relation term* is a term of the form Rxy where x and y are variables. A (possibly empty) set of relations terms is called a *relation set.* A *labelled sequent* (denoted $\mathcal{R}, X \Rightarrow Y$) is the ordered triple $(\mathcal{R}, X, Y)$ where $\mathcal{R}$ is a relation set and X ('antecedent') and Y ('succedent') are multisets of labelled formulae.

Definition 7.5 (labelled sequent calculus) *Obtained from Definition 7.1 with the phrase 'traditional sequent' replaced by 'labelled sequent'.*

Remark 7.6 (side conditions) *Inference rules may contain additional conditions that need to be satisfied in order to apply the rule. Such conditions are called* side conditions. *A common side condition for a labelled sequent inference rule is* *a restriction of the form "z does not appear in the conclusion sequent of the rule" for some schematic variable z. Let us call this type of side condition a* standard variable restriction.

A frame is a pair (W, R) where W is a set of states and R is a binary relation on W (see Definition 4.1). A frame is said to be *rooted* if it is generated (see the discussion preceding Lemma 6.3) by some $\{x\} \subseteq W$. In this case, x is called the *root* of F. A rooted frame whose underlying *undirected* graph does not contain a path from any node back to itself (ie. no cycles) is called a *tree*. For example, a frame containing a reflexive state is not a tree. An empty frame is trivially a tree. Due to the prohibition of cycles, a non-empty tree has exactly one root.

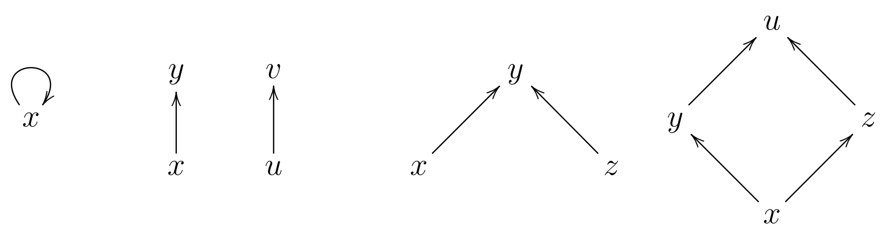
For some relation set $\mathcal{R}$, the set $W_{\mathcal{R}}$ consists of all those states that appear in $\mathcal{R}$. That is,

$$W_{\mathcal{R}} = \{x \,|\, Rxv \in \mathcal{R} \text{ or } Rvx \in \mathcal{R} \text{ for some state } v\}$$

Let $R_{\mathcal{R}}$ be the binary relation on $W_{\mathcal{R}}$ given by $(x, y) \in R_{\mathcal{R}}$ iff $Rxy \in \mathcal{R}$. Then we say that the frame $F_{\mathcal{R}} = (W_{\mathcal{R}}, R_{\mathcal{R}})$ is defined by the relation set $\mathcal{R}$.

We say that a relation set $\mathcal{R}$ is *treelike* if the frame defined by $\mathcal{R}$ is a tree. For a non-empty relation set $\mathcal{R}$ that is treelike, let root$(\mathcal{R})$ denote the root of this tree.

To illustrate this definition, consider the relation sets $\{Rxx\}$, $\{Rxy, Ruv\}$, $\{Rxy, Rzy\}$, and $\{Rxy, Rxz, Ryu, Rzu\}$. The frames defined by these sets are, respectively,



None of the above relation sets are treelike because the frames defined by their relation sets are not trees. In the above frames from left-to-right, frame 1 contains a reflexive state (and hence a cycle); frame 2 and frame 3 are not rooted. Finally, frame 4 is not a tree because the underlying undirected graph contains a cycle.

Definition 7.7 (labelled tree sequent) *A* labelled tree-sequent *is a labelled sequent of the form $\mathcal{R}, X \Rightarrow Y$ where*

(i) $\mathcal{R}$ is treelike, and

(ii) *if* $\mathcal{R} = \emptyset$ *then* X *has the form* $\{x : A_1, \ldots, x : A_n\}$ *and* Y *has the form* $\{x : B_1, \ldots, x : B_m\}$ *for some state variable* x *(ie. each labelled formula in* X *and* Y *has the same label), and*

(iii) *if* $\mathcal{R} \neq \emptyset$ *then every state variable* x *that occurs in either* X *or* Y *(as a labelled formula* $x : A$ *for some formula* A*) also occurs in* $\mathcal{R}$ *(ie as a term* Rxu *or* Rux *for some state* u*).*

For example, each of the following is a labelled tree sequent:

$$x : A \Rightarrow x : A \qquad\qquad \Rightarrow y : A \qquad\qquad Rxy, Rxz, x : A \Rightarrow y : A$$

Notice that it is possible for a state variable to occur in the relation set and not in the X, Y multisets (this is what happens with the state variable z in the example above right). The following are *not* labelled tree sequents:

$$x : A \Rightarrow x : A, z : A \qquad\quad Rxy, x : A \Rightarrow z : A \qquad\quad Rxy, Ryz, Rxz \Rightarrow$$

From left-to-right above, the first labelled sequent is not a labelled tree sequent because the relation set is empty and yet two distinct state variables occur in the sequent (violating condition (ii)). The next sequent violates condition (iii) because the state variable z appears in the succedent (as $z : A$) but it does not appear in the relation set. The final sequent violates condition (i) because the relation set is not treelike.

Definition 7.8 (labelled tree sequent calculus) *A* labelled tree sequent calculus *is a labelled sequent calculus where only labelled tree sequents may occur.*

For brevity, from here onwards we will write LTS to stand for the words 'labelled tree sequent(s)'.

Negri [52] uses the following cut-rule *cut* for labelled sequent calculi:

$$\frac{\mathcal{R}_1, X \Rightarrow Y, x : A \qquad \mathcal{R}_2, x : A, U \Rightarrow V}{\mathcal{R}_1 \cup \mathcal{R}_2, X, U \Rightarrow Y, V}\ cut$$

We cannot use this rule directly in a labelled tree sequent calculus because $\mathcal{R}_1 \cup \mathcal{R}_2$ need not be treelike even if $\mathcal{R}_1$ and $\mathcal{R}_2$ are treelike. Instead of placing additional conditions on the cut-rule, we define an 'additive' cut-rule for labelled tree sequent calculi as follows:

$$\frac{\mathcal{R}, X \Rightarrow Y, x : A \qquad \mathcal{R}, x : A, X \Rightarrow Y}{\mathcal{R}, X \Rightarrow Y}\ cut_{LTS}$$

We will close this section by revising standard terminology. The terminology is applicable to Gentzen/THS/nested sequent/LTS calculi, so in the following we use the term 'calculus' to refer to any one of these systems. Similarly the term 'sequent' here refers to a traditional/tree-hyper/nested/labelled tree sequent.

Syntactic cut-admissibility means that the cut-rule is syntactically admissible — ie. if the premises of a cut-rule instance are derivable (in the calculus in question), then there is an effective transformation of the premise derivations leading to a derivation of the conclusion sequent. Cut-admissibility is an important proof-theoretic result for a calculus (see Section 1.1). Of course, if the calculus already contains the cut-rule then syntactic admissibility is obvious. When the calculus contains the cut-rule, the more meaningful concept is *syntactic cut-elimination*: if δ is a derivation in $\mathcal{C} + cut$ then there is an effective transformation to a derivation δ' in $\mathcal{C}$ with identical endsequent. Syntactic cut-elimination is equivalent to the statement that syntactic cut-admissibility holds for the calculus without cut. For example, suppose that we have cut-admissibility for a calculus $\mathcal{C}$. To show cut-elimination for $\mathcal{C} + cut$, take an arbitrary derivation δ in $\mathcal{C} + cut$. Argue by induction on the number of cut-rule instances in δ. To prove the inductive case, choose a topmost cut-rule in δ and invoke the syntactic cut-admissibility result. In this chapter we deal with cutfree calculi so we focus on syntactic cut-admissibility.

General cut-admissibility is the statement that the conclusion sequent is derivable whenever the premises of a cut-rule are derivable. There is no guarantee here of an effective transformation for obtaining the conclusion sequent. An example of such a result would be a semantic proof of cut-admissibility. Clearly syntactic cut-admissibility implies general cut-admissibility. However it need not be the case that the reverse direction holds. For, even in a cutfree calculus for a logic, backward proof search may not necessarily terminate. In other instances, further argument may be required to avoid loops. Even when general cut-admissibility implies an algorithm for obtaining a cutfree derivation of a given sequent, from a proof-theoretical perspective, we observe that the implicit interest is in an algorithm that reveals insights into the proof calculus, for example by opening up the possibility of generalisation to new logics and new rules. Furthermore, another implicit motivation for an algorithm witnessing a syntactic proof of cut-elimination is the ability to obtain bounds on the length of a cutfree derivation with respect to the original derivation. Note that if we use an exhaustive search to induce a cutfree derivation, there would be no connection between this derivation and the original derivation.

A *rule instance* in the calculus $\mathcal{C}$ is a substitution instance of formulae (and state variables, if applicable) of one of the inference rules from $\mathcal{C}$. An *initial sequent instance* in the calculus $\mathcal{C}$ is a substitution instance of formulae (and state variables, if applicable) of an initial sequent from $\mathcal{C}$. A *derivation* in the calculus $\mathcal{C}$ is defined in the usual way, as either an initial sequent instance, or an application of a rule instance to derivations of the premises of the rule. If there is a derivation of some sequent $\mathcal{S}$ in $\mathcal{C}$, then we say that $\mathcal{S}$ is derivable in $\mathcal{C}$. The *height* of a derivation is defined in the usual way as the maximum depth of the derivation tree. We write $\vdash^{\delta}_{\mathcal{C}} \mathcal{S}$ to mean that there is a derivation δ of the sequent $\mathcal{S}$ in $\mathcal{C}$. To avoid having to name the derivation we simply write $\vdash_{\mathcal{C}} \mathcal{S}$.

We say that an inference rule ρ is *admissible* in $\mathcal{C}$ if whenever premises of any rule instance of ρ is derivable in $\mathcal{C}$, then so is the conclusion of the rule instance. The word *effective* signifies the presence of an algorithm. If there is an effective transformation that witnesses the admissibility, then we say that ρ is *syntactically admissible*. If a calculus $\mathcal{C}$ derives precisely those formulae belonging to the logic L (ignoring any extraneous information such as labels) we say that $\mathcal{C}$ *presents* L.

7.2 Maps between THS and LTS

Although the fact that THS and LTS are notational variants is quite apparent, we will present concrete definitions for the maps witnessing this fact for use in later sections. The reader who wishes to omit the algorithmic and technical details concerning the mapping functions and the calculi induced by these functions may safely proceed to Definition 7.19 (perhaps after reviewing Definition 7.9 and Definition 7.11).

If $X \Rightarrow Y$ and $U \Rightarrow V$ are traditional sequents, recall that we defined $X \Rightarrow Y \otimes U \Rightarrow V$ to be the traditional sequent $X, U \Rightarrow Y, V$. Overloading the operator, if $\mathcal{R}_1, X \Rightarrow Y$ and $\mathcal{R}_2, U \Rightarrow V$ are two labelled sequents, then define $\mathcal{R}_1, X \Rightarrow Y \otimes \mathcal{R}_2, U \Rightarrow V$ to be the labelled sequent $\mathcal{R}_1 \cup \mathcal{R}_2, X, U \Rightarrow Y, V$. Because the order of elements in a multiset is irrelevant, in each case $\otimes$ is associative and commutative.

Definition 7.9 *Define the function $\mathbb{TL}_x$ taking a THS to a labelled sequent as* *follows:*

$$\mathbb{TL}_x(X \Rightarrow Y) = x : X \Rightarrow x : Y$$
$$\mathbb{TL}_x(X \Rightarrow Y/G_1; \ldots; G_n) = \left(\otimes_{j=1}^{n} \mathbb{TL}_{y_j}(G_j)\right) \otimes (Rxy_1, \ldots, Rxy_n, x : X \Rightarrow x : Y)$$

where $y_1, \ldots, y_n$ are variables that have not been used already (fresh variables).[1]

By inspection of the function, it is easy to check that the image of $\mathbb{TL}_x$ is a labelled tree sequent. We will sometimes suppress the subscript, writing $\mathbb{TL}$ for the sake of clarity when the state variable that is used is not important. Observe that $\mathbb{TL}G$ assigns a unique state variable to each traditional sequent $\mathcal{S}$ appearing in G. Moreover, given the THS $G_1\{\mathcal{S}_1\}$ and $G_2\{\mathcal{S}_2\}$ such that $S_1 \sim S_2$ (traditional sequents in an equivalent position), without loss of generality we may assume that the state variable assigned to $\mathcal{S}_1$ in $\mathbb{TL}(G_1\{\mathcal{S}_1\})$ and $\mathcal{S}_2$ in $\mathbb{TL}(G_2\{\mathcal{S}_2\})$ is identical.

Before introducing the map from a LTS to a THS let us introduce some notation. Let $\mathcal{R}$ be a relation set, and let Γ be a multiset of labelled formulae. Define the following sets:

$$\mathcal{R}_x = \{Rxv \,|\, Rxv \in \mathcal{R} \text{ for some state } v\}$$
$$\Gamma_x = \{x : A \,|\, x : A \in \Gamma \text{ for some formula } A\}$$

So Γ_x consists of those labelled formulae in Γ that are labelled with the state x. Notice that if $\mathcal{R}$ is treelike and $\mathcal{R} \neq \emptyset$ then $\mathcal{R}_{\mathrm{root}(\mathcal{R})} \neq \emptyset$.

For a relation set $\mathcal{R}$, we define the set $\mathcal{R}_{x\uparrow}$ to be the relation set defining the subframe (of the frame defined by $\mathcal{R}$) generated by $\{x\}$ (when x occurs in $\mathcal{R}$) and the empty set otherwise. In notation,

$$Ruv \in \mathcal{R}_{x\uparrow} \text{ iff } Ruv \in \mathcal{R}_x \text{ or } \exists v_1, \ldots, v_n.\{Rxv_1, \ldots, Rv_nu, Ruv\} \subseteq \mathcal{R}$$

Finally, if Γ is a set of labelled formulae, let $\Gamma_{\mathcal{R}}$ be the labelled formulae that are labelled with states occurring in $\mathcal{R}$.

Example 7.10 *Consider the labelled tree sequent $\mathcal{R}, X \Rightarrow Y$, where*

$$\mathcal{R} = \{Rwu, Rwv, Rus, Rut\}$$
$$X = \{w : A, w : B, u : C\}$$
$$Y = \{u : A, v : B, s : C\}$$

[1]From a technical perspective we should be more precise here. To ensure that a variable is not reused, the stack of variables available for use needs to be passed to the function at each call. This can certainly be done — we omit the details here.

Let us illustrate the use of the functions we introduced above. For example, we have $\mathcal{R}_w = \{Rwu, Rwv\}$ *and* $\mathcal{R}_v = \emptyset$*;* $X_w = \{w : A, w : B\}$ *and* $Y_w = \emptyset$*. Also,* $\mathcal{R}_{w\uparrow} = \mathcal{R}$*,* $\mathcal{R}_{u\uparrow} = \{Rus, Rut\}$ *and* $\mathcal{R}_{s\uparrow} = \emptyset$*. Finally,* $Y_{\mathcal{R}_{w\uparrow}} = Y$*,* $Y_{\mathcal{R}_{u\uparrow}} = \{u : A, s : C\}$ *and* $Y_{\mathcal{R}_{s\uparrow}} = \emptyset$*.*

Definition 7.11 *Define the function* $\mathbb{LT}$ *taking a* LTS $\mathcal{R}, X \Rightarrow Y$ *to a* THS *as follows:*

If $\mathcal{R} = \emptyset$ *then* $\mathcal{R}, X \Rightarrow Y$ *must have the form* $x : U \Rightarrow x : V$ *for some state variable* x*. Set* $\mathbb{LT}(x : U \Rightarrow x : V) = U \Rightarrow V$*.*

Otherwise, for $x = root(\mathcal{R})$ *and* $\mathcal{R}_x = \{Rxy_1, \ldots, Rxy_n\}$*:*

$$\mathbb{LT}(\mathcal{R}, X \Rightarrow Y) = \\ X_x \Rightarrow Y_x / (\mathbb{LT}(\mathcal{R}_{y_1\uparrow}, X_{\mathcal{R}_{y_1\uparrow}} \Rightarrow Y_{\mathcal{R}_{y_1\uparrow}}); \ldots ; \mathbb{LT}(\mathcal{R}_{y_n\uparrow} X_{\mathcal{R}_{y_n\uparrow}} \Rightarrow Y_{\mathcal{R}_{y_n\uparrow}}))$$

Recall that $\mathbb{SV}$ denotes the set of state variables. Let $Var(\mathcal{S}) \subset \mathbb{SV}$ denote the finite set of state variables occurring in the labelled sequent S. A *renaming of* $\mathcal{S}$ is a one-to-one function $f_{\mathcal{S}} : Var(\mathcal{S}) \mapsto \mathbb{SV}$ (by one-to-one we mean that if $f_{\mathcal{S}}(x) = f_{\mathcal{S}}(y)$ then $x = y$). We write $Dom(f_{\mathcal{S}})$ and $Im(f_{\mathcal{S}})$ to denote the domain and image of $f_{\mathcal{S}}$ respectively.

For any labelled sequent $\mathcal{S}'$ and renaming $f_{\mathcal{S}}$ of the labelled sequent $\mathcal{S}$, let $\mathcal{S}'_{f_{\mathcal{S}}}$ be the labelled sequent obtained from $\mathcal{S}'$ by the simultaneous and uniform substitution $x \mapsto f_{\mathcal{S}}(x)$ for all $x \in Dom(f_{\mathcal{S}}) \cap Var(\mathcal{S}')$. Notice that $\mathcal{S}'_{f_{\mathcal{S}}}$ need not be an LTS even if $\mathcal{S}'$ is a LTS.

Example 7.12 *Consider the following* LTS $\mathcal{S}$ *(below left) and* $\mathcal{S}'$ *(below right):*

$$x : A \Rightarrow x : A \qquad\qquad Rxy, x : A \Rightarrow y : B$$

Let the renaming $f_{\mathcal{S}}$ *of* $\mathcal{S}$ *be the function mapping* $x \mapsto y$*. Then* $\mathcal{S}'_{f_{\mathcal{S}}}$ *is the labelled sequent* $Ryy, y : A \Rightarrow y : B$*. Clearly this sequent is not a* LTS*.*

It is easy to check that if $\mathcal{S}$ is a LTS, then for any renaming $f_{\mathcal{S}}$ of $\mathcal{S}$, it is the case that $\mathcal{S}_{f_{\mathcal{S}}}$ must be a LTS.

Lemma 7.13 *Let* G *denote a* THS *and let* $\mathcal{S}$ *denote a* LTS*. Then*

(i) $\mathbb{TL}G$ *is a labelled tree-sequent.*

(ii) $\mathbb{LT}\mathcal{S}$ *is a* THS*.*

(iii) $\mathbb{LT}(\mathbb{TL}G) \equiv G$*, and* $\mathbb{TL}(\mathbb{LT}\mathcal{S}) \equiv \mathcal{S}_{f_{\mathcal{S}}}$ *for some renaming* $f_{\mathcal{S}}$ *of* $\mathcal{S}$*.*

Proof. The proofs of (i) and (ii) are straightforward, following from an inspection of the functions $\mathbb{TL}$ and $\mathbb{LT}$. In the case of (iii), observe that Definition 7.7(iii) ensures that no labelled formulae are 'lost' when passing from $\mathcal{S}$ to $\mathbb{LT}\mathcal{S}$. However, since the $\mathbb{TL}$ function *assigns* state variables, it may be necessary to 'swap' label names in order to obtain equality of the LTS $\mathbb{TL}(\mathbb{LT}\mathcal{S})$ and $\mathcal{S}$. In other words, there is some renaming $f_{\mathcal{S}}$ of $\mathcal{S}$ such that $\mathbb{TL}(\mathbb{LT}\mathcal{S}) \equiv \mathcal{S}_{f_{\mathcal{S}}}$. Q.E.D.

Lemma 7.14 (substitution lemma) *Suppose that $\mathcal{C}$ is a LTS calculus and $\mathcal{S}$ is a LTS. Also suppose that the inference rules in $\mathcal{C}$ have no side conditions, or a standard variable restriction (see Remark 7.6). Let $f_{\mathcal{S}}$ be an arbitrary renaming of $\mathcal{S}$. If $\vdash^{\delta}_{\mathcal{C}} \mathcal{S}$ then there is an effective transformation to a derivation δ' such that $\vdash^{\delta'}_{\mathcal{C}} \mathcal{S}_{f_{\mathcal{S}}}$.*

Proof. Induction on the height of δ. If the height is one, then δ must be an initial sequent. It is easy to see that $\mathcal{S}_{f_{\mathcal{S}}}$ is also an initial sequent.

Now suppose that the last rule in δ is the LTS inference rule ρ, with premises $\mathcal{S}_1, \ldots, \mathcal{S}_n$. Although each $\mathcal{S}_i$ is a LTS, if $\cup_i Var(\mathcal{S}_i)$ contains state variables not in $\mathcal{S}$ (for example, due to a standard variable restriction on ρ), it is possible that $(\mathcal{S}_i)_{f_{\mathcal{S}}}$ is not a LTS even if $\mathcal{S}_i$ is a LTS.

For example, suppose that f is the renaming $x \mapsto y$ of the $LTS \Rightarrow x : \Box A$, and consider the following rule instance of ρ:

$$\frac{\mathcal{S}_1 = Rxy, y : \Box A \Rightarrow y : A}{\mathcal{S} = \qquad\qquad \Rightarrow x : \Box A}$$

Then $(\mathcal{S}_1)_f$ is the labelled sequent $Ryy, y : \Box A \Rightarrow y : A$ which is not a LTS (because the relation set contains the cycle Ryy) although $\mathcal{S}_1$ is a LTS.

Returning to the proof, notice that we can always define a one-to-one function g from $\cup_i Var(\mathcal{S}_i) \setminus Dom(f_{\mathcal{S}})$ to fresh state variables (in particular, to variables outside $Im(f_{\mathcal{S}})$). Then $((\mathcal{S}_i)_g)_{f_{\mathcal{S}}}$ is a LTS. Then $((\cdot)_g)_{f_{\mathcal{S}}}$ implicitly defines a renaming $Var(\mathcal{S}_i) \mapsto \mathbb{SV}$ for $\mathcal{S}_i$ for each i. So $((\mathcal{S}_i)_g)_{f_{\mathcal{S}}}$ is a LTS.

Continuing the example above, set g as the map $y \mapsto z$, so

$$\frac{((\mathcal{S}_1)_g)_f = Ryz, z : \Box A \Rightarrow z : A}{(\mathcal{S}_g)_f = \qquad\qquad \Rightarrow y : \Box A}$$

and this is a legal rule instance of ρ.

Once again, returning to the proof, by the induction hypothesis we can obtain derivations of $((\mathcal{S}_i)_g)_{f_{\mathcal{S}}}$ in $\mathcal{C}$. Moreover, observe that

$$\frac{((\mathcal{S}_1)_g)_{f_\mathcal{S}} \ldots ((\mathcal{S}_n)_g)_{f_\mathcal{S}}}{(\mathcal{S}_g)_{f_\mathcal{S}}}$$

is a rule instance of ρ. Hence we have a derivation of $(\mathcal{S}_g)_f$. Since $Var(\mathcal{S}) \cap Dom(g) = \emptyset$, it follows that $(\mathcal{S}_g)_{f_\mathcal{S}} = \mathcal{S}_{f_\mathcal{S}}$. Q.E.D.

We remind the reader that this substitution lemma pertains to LTS calculi as given in Definition 7.8. In particular, this lemma may not apply to calculi containing pathological rules that are not invariant under renaming, such as the following rule:

$$\frac{x \neq a}{x : A \Rightarrow x : B} \text{ (a is some fixed state variable)}$$

Inference rules induced by $\mathbb{TL}$ and $\mathbb{LT}$

We will now look at how to construct an inference rule for THS from an inference rule for LTS and *vice versa*.

1. *Let ρ be an inference rule built from LTS.* Following standard terminology (for example, see [70, 52]), any labelled formula that 'appears' or 'disappears' when moving from a premise to the conclusion of the inference rule is called a *principal formula*. A *principal element* in an inference rule ρ is either a principal formula of ρ or an occurrence of a term Ruv such that ρ contains a principal formula with label u or v. The remaining terms are referred to as *context elements*. The image $\mathbb{LT}\rho$ of ρ under $\mathbb{LT}$ is obtained as follows. Each LTS $\mathcal{S}_i$ occurring in ρ is mapped to a THS of the form $G_i\{(X \Rightarrow Y/\underline{X}) + H_i\}$ where the structure $G_i\{(X \Rightarrow Y/\underline{X})\}$ corresponds to the context in $\mathcal{S}_i$ (and any standard variable restriction), the structure of H_i is computed by an analysis of the principal formulae in $\mathcal{S}_i$, and the symbol $+$ is used informally to mean that the terms in H_i should be 'appropriately' inserted into the $G_i\{(X \Rightarrow Y/\underline{X})\}$. Finally, $\mathbb{LT}\rho$ is obtained by substituting $G_i\{(X \Rightarrow Y/\underline{X}) + H_i\}$ for each $\mathcal{S}_i$ in ρ. This procedure is best illustrated by an example.

Example 7.15 *We will first look at an LTS inference rule without any standard variable restrictions. Consider the inference rule $L\Box$:*

$$\frac{\mathcal{R}, \overbrace{Rxy}^{principal}, x : \Box A, \Gamma \Rightarrow \Delta, \overbrace{y : \Box A}^{principal} \qquad \mathcal{R}, x : \Box A, \overbrace{Rxy, y : A}^{principal}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \underbrace{Rxy}_{principal}, x : \Box A, \Gamma \Rightarrow \Delta} L\Box$$

Consider the left premise of $L\Box$. This LTS can be written

$$\underbrace{\mathcal{R}, x : \Box A, \Gamma \Rightarrow \Delta}_{context} \otimes \underbrace{Rxy \Rightarrow y : \Box A}_{principal}$$

Since there are no restrictions on Γ, Δ or $\mathcal{R}$ (apart from it being treelike) — we can write its image under $\mathbb{LT}$ as THS $G\{\Box A, X \Rightarrow Y/\underline{X}\}$ where the variable corresponding to $X \Rightarrow Y$ (under $\mathbb{LT}$) is x and G and $\underline{X}$ are arbitrary THS, and X and Y are arbitrary labelled formula multisets. The image H of the principal elements is $\mathbb{LT}(Rxy \Rightarrow y : \Box A)$ which is simply $\Rightarrow / \Rightarrow \Box A$. Then $G\{(\Box A, X \Rightarrow Y/\underline{X}) + H\}$ becomes

$$G\{(\Box A, X \Rightarrow Y/\underline{X}) + (\Rightarrow / \Rightarrow \Box A)\}$$

which can be written $G\{\Box A, X \Rightarrow Y/(U \Rightarrow V, \Box A/\underline{X}')\}$. Because G was arbitrary, we can simplify the notation by absorbing the $\underline{X}'$ term to get $G\{A, X \Rightarrow Y/U \Rightarrow V, \Box A\}$. Applying this procedure to the remaining premise sequent and the conclusion we obtain the rule $\mathbb{LT}(L\Box)$:

$$\frac{G\{\Box A, X \Rightarrow Y/(U \Rightarrow V, \Box A/\underline{X})\} \qquad G\{\Box A, X \Rightarrow Y/(A, U \Rightarrow V/\underline{X})\}}{G\{\Box A, X \Rightarrow Y/(U \Rightarrow V/\underline{X})\}}\ \mathbb{LT}(L\Box)$$

Example 7.16 *Now let us look at an inference rule with a standard variable restriction. Consider the following inference rule $R\Box$ constructed from LTS:*

$$\frac{\mathcal{R}, \overbrace{Rxy, y : \Box A}^{principal}, \Gamma \Rightarrow \Delta, \overbrace{y : A}^{principal}}{\mathcal{R}, \Gamma \Rightarrow \Delta, \underbrace{x : \Box A}_{principal}}\ R\Box$$

where y does not appear in the conclusion of the rule.

We can write the premise sequent as a product of context elements and principal elements:

$$\underbrace{\mathcal{R}, \Gamma \Rightarrow \Delta}_{context} \otimes \underbrace{Rxy, y : \Box A \Rightarrow y : A}_{principal}$$

Similarly we can write the conclusion sequent as follows:

$$\underbrace{\mathcal{R}, \Gamma \Rightarrow \Delta}_{context} \otimes \underbrace{\Rightarrow x : \Box A}_{principal}$$

Notice that $\mathcal{R}, \Gamma \Rightarrow \Delta$ is no longer completely arbitrary because of the variable restriction. The image of $\mathcal{R}, \Gamma \Rightarrow \Delta$ under $\mathbb{LT}$ can be written $G\{X \Rightarrow Y/\underline{X}\}$

where the variable assigned to $X \Rightarrow Y$ *is taken to be* x*, and* no traditional sequent in $\underline{X}$ is assigned the variable y. *Meanwhile we have*

$$\mathbb{LT}(Rxy, y : \Box A \Rightarrow y : A) = \overbrace{\quad \Rightarrow \quad}^{x} / \overbrace{\Box A \Rightarrow A}^{y}$$
$$\mathbb{LT}(\Rightarrow x : \Box A) = \underbrace{\quad \Rightarrow \Box A}_{x}$$

(the braces identify traditional sequents corresponding to the state variables). Then the image of the premise sequent is $G\{(X \Rightarrow Y/\underline{X}) + (\quad \Rightarrow \quad /\Box A \Rightarrow A)\}$ *which simplifies to* $G\{X \Rightarrow Y/(\Box A \Rightarrow A; \underline{X})\}$ *(notice that there is no 'mixing' of the sequent* $\Box A \Rightarrow A$ *and* $\underline{X}$ *because no traditional sequent in* $\underline{X}$ *is assigned the variable* y*). We can simplify the notation by absorbing the* $\underline{X}$ *to get* $G\{X \Rightarrow Y/\Box A \Rightarrow A\}$*, where it is implicit that* $G\{X \Rightarrow Y\}$ *does not contain a sequent corresponding to the variable* y*. Then the image of the conclusion sequent is simply* $G\{X \Rightarrow Y + (\quad \Rightarrow \Box A)\}$ *which becomes* $G\{X \Rightarrow Y, \Box A\}$*. So we ultimately obtain the rule* $\mathbb{LT}(R\Box)$*:*

$$\frac{G\{X \Rightarrow Y/\Box A \Rightarrow A\}}{G\{X \Rightarrow Y, \Box A\}}\ \mathbb{LT}(R\Box)$$

2. *Let* ρ *be a rule constructed from LTS.* The image of ρ under $\mathbb{TL}$ is obtained in a reverse of the procedure described above. We will also need to add a standard variable restriction of the form "z does not appear in the conclusion of $\mathbb{TL}\rho$" whenever a traditional sequent in ρ (corresponding to the state variable z under $\mathbb{TL}$) 'disappears' when moving from premise to conclusion. This definition requires further explication. For the purposes of clarity, rather than providing a technical definition, we will illustrate with an example.

Example 7.17 *Consider the THS rule* $\Box K_{gl}$*.*

$$\frac{G\{X \Rightarrow Y/\Box A \Rightarrow A\}}{G\{X \Rightarrow Y, \Box A\}}\ \Box K_{gl}$$

Notice that the sequent $\Box A \Rightarrow A$ *in the premise disappears in the conclusion. Formally, there is no traditional sequent* $\mathcal{S}$ *in the equivalent position* ($\sim$) *to* $\Box A \Rightarrow A$*. Let us compute* $\mathbb{TL}\Box K_{gl}$*. For the premise we have*

$$\mathbb{TL}_x G\{X \Rightarrow Y/\Box A \Rightarrow A\} = \mathbb{TL}_x G\{X \Rightarrow Y/\emptyset\} \otimes (v : \Box A \Rightarrow v : A)$$

where u *is the variable corresponding to the traditional sequent* $X \Rightarrow Y$ *in* $G\{X \Rightarrow Y/\emptyset\}$*, and* v *is the variable corresponding to* $\emptyset$*. Remember that in general* $x \neq$

u since x corresponds to the 'root sequent' in $G\{\emptyset\}$ and u corresponds to the occurrence of $\emptyset$. For the conclusion sequent we have

$$\mathbb{TL}_x G\{X \Rightarrow Y, \Box A\} = \mathbb{TL}_x G\{X \Rightarrow Y\} \otimes (u : \Box A)$$

where once again u is the variable corresponding to the traditional sequent $X \Rightarrow Y$ in $G\{X \Rightarrow Y\}$. However, notice that there is no traditional sequent in $G\{X \Rightarrow Y\}$ corresponding to the variable v.

Now, $\mathbb{TL}_x G\{X \Rightarrow Y/\emptyset\}$ can be written $\mathcal{R}, Ruv, \Gamma \Rightarrow \Delta$ where $\mathcal{R}$ is a relation set with root x and there is no occurrence of the variable v in $\mathcal{R}$, Γ or Δ. Then $\mathbb{TL}_x G\{X \Rightarrow Y\}$ becomes $\mathcal{R}, \Gamma \Rightarrow \Delta$. We thus obtain the LTS rule $\mathbb{TL}\Box K_{gl}$.

$$\frac{\mathcal{R}, Ruv, v : \Box A, \Gamma \Rightarrow \Delta, v : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, u : \Box A}\ \mathbb{TL}\Box K_{gl}$$

with the standard variable restriction "v does not occur in the the conclusion of $\mathbb{TL}\Box K_{gl}$".

We advise the reader that these procedures can usually be executed on sight.

Notice that the image $\mathbb{LT}(R\Box)$ of the LTS inference rule $R\Box$ (Example 7.16) is precisely the THS inference rule $\Box K_{gl}$. Also the image $\mathbb{TL}\Box K_{gl}$ of the THS inference rule $\Box K_{gl}$ (Example 7.17)) is precisely the LTS inference rule $R\Box$ (upto variable renaming). Thus $\mathbb{TL}(\mathbb{LT}(R\Box))$ is identical to $R\Box$ (upto variable renaming), and $\mathbb{LT}(\mathbb{TL}\Box K_{gl})$ is identical to $\Box K_{gl}$. An inspection of the above procedures reveal that this is true for arbitrary inference rules. That is, for any THS inference rule ρ we have that $\mathbb{TL}(\mathbb{LT}\rho)$ is identical to ρ (upto variable renaming) and for any LTS inference rule ρ we have $\mathbb{LT}(\mathbb{TL}\rho) \equiv \rho$.

Remark 7.18 *The THS inference rules we discussed above contain no side condition. The LTS inference rules we looked at were permitted to contain only the standard variable restrictions. Although we have defined the THS and LTS calculi to comprise of inference rules of these forms only, it is clear that inference rules for THS and LTS calculi may well contain other types of side conditions as well. For example, the cut-rule cut for THS we introduced in Section 7.1.2 imposes the condition that the sequents containing the cut-formulae must be in an equivalent position. Similarly, the cut-rule $\mathbb{TL}$cut for LTS imposes the condition that the the union of the relation sets of the premise sequents is treelike.*

Since the variety of side conditions is seemingly limitless, we have looked only at the conditions that we will encounter later. We do observe that the $\mathbb{TL}$cut rule can be obtained from the rule cut by a similar analysis to the above.

Calculi induced by $\mathbb{TL}$ and $\mathbb{LT}$

We can now construct a THS calculus from a LTS calculus and *vice versa*.

Definition 7.19 (calculi induced by $\mathbb{TL}$ and $\mathbb{LT}$) *If $\mathcal{C}$ is a THS calculus, then let $\mathbb{TL}\mathcal{C}$ denote the calculus consisting of the image of every initial sequent and inference rule in $\mathcal{C}$ under $\mathbb{TL}$.*

Next, if $\mathcal{C}$ is a LTS calculus, then let $\mathbb{LT}\mathcal{C}$ denote the calculus consisting of the image of every initial sequent and inference rule in $\mathcal{C}$ under $\mathbb{LT}$.

We are ready to prove some properties of these calculi.

Lemma 7.20 *Let $\mathcal{C}$ be a THS calculus. Then,*

(i) for any THS G, $\vdash_{\mathcal{C}} G$ iff $\vdash_{\mathcal{C}} \mathbb{LT}(\mathbb{TL}G)$

(ii) for any LTS $\mathcal{S}$, $\vdash_{\mathbb{TL}\mathcal{C}} \mathcal{S}$ iff $\vdash_{\mathcal{C}} \mathbb{TL}(\mathbb{LT}\mathcal{S})$

Proof. Immediate from Lemma 7.13 and Lemma 7.14. Q.E.D.

Lemma 7.21 *(i) The THS calculi $\mathcal{C}$ and $\mathbb{LT}(\mathbb{TL}\mathcal{C})$ derive exactly the same set of THS. (ii) The LTS calculi $\mathcal{C}$ and $\mathbb{TL}(\mathbb{LT}\mathcal{C})$ derive exactly the same set of LTS.*

Proof. Proof of (i). From Lemma 7.13 we know that $\mathbb{LT}(\mathbb{TL}G) \equiv G$ for any THS G, so the calculi $\mathcal{C}$ and $\mathbb{LT}(\mathbb{TL}\mathcal{C})$ are identical.

Proof of (ii). Consider the LTS calculi $\mathcal{C}$ and $\mathbb{TL}(\mathbb{LT}\mathcal{C})$. By the way we defined the mapping functions, it should be clear that the two calculi are identical modulo some renaming of state variables in the initial sequents, inference rules and standard variable restrictions. Lemma 7.14 assures us that the identical LTS are derivable so we are done. Q.E.D.

Lemma 7.22 *Let $\mathcal{C}$ be a THS calculus. Then,*

(i) for any THS G, $\vdash_{\mathcal{C}} G$ iff $\vdash_{\mathbb{TL}\mathcal{C}} \mathbb{TL}G$

(ii) for any LTS $\mathcal{S}$, $\vdash_{\mathbb{TL}\mathcal{C}} \mathcal{S}$ iff $\vdash_{\mathcal{C}} \mathbb{LT}\mathcal{S}$

In each case, the respective derivations in $\mathcal{C}$ and $\mathbb{TL}\mathcal{C}$ have identical height.

Proof. Proof of (i). Suppose that $\vdash^{\delta}_{\mathcal{C}} G$. We need to show that $\mathbb{TL}G$ is derivable in $\mathbb{TL}\mathcal{C}$. We can obtain a derivation δ' of $\mathbb{TL}G$ from δ by replacing every THS G' appearing in δ with $\mathbb{TL}G'$, and every rule ρ with $\mathbb{TL}\rho$ — by the definition of $\mathbb{TL}\mathcal{C}$, the resulting object is a derivation in the calculus $\mathbb{TL}\mathcal{C}$ with endsequent $\mathbb{TL}G$. In particular, notice that if ρ is a legal rule instance in $\mathcal{C}$, then $\mathbb{TL}\rho$ will obey any relevant standard variable restrictions in $\mathbb{TL}\mathcal{C}$. Moreover, by construction, δ and δ' have identical height.

Proof of (ii) is analogous to the above. Q.E.D.

Corollary 7.23 *For any THS calculus $\mathcal{C}$ and modal formula A we have $\vdash_{\mathcal{C}} \Rightarrow A$ iff $\vdash_{\mathbb{TL}\mathcal{C}} \Rightarrow x : A$. Moreover the translation from one derivation to the other is effective.*

Proof. Immediate from Lemma 7.22. Effectiveness of the translation follows from an inspection of the algorithm given there. Q.E.D.

7.3 Poggiolesi's $CSGL$ and Negri's $G3GL$

Recently Negri [52] presented a labelled sequent calculus $G3GL$ for provability logic GL as part of a systematic program to present labelled sequent calculi for modal logics. Subsequently Poggiolesi [58] presented the THS calculus $CSGL$ and proved soundness and completess for GL as well as syntactic cut-admissibility.

In this section, we begin by presenting the THS calculus $CSGL$. Using the procedures we described in the previous section, we then obtain the LTS calculus $\mathbb{TL}CSGL$. Next we present Negri's [52] labelled sequent calculus $G3GL$. A key aspect of this work is showing that we can import results from $G3GL$ to the LTS calculus $\mathbb{TL}CSGL$ (Theorem 7.27). Using Corollary 7.23 we then establish the following:

(i) We answer in full a question raised by Poggiolesi [58] regarding the exact relationship between the calculi $G3GL$ and $CSGL$. In particular, we demonstrate a translation between the two systems.

(ii) We show that $CSGL$ is sound and complete for provability logic GL and prove syntactic cut-admissibility. Poggiolesi [58] needs to invoke the semantics of the logic to prove soundness, and has to consider the many cases that arise in the cut-admissibility proof. The novelty of our proof is that we show how to use the existing soundness and completeness result and the

Initial THS**:** $G\{p, X \Rightarrow Y, p\}$ $\qquad$ $G\{\Box A, X \Rightarrow Y, \Box A\}$

Propositional rules:

$$\frac{G\{X \Rightarrow Y, A\}}{G\{\neg A, X \Rightarrow Y\}}\neg A \qquad \frac{G\{A, X \Rightarrow Y\}}{G\{X \Rightarrow Y, \neg A\}}\neg K$$

$$\frac{G\{A, B, X \Rightarrow Y\}}{G\{A \wedge B, X \Rightarrow Y\}}\wedge A \qquad \frac{G\{X \Rightarrow Y, A\} \qquad G\{X \Rightarrow Y, B\}}{G\{X \Rightarrow Y, A \wedge B\}}\wedge K$$

Modal rules:

$$\frac{G\{\Box A, X \Rightarrow Y/(U \Rightarrow V, \Box A/\underline{X})\} \qquad G\{\Box A, X \Rightarrow Y/(A, U \Rightarrow V/\underline{X})\}}{G\{\Box A, X \Rightarrow Y/(U \Rightarrow V/\underline{X})\}}\Box A_{gl}$$

$$\frac{G\{X \Rightarrow Y/\Box A \Rightarrow A\}}{G\{X \Rightarrow Y, \Box A\}}\Box K_{gl}$$

Special logical rule:

$$\frac{G\{\Box A, X \Rightarrow Y/(\Box A, U \Rightarrow V/\underline{X})\}}{G\{\Box A, X \Rightarrow Y/(U \Rightarrow V/\underline{X})\}}4$$

Table 7.1: The THS calculus $CSGL$ [58]

syntactic cut-admissibility result for $G3GL$. Since many proof-theoretical properties (invertibility of inference rules, for example) are preserved under the notational variants translation we can get these results for 'free' as well, alleviating the need for independent proofs of these results.

7.3.1 The calculus $CSGL$ and $\mathbb{TL}CSGL$

Poggiolesi's THS calculus $CSGL$ [58] is presented in Table 7.1. From this calculus we can construct the LTS calculus $\mathbb{TL}CSGL$ (Table 7.2).

For a relation term or labelled formula α, we introduce the left and right weakening rules:

$$\frac{\mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \alpha, \Gamma \Rightarrow \Delta}LW_{LTS} \qquad \frac{\mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta, \alpha}RW_{LTS}$$

with a restriction on α in each case to ensure that the conclusion sequent of each rule is a LTS whenever the premise sequent is a LTS.

Next we introduce the left and right contraction rules:

Initial *LTS***:** $\mathcal{R}, x : p, \Gamma \Rightarrow \Delta, x : p \qquad \mathcal{R}, x : \Box A, \Gamma \Rightarrow \Delta, x : \Box A$

Propositional rules:

$$\frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A}{\mathcal{R}, x : \neg A, \Gamma \Rightarrow \Delta} \neg A \qquad \frac{\mathcal{R}, x : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \neg A} \neg K$$

$$\frac{\mathcal{R}, x : A, x : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A \wedge B, \Gamma \Rightarrow \Delta} \wedge A \qquad \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \qquad \mathcal{R}, \Gamma \Rightarrow \Delta, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \wedge B} \wedge K$$

Modal rules:

$$\frac{\mathcal{R}, Rxy, x : \Box A, \Gamma \Rightarrow \Delta, y : \Box A \qquad \mathcal{R}, Rxy, x : \Box A, y : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, Rxy, x : \Box A, \Gamma \Rightarrow \Delta} \mathbb{TL}\Box A_{gl}$$

$$\frac{\mathcal{R}, Rxy, y : \Box A, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A} \mathbb{TL}\Box K_{gl}$$

where y does not appear in the conclusion of $\mathbb{TL}\Box K_{gl}$.

Special logical rule:

$$\frac{\mathcal{R}, Rxy, x : \Box A, y : \Box A, \Gamma \Rightarrow \Delta}{\mathcal{R}, Rxy, x : \Box A, \Gamma \Rightarrow \Delta} \mathbb{TL}4$$

Table 7.2: The *LTS* calculus $\mathbb{TL}CSGL$

$$\frac{\mathcal{R}, x : A, x : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A, \Gamma \Rightarrow \Delta} LC \qquad \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A, x : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A} RC$$

Lemma 7.24 *The rules LW_{LTS} and RW_{LTS} for weakening and the rules LC and RC for contraction are height-preserving syntactically admissible in $\mathbb{TL}CSGL$.*

Proof. Poggiolesi [58] shows that the corresponding *THS* rules (ie. the weakening and contraction rules under $\mathbb{LT}$) are height-preserving syntactically admissible in *CSGL*. By Theorem 7.22, the mapping between derivations in *CSGL* and $\mathbb{TL}CSGL$ is height-preserving. Hence the analogous results apply to $\mathbb{TL}CSGL$ too, so we are done. Q.E.D.

7.3.2 Negri's calculus $G3GL$

If we compare the $\mathbb{TL}CSGL$ calculus with Negri's labelled sequent calculus $G3GL$ for provability logic GL, the only differences are that

(i) The condition that the relation set of every labelled sequent is treelike is removed, and

(ii) $G3GL$ does not contain the 'Special logical rule' $\mathbb{TL}4$, and

(iii) $G3GL$ contains the following initial sequent (below left) and inference rule (below right):

$$\mathcal{R}, Rxx, \Gamma \Rightarrow \Delta \ (\mathit{Irref}) \qquad\qquad \frac{\mathcal{R}, Rxz, Rxy, Ryz, \Gamma \Rightarrow \Delta}{\mathcal{R}, Rxy, Ryz, \Gamma \Rightarrow \Delta}\ (\mathit{Trans})$$

For those rules in $G3GL$ that also occur in $\mathbb{TL}CSGL$, we will use the rule labelling of $\mathbb{TL}CSGL$. For example, we write $\mathbb{TL}\Box K_{gl}$ instead of the label $R\Box{-}L$ used in [52]. Strictly speaking, the calculus $G3GL$ also contains rules for the disjunction and implication connectives. Since these connectives can be written in terms of negation and conjunction, for our purposes there is no harm in this omission. The weakening rules LW and RW are obtained from LW_{LTS} and RW_{LTS} by dropping the term restriction on α. The rules LW and RW as well as the contraction rules LC and RC are height-preserving admissible in $G3GL$ [52].

Theorem 7.25 (Negri) *The labelled sequent calculus $G3GL$ (i) has syntactic cut-admissibility, and (ii) presents the logic GL.*

Proof. See Negri [52]. Q.E.D.

7.3.3 Results

Poggiolesi [58] states (in the "Conclusions and further work" section):

> "As it has probably already emerged in the previous sections, $CSGL$ is quite similar to Negris calculus $G3GL$ [see [52]]: indeed, except for the rule 4 that only characterizes $CSGL$, the propositional and modal rules of the two calculi seem to be based on a same intuition. Given this situation, a question naturally arises: what is the exact relation between the two calculi? Is it possible to find a translation from the THS calculi to the labeled calculi and vice versa?"

We will need the following lemma.

Lemma 7.26 *The following rule is syntactically admissible in $\mathbb{TL}CSGL$:*

$$\frac{Rxy, Ryz, Rxu, \mathcal{R}, X \Rightarrow Y}{Rxy, Ryz, \mathcal{R}[z/u], X[z/u] \Rightarrow Y[z/u]}\ (\mathit{Trans}')$$

where u does not appear in the premise sequent.

Proof. The argument is a standard induction on the height of the premise derivation. For example, suppose that the last rule in the premise derivation is $\mathbb{TL}4$, so we have

$$\dfrac{\begin{array}{c}\vdots\\ Rxy, Ryz, Rxu, \mathcal{R}, x : \Box A, u : \Box A, X \Rightarrow Y\end{array}}{\mathcal{R}, Rxy, Ryz, Rxu, \mathcal{R}, x : \Box A, X \Rightarrow Y}\ \mathbb{TL}4$$

From $Rxy, Ryz, Rxu, \mathcal{R}, x : \Box A, u : \Box A, X \Rightarrow Y$, by the induction hypothesis we can obtain a derivation of $Rxy, Ryz, \mathcal{R}[z/u], x : \Box A, z : \Box A, X[z/u] \Rightarrow Y[z/u]$. Then

$$\dfrac{\dfrac{\dfrac{Rxy, Ryz, \mathcal{R}[z/u], x : \Box A, z : \Box A, X[z/u] \Rightarrow Y[z/u]}{Rxy, Ryz, \mathcal{R}[z/u], x : \Box A, y : \Box A, z : \Box A, X[z/u] \Rightarrow Y[z/u]}\ LW_{LTS}}{Rxy, Ryz, \mathcal{R}[z/u], x : \Box A, y : \Box A, X[z/u] \Rightarrow Y[z/u]}\ \mathbb{TL}4}{Rxy, Ryz, \mathcal{R}[z/u], x : \Box A, X[z/u] \Rightarrow Y[z/u]}\ \mathbb{TL}4$$

This is the required derivation.

Next, suppose that the last rule in the premise derivation is $\mathbb{TL}\Box A_{gl}$, so

$$\dfrac{\begin{array}{c}\vdots\\ Rxy, Ryz, Rxu, \mathcal{R}, x : \Box A, X \Rightarrow Y, u : \Box A\end{array} \qquad \begin{array}{c}\vdots\\ Rxy, Ryz, Rxu, \mathcal{R}, x : \Box A, u : A, X \Rightarrow Y\end{array}}{Rxy, Ryz, Rxu, \mathcal{R}, x : \Box A, X \Rightarrow Y}\ \mathbb{TL}\Box A_{gl}$$

From the premise sequents of $\mathbb{TL}\Box A_{gl}$, by the induction hypothesis we can obtain derivations of the following two sequents:

$$Rxy, Ryz, \mathcal{R}[z/u], x : \Box A, X[z/u] \Rightarrow Y[z/u], z : \Box A$$
$$Rxy, Ryz, \mathcal{R}[z/u], x : \Box A, z : A, X[z/u] \Rightarrow Y[z/u]$$

Applying $LW_{LTS}(y : \Box A)$ to each sequent we get, respectively,

$$Rxy, Ryz, \mathcal{R}[z/u], x : \Box A, y : \Box A, X[z/u] \Rightarrow Y[z/u], z : \Box A$$
$$Rxy, Ryz, \mathcal{R}[z/u], x : \Box A, y : \Box A, z : A, X[z/u] \Rightarrow Y[z/u]$$

Applying $\mathbb{TL}\Box A_{gl}$ to these sequents we get

$$\dfrac{\dfrac{\dots}{Rxy, Ryz, \mathcal{R}[z/u], x : \Box A, y : \Box A, X[z/u] \Rightarrow Y[z/u]}\ \mathbb{TL}\Box A_{gl}}{Rxy, Ryz, \mathcal{R}[z/u], x : \Box A, X[z/u] \Rightarrow Y[z/u]}\ \mathbb{TL}4$$

This is the required derivation. The other cases are similar. Q.E.D.

The LTS rule (*Trans'*) corresponds to the following tree-hypersequent rule under the mapping $\mathbb{LT}$:

$$\frac{G\{X \Rightarrow Y/(U \Rightarrow V/S \Rightarrow T/\underline{X});(L \Rightarrow M/\underline{X}')\}}{G\{U \Rightarrow V/(U, L \Rightarrow V, M/\underline{X};\underline{X}')\}}$$

The main result in this section is the following result which connects Negri's labelled sequent calculus $G3GL$ and the labelled tree sequent calculus $\mathbb{TL}CSGL$. Together with Corollary 7.23, this completely answers the question posed in Poggiolesi [58].

Theorem 7.27 *For any modal formula A, $\vdash_{\mathbb{TL}CSGL} \Rightarrow x : A$ iff $\vdash_{G3GL} \Rightarrow x : A$. Moreover the translation between the corresponding derivations is effective.*

Proof. The translation is effective due to the constructive transformations that we will use.

For the left-to-right direction it suffices to show that $\mathbb{TL}4$ is syntactically admissible in $G3GL$. First, working in $G3GL$ (note that because we are working in a labelled sequent calculus but not a LTS calculus, the relation sets that occur in the derivation need not be treelike), observe that:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{z : \Box A \Rightarrow z : \Box A}{Rxz, x : \Box A, z : \Box A \Rightarrow z : A, z : \Box A} \qquad \dfrac{z : A \Rightarrow z : A}{Rxz, z : A, x : \Box A, z : \Box A \Rightarrow z : A}}{Rxz, x : \Box A, z : \Box A \Rightarrow z : A}\,\mathbb{TL}\Box A_{gl}}{Rxy, Ryz, Rxz, x : \Box A, z : \Box A \Rightarrow z : A}\,LW}{Rxy, Ryz, x : \Box A, z : \Box A \Rightarrow z : A}\,(Trans)}{Rxy, x : \Box A \Rightarrow y : \Box A}\,\mathbb{TL}\Box K_{gl}$$

Suppose that we are given a derivation of the premise $\mathcal{R}, Rxy, y : \Box A, x : \Box A, X \Rightarrow Y$ of $\mathbb{TL}4$. From the cut-rule and the above derivation we get a derivation of $\mathcal{R}, Rxy, x : \Box A, x : \Box A, X \Rightarrow Y$. By Theorem 7.25 we can obtain a cutfree derivation of this sequent. Since the left contraction rule LC is admissible in $G3GL$ [52], we get $Rxy, x : \Box A, X \Rightarrow Y$ and thus $\mathbb{TL}4$ is syntactically admissible in $G3GL$.

For the right-to-left direction, suppose that $\vdash^{\delta}_{G3GL} \Rightarrow x : A$ for some derivation δ. We will show how to transform δ into a derivation in $\mathbb{TL}CSGL$ of the same sequent.

We observe first that δ does not contain any occurrences of the initial sequent (*Irref*). To see this, observe that in any $G3GL$ derivation, viewed downwards, a state variable occurrence y can disappear from premise sequent to conclusion sequent only via the $\mathbb{TL}\Box K_{gl}$ rule — all the other rules preserve the set of state variables in the relation set. The side condition of $\mathbb{TL}\Box K_{gl}$ enforces that the variable y appears exactly *once* in the relation set of the premise of $\mathbb{TL}\Box K_{gl}$ (in a term of the form Rxy for some variable x distinct from y). Now, if δ contains the initial sequent (*Irref*) $\mathcal{R}, Ryy, X \Rightarrow Y$, then the relation set of the initial sequent contains at least two occurrences of y. It follows that the relation set of every sequent below this initial sequent in δ will contain at least two occurrences of y, contradicting the fact that the endsequent of δ has the form $\Rightarrow x : A$.

To obtain a derivation of $\Rightarrow x : A$ in $\mathbb{TL}CSGL$, it suffices to show that we can transform δ by eliminating all instances of (*Trans*) while ensuring that every sequent in the transformed derivation is a LTS.

Obviously the endsequent $\Rightarrow x : A$ of δ is a LTS. Working upwards from the endsequent towards the initial sequents, by inspection, every rule with the exception of the (*Trans*) rule has the property that if the conclusion sequent is a LTS, then so are the premise sequents. Working upwards from the endsequent, suppose we encounter the following occurrence of the (*Trans*) rule, where the conclusion sequent $\mathcal{R}, Rul, Rlv, X \Rightarrow Y$ is a LTS:

$$\begin{array}{c} \vdots \\ \dfrac{Rul, Rlv, Ruv, \mathcal{R}, X \Rightarrow Y}{Rul, Rlv, \mathcal{R}, X \Rightarrow Y}\ (\textit{Trans}) \\ \vdots \\ \Rightarrow x : A \end{array}$$

Clearly the premise sequent $\mathcal{R}, Rul, Rlv, Ruv, X \Rightarrow Y$ of (*Trans*) is *not* a LTS because the relation set is not treelike. We *claim* that we can transform the derivation of $\mathcal{R}, Rul, Rlv, Ruv, X \Rightarrow Y$ into a derivation of $\mathcal{R}, Rul, Rlv, Rus, X \Rightarrow Y$ where s is a state variable not appearing in the original derivation. Then applying (*Trans'*) we can obtain the conclusion sequent $\mathcal{R}, Rul, Rlv, X \Rightarrow Y$ having at the same time deleted the original occurrence of (*Trans*). By repeatedly applying this argument we can obtain a derivation of $\Rightarrow x : A$ containing no occurrences of (*Trans*). Because the (*Trans'*) rule is syntactically admissible in $\mathbb{TL}CSGL$ the result follows.

Proof of the claim. The proof is by induction on the height of the premise derivation of $Rul, Rlv, Ruv, \mathcal{R}, X \Rightarrow Y$. If the premise derivation is an initial sequent of the form $Rul, Rlv, Ruv, \mathcal{R}, q : p, X \Rightarrow Y, q : p$ then we can replace this with $Rul, Rlv, Rus, \mathcal{R}, q : p, X \Rightarrow Y, q : p$ for some fresh state variable s. This is clearly a LTS so we are done.

Now suppose that the last rule in this derivation is $\mathbb{TL}\Box A_{gl}$ so we have

$$\dfrac{Rul, Rlv, Ruv, \mathcal{R}, u : \Box C, X \Rightarrow Y, v : \Box C \qquad Rul, Rlv, Ruv, \mathcal{R}, u : \Box C, v : C, X \Rightarrow Y}{Rul, Rlv, Ruv, \mathcal{R}, u : \Box C, X \Rightarrow Y}\ \mathbb{TL}\Box A_{gl}$$

By the induction hypothesis we can obtain derivations of the following sequents:

$$Rul, Rlv, Rus, \mathcal{R}, u : \Box C, X \Rightarrow Y, v : \Box C$$
$$Rul, Rlv, Rus, \mathcal{R}, u : \Box C, v : C, X \Rightarrow Y$$

for a variable s not appearing in the original derivation. Applying $LW_{LTS}(l : \Box C)$ to each sequent we get, respectively

$$Rul, Rlv, Rus, \mathcal{R}, u : \Box C, l : \Box C, X \Rightarrow Y, v : \Box C$$
$$Rul, Rlv, Rus, \mathcal{R}, u : \Box C, l : \Box C, v : C, X \Rightarrow Y$$

Applying $\mathbb{TL}\Box A_{gl}$ to these sequents we get

$$\dfrac{\dfrac{\ldots}{Rul, Rlv, Rus, \mathcal{R}, u : \Box C, l : \Box C, X \Rightarrow Y}\ \mathbb{TL}\Box A_{gl}}{Rul, Rlv, Rus, \mathcal{R}, u : \Box C, X \Rightarrow Y}\ \mathbb{TL}4$$

By admissibility of (*Trans′*) we obtain a derivation of $Rul, Rlv, \mathcal{R}, u : \Box C, X \Rightarrow Y$ as required. The other cases are straightforward. Q.E.D.

Negri uses the initial sequent $\mathcal{R}, Rxx, \Gamma \Rightarrow \Delta$ (*Irref*) in the proof of cut-admissibility for $G3GL$ to argue that there cannot be a labelled sequent with a relation set (in our terminology) containing $\{Rxx_1, Rx_1x_2, \ldots Rx_nx\}$ (a 'loop'). We saw above that (*Irref*) cannot occur in any $G3GL$ derivation of a sequent of the form $\Rightarrow x : A$. By definition, the relation set of a LTS can never contain such a loop so there is no initial LTS in $\mathbb{TL}CSGL$ corresponding to (*Irref*) in $G3GL$.

Theorem 7.28 *The calculus CSGL (i) presents the logic GL, and (ii) has syntactic cut-admissibility.*

Proof. Follows from Theorem 7.25 using Corollary 7.23 and Theorem 7.27. Q.E.D.

Note that although the above proofs make use of the results for $G3GL$ [52], these results are syntactic because the proofs for $G3GL$ are syntactic.

It is not clear if the cutfree LTS calculus $(\mathbb{TL}CSGL)^s$ obtained from $\mathbb{TL}CSGL$ by substituting the rule $\mathbb{TL}4$ with the rule (*Trans′*) is sound and complete for GL.

7.4 Generalised Hein's scheme

A cutfree system is said to be *modular* (for some set) if there are inference rules corresponding to each modal axiom and the addition of any combination of these rules to the base calculus is sound and complete for the corresponding logic. Kracht's Display Theorem I [39] (see Section 5.1) specifies how to construct for every primitive tense formula t_i a set $\{\rho_{ij}\}_j$ of display rules, so that any extension $Kt \oplus \{t_i\}_{i \in T}$ of the basic tense logic Kt by primitive tense formulae $\{t_i\}_{i \in T}$ is properly displayed by the display calculus $DLM + \{\rho_{ij}\}_{ij}$. It follows that these calculi are modular for the set of primitive tense formulae. In a similar vein, Negri [52] has shown how to construct a labelled sequent inference rule corresponding to a given geometric formulae such that any modal logic with frame semantics described by some finite number of geometric formulae can be presented by extending the labelled sequent calculus $G3K$ for K with the corresponding rules. Note that although Negri's characterisation is semantic in the sense that the applicable logics need to be specified in terms of their frame semantics, for geometric formulae belonging to the Kracht fragment — such as the class $\mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$ (see Section 5.2.3), for example — we could just as easily characterise these logics syntactically using the modal (Sahlqvist) correspondent of these formulae (see Section 4.2). Indeed, in the reverse direction, should we wish, we could express Display Theorem I in semantic terms making use of the semantic characterisation of primitive tense formulae (see Section 5.1.2).

These results aside, the construction of modular cutfree systems for Gentzen sequent calculi and its variants is generally regarded as a challenging problem. In the context of nested sequents, Brünnler and Straßburger [12] have suggested that the key to obtaining modular systems might be the use of nested sequent *structural rules* (these are nested sequent inference rules containing no occurrences of formulae). Using such rules, Brünnler and Straßburger have presented modular cutfree nested sequent systems for the logics $K \oplus X$ where $X \subseteq \{T, 4, B, 5, D\}$ (see Table 7.3).

Since nested sequents are a notational variant of LTS, these results can be viewed from a LTS perspective. Let us use the term *relation rule*[2] to mean a labelled sequent inference rule containing no schematic formula variables. Then it is easy to verify that the class of nested sequent structural rules corresponds exactly to the class of relation rules. Hein [35] has proposed a method for ob-

[2] We have reserved the term 'structural rule' for the contraction and weakening rules of the calculus.

Name	Axiom	Frame property
T	$\Box p \supset p$	$\forall x.Rxx$ reflexivity
4	$\Box p \supset \Box\Box p$	$\forall xyz(Rxy \wedge Ryz \rightarrow Rxz)$ transitivity
5	$\Diamond\Box p \supset \Box p$	$\forall xyz(Rxy \wedge Rxz \rightarrow Ryz)$ euclideanness
B	$p \supset \Box\Diamond p$	$\forall xy(Rxy \rightarrow Ryx)$ symmetry
3	$\Box(\Box p \supset q) \vee \Box(\Box q \supset p)$	$\forall xyz(Rxy \wedge Rxz \rightarrow (Ryz \vee Rzy))$ connectedness
D	$\Box p \supset \Diamond p$	$\forall x\exists y.Rxy$ seriality
2	$\Diamond\Box p \supset \Box\Diamond p$	$\forall xyz(Rxy \wedge Rxz \rightarrow \exists w(Ryw \wedge Rzw))$ directedness

Table 7.3: Some modal axioms and their global first-order frame correspondents. The formulae T, 4, B and 5 are 3/4 Lemmon-Scott axioms.

taining LTS inference rules for formulae belonging to the class of so-called 3/4 Lemmon-Scott axioms, by 'unfolding' the problematic relation sets in Negri's rules. Hein then shows how to obtain LTS calculi for logics over K axiomatised by these formulae. However, these calculi contain the cut-rule and no proof of cut-elimination is presented.

Here we first generalise Hein's inference rules and then obtain cut-elimination and a proof of modularity for some of the resulting calculi, making use of the results of Brünnler and Straßburger [12]. This indicates that the Generalised Hein's scheme for obtaining LTS inference rules from 3/4 Lemmon-Scott axioms seems to 'work' in the sense that it leads to cutfree systems for at least some of these axioms. This is interesting because it raises the possibility that we may be able to obtain an algorithm for obtaining LTS calculi (and hence nested sequent, THS calculi) for 3/4 Lemmon-Scott logics through a suitable generalisation of the cut-elimination proof. In particular, observe that Brünnler and Straßburger do not explain how to obtain the inference rules for logics outside $\{T, 4, B, 5, D\}$ (see Table 7.3). Thus the incentive for working in the LTS setting is Negri's general result which identifies those logics that can be presented via cutfree labelled sequent calculi, and the Generalised Hein's scheme which shows how to construct LTS inference rules for the 3/4 Lemmon-Scott axioms. Although it is true that the Generalised Hein's scheme (though not necessarily Negri's result) can be ported to the nested sequent setting, because this approach is heavily reliant on the correspondence theory between modal formulae and their first-order correspondents, in terms of argument and notation it is helpful here that the LTS notation makes more visible the semantic content in the inference rule, as compared to nested sequents where the semantic content can be obscured by the [ ] notation.

Following Negri [52], let us examine how to construct a relation rule from a first-order formula.[3] Recall the notation we introduced in Section 4.2.1.

$$\mathcal{R}^1uv := Ruv$$
$$\mathcal{R}^{k+1}uv = (\exists u_{k+1} \rhd u)\mathcal{R}^k u_{k+1}v \quad (k \geq 1)$$

Let α be a formula in $\mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$ (see Section 5.2.3). Then α has the form

$$(\forall u_1 \rhd v_1) \dots (\forall u_m \rhd v_m)(\exists^{r'} \bar{y}_1\, M_1 \vee \dots \vee \exists^{r'} \bar{y}_n\, M_n) \tag{7.1}$$

where, the variables in $\{x\} \cup \{u_i\}$ are called the inherently universal variables, and by definition the restrictor v_i of each u_i is either x or some u_j $(j < i)$, and each M_i is a conjunction of terms of the form $l = l$, $l = m$ and $\mathcal{R}^k lm$ where in $l = m$ and $\mathcal{R}^k lm$ at least one of l, m is an inherently universal variable.

For simplicity of notation, let us look at a special case of (7.1) where $\bar{y}_i$ is a single variable y_i for each i so $\exists^{r'} \bar{y}_i$ denotes $(\exists y_i \rhd u'_i)$ for some $u'_i \in \{u_i\}$, and M_i is a conjunction of terms $\bar{Q}_i = \{Q_{i1}, \dots, Q_{ik_i}\}$, where each Q_{ij} has the form $R^k lm$ where at least one of l, m is an inherently universal variable. Moreover, let P_i denote the term Ru_iv_i and $\bar{P} = \{P_i\}$. By expanding the restricted quantifier notation, (7.1) can be written as the following first-order formula

$$\forall u_1 \dots \forall u_m (P_1 \wedge \dots P_m \rightarrow \exists y_1 (Ru'_1 y_1 \wedge M_1) \vee \dots \exists y_n (Ru'_n y_n \wedge M_n)) \tag{7.2}$$

Example 7.29 *Consider the formula* $(\forall x \rhd y)(\forall x \rhd z)(\exists w \rhd y)(Rzw) \in \mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$. *We can write this formula as*

$$\forall yz(Rxy \wedge Rxz \rightarrow \exists w(Ryw \wedge Rzw))$$

For a formula of the form (7.2), let R_α denote the following labelled sequent schematic rule, where $\mathcal{R}$ is a schematic relation set variable, and X and Y are schematic labelled multiset variables.

$$\frac{\bar{Q}_1[z_1/y_1], \bar{P}, \mathcal{R}, X \Rightarrow Y \qquad \dots \qquad \bar{Q}_n[z_n/y_n], \bar{P}, \mathcal{R}, X \Rightarrow Y}{\bar{P}, \mathcal{R}, X \Rightarrow Y}\, R_\alpha$$

with the standard variable restriction that the variables $z_1, \dots z_n$ do not appear in the conclusion sequent. Since $\mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$ is a subclass of the modal Kracht

[3] Although Negri's result applies to the class of geometric formulae ($\supseteq \mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$), we will work with $\mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$ because we know that this class belongs to the Kracht fragment.

Initial LTS**:** $\mathcal{R}, x : p, \Gamma \Rightarrow \Delta, x : p$

Propositional rules:

$$\frac{\mathcal{R}, x : A, \Gamma \Rightarrow \Delta \qquad \mathcal{R}, x : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A \vee B, \Gamma \Rightarrow \Delta} L\vee \qquad \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \vee B} R\vee$$

$$\frac{\mathcal{R}, x : A, x : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A \wedge B, \Gamma \Rightarrow \Delta} L\wedge \qquad \frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \qquad \mathcal{R}, \Gamma \Rightarrow \Delta, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \wedge B} R\wedge$$

$$\frac{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \qquad \mathcal{R}, x : B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : A \supset B, \Gamma \Rightarrow \Delta} L\supset \qquad \frac{\mathcal{R}, x : A, \Gamma \Rightarrow \Delta, x : B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A \supset B} R\supset$$

Modal rules:

$$\frac{\mathcal{R}, Rxy, x : \Box A, y : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, Rxy, x : \Box A, \Gamma \Rightarrow \Delta} L\Box \qquad \frac{\mathcal{R}, Rxy, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A} R\Box$$

$$\frac{\mathcal{R}, Rxy, y : A, \Gamma \Rightarrow \Delta}{\mathcal{R}, x : \Diamond A, \Gamma \Rightarrow \Delta} L\Diamond \qquad \frac{\mathcal{R}, Rxy, \Gamma \Rightarrow \Delta, x : \Diamond A, y : A}{\mathcal{R}, Rxy, \Gamma \Rightarrow \Delta, x : \Diamond A} R\Diamond$$

Table 7.4: The labelled sequent calculus $G3K$ [52]. The rules $\Box R$ and $\Diamond L$ have the condition that y does not appear in the conclusion sequent.

formulae, $\forall x \alpha$ globally corresponds to some modal (Sahlqvist) formula M_α. In Table 7.3 we present some well-known modal axioms and the global first-order frame correspondents (expanded from their Kracht form).

The labelled cutfree sequent calculus $G3K$ for K is presented in Table 7.4.

Theorem 7.30 (Negri) *Suppose that* $\{\alpha_i\} \subset \mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$ *and let* M_{α_i} *denote the modal correspondent of* α_i*. Then* $G3K + \{R_{\alpha_i}\}$[4] *is sound and complete for the logic* $K \oplus \{M_{\alpha_i}\}$ *and the cut-rule is syntactically admissible. In notation,*

$$\vdash_{G3K+\{R_{\alpha_i}\}} \Rightarrow x : A \textit{ iff } A \in K \oplus \{M_{\alpha_i}\}$$

Proof. See Negri [52]. Q.E.D.

Here are the labelled sequent rules for reflexivity (*Ref*), transitivity (*Trans*), symmetry (*Sym*) and seriality (*Ser*) obtained according to Negri's scheme:

$$\frac{Rxx, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} (\mathit{Ref}) \qquad \frac{Rxy, Ryz, Rxz, \mathcal{R}, \Gamma \Rightarrow \Delta}{Rxy, Ryz, \mathcal{R}, \Gamma \Rightarrow \Delta} (\mathit{Trans})$$

$$\frac{Rxy, Ryx, \mathcal{R}, \Gamma \Rightarrow \Delta}{Rxy, \mathcal{R}, \Gamma \Rightarrow \Delta} (\mathit{Sym}) \qquad \frac{Rxy, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta} (\mathit{Ser})$$

[4] Negri also stipulates the addition of more rules to the calculus in order to obtain height-preserving admissibility of the contraction rule in the calculus — each additional rule corresponds to a contraction instance for any duplication of terms in $\bar{P}$ appearing in the premise sequents of R_{α_i}.

where (Ser) has the condition that y does not appear in the conclusion sequent.

In general it is not clear how to obtain a LTS calculus for the logic $K \oplus \{M_{\alpha_i}\}$ from the labelled sequent calculus $G3K + \{R_{\alpha_i}\}$ presenting this logic. This is because the relation rules $\{R_{\alpha_i}\}$ obtained according to Negri's scheme either becomes redundant in a LTS calculus or fail to preserve LTS from premise to conclusion. For example, the rule (Ref) is redundant in a LTS calculus because the premise sequent can never be a LTS due to the presence of the Rxx term. Hence the LTS calculus $G3K + (Ref)$ presents the logic K (and not $K \oplus \Box p \supset p$). An exception is the rule (Ser). Clearly the LTS calculus $G3K + (Ser)$ is sound for $K \oplus \Box p \supset \Diamond p$ because of soundness of the labelled sequent calculus $G3K + (Ser)$. To prove completeness, we can argue that whenever the sequent $\Rightarrow x : A$ is derivable in the labelled sequent calculus $G3K + (Ser)$ we can transform that derivation into a derivation of $\Rightarrow x : A$ in the LTS calculus. Furthermore, syntactic cut-admissibility for the LTS calculus then follows immediately from syntactic cut-admissibility for the labelled sequent calculus (Theorem 7.30). We omit the details.

Hein [35] has proposed a method for constructing LTS inference rules corresponding to the formula $\forall x \alpha(x)$ for $\alpha \in \mathbf{A}^{\mathbf{r}'}\mathbf{f}\exists^{\mathbf{r}'}\mathbf{x}$ when $\forall x\alpha(x)$ can be written in the following form $(h, i, j \geq 0)$:

$$\forall xyz(\mathcal{R}^h xy \wedge \mathcal{R}^j xz \to \mathcal{R}^i yz) \tag{7.3}$$

It is well-known [43] that a Kracht formula of the above form corresponds to the following modal formula, which is a special case of the Lemmon-Scott axioms $\{\Diamond^h \Box^i p \supset \Box^j \Diamond^k p \mid h, i, j, k \geq 0\}$:

$$\Diamond^h \Box^i p \supset \Box^j p \tag{7.4}$$

For this reason, Hein refers to the class of formulae of the form (7.4) as the *3/4 Lemmon-Scott axioms*. The formulae T, 4, B and 5 in Table 7.3 are examples of 3/4 Lemmon-Scott axioms.

For any 3/4 Lemmon-Scott axiom M, let M_H be the following LTS rule where $R^0 lm$ abbreviates $l = m$ and $R^{k+1} lm$: abbreviates $Rll_1, \ldots, Rl_{k-1}l_k, Rl_k m$.

$$\frac{R^h xy, R^j xz, R^i yu, \mathcal{R}, X \Rightarrow Y}{R^h xy, R^j xz, \mathcal{R}, X[z/u] \Rightarrow Y[z/u]}\, M_H$$

Then the rule M_H has the condition that the variables $u, y_1, \ldots, y_{i-1}$ in $R^i yu$ do not appear in the conclusion sequent. A word about the notation here: $\mathcal{R}^k lm$ abbreviates a term from the first-order frame language and possibly contains the existential operator $\exists$ whereas $R^k lm$ is a set of relation terms.

Theorem 7.31 (Hein) *Let $\{M_i\}_{i\in\mathcal{M}}$ be a set of 3/4 Lemmon-Scott axioms, and let $\{M_{iH}\}_{i\in\mathcal{M}}$ be the corresponding LTS rules obtained as described above. Then the LTS calculus $G3K + \{M_{iH}\}_{i\in\mathcal{M}} + cut_{LTS}$ is sound and complete for $K \oplus \{M_i\}_{i\in\mathcal{M}}$.*

Proof. See Hein [35]. Q.E.D.

Notice that in the LTS rule M_H above, the condition that u does not appear in the conclusion sequent enforces that u does not appear in $\mathcal{R}$. This condition appears to be unnecessarily strong and so we propose the following *Generalised Hein's scheme* for obtaining the following LTS rule $M_{H'}$ from a 3/4 Lemmon-Scott axiom M of the form (7.3):

$$\frac{R^h xy, R^j xz, R^i yu, \mathcal{R}, X \Rightarrow Y}{R^h xy, R^j xz, \mathcal{R}[z/u], X[z/u] \Rightarrow Y[z/u]}\ M_{H'}$$

once again with the condition that the variables $u, y_1, \ldots, y_{i-1}$ in $R^i yu$ do not appear in the conclusion sequent. Notice that, unlike M_H, the conclusion sequent of $M_{H'}$ is a LTS even if $\mathcal{R}$ contains an occurrence of u.

Theorem 7.32 *Let $\{M_i\}_{i\in\mathcal{M}}$ be a set of 3/4 Lemmon-Scott axioms, and let $\{M_{iH'}\}_{i\in\mathcal{M}}$ be the corresponding LTS rules obtained by the Generalised Hein's scheme. Then the LTS calculus $G3K + \{M_{iH'}\}_{i\in\mathcal{M}} + cut_{LTS}$ is sound and complete for $K \oplus \{M_i\}_{i\in\mathcal{M}}$.*

Proof. Since each LTS rule $M_{iH'}$ subsumes M_{iH}, completeness is immediate from Theorem 7.31. We could prove soundness following Hein's [35] model-theoretic argument for M_{iH}. However it is simpler to observe that the calculus $G3K + \{M_{iH'}\}_{i\in\mathcal{M}}$ can be embedded in the labelled sequent calculus $G3K + \{R_i\}_{i\in\mathcal{M}}$ where R_i is the relation rule corresponding to M_i and obtained using Negri's scheme. The result follows from Theorem 7.30. Q.E.D.

By itself, this result is really not that strong because the theorem does not resolve the central issue of whether the cut-rule can be eliminated from $G3K+\{\rho_{M_i}\}_{i\in\mathcal{M}}+cut_{LTS}$. Hein conjectures that cut-elimination holds but does not present a proof. In the remainder of this section we will examine this issue further.

For the formulae T, 4, B and 5, the LTS rules obtained according to the Generalised Hein's scheme are denoted, respectively, as $T_{H'}$, $4_{H'}$, $B_{H'}$ and $5_{H'}$ (Table 7.5). For each of the formulae in T, 4, B, 5 and D, the corresponding

$$\frac{Rxy, \mathcal{R}, X \Rightarrow Y}{\mathcal{R}[x/y], X[x/y] \Rightarrow Y[x/y]}\, T_{H'} \qquad \frac{Rxy, Ryz, Rxu, \mathcal{R}, X \Rightarrow Y}{Rxy, Ryz, \mathcal{R}[z/u], X[z/u] \Rightarrow Y[z/u]}\, 4_{H'}$$

$$\frac{Rxy, Ryz, \mathcal{R}, X \Rightarrow Y}{Rxy, \mathcal{R}[x/z], X[x/z] \Rightarrow Y[x/z]}\, B_{H'} \qquad \frac{Rxy, Rxz, Ryu, \mathcal{R}, X \Rightarrow Y}{Rxy, Rxz, \mathcal{R}[z/u], X[z/u] \Rightarrow Y[z/u]}\, 5_{H'}$$

Table 7.5: The Generalised Hein's scheme LTS rules for some 3/4 Lemmon-Scott axioms.

Nested sequent rule	Corresponding LTS rule
$\dfrac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}}\, B\dot{S}$	$\dfrac{Rxy, \mathcal{R}, X \Rightarrow Y}{\mathcal{R}[x/y], X[x/y] \Rightarrow Y[x/y]}\, T_{BS}$
$\dfrac{\Gamma\{[\Delta], [\Sigma]\}}{\Gamma\{[[\Delta], \Sigma]\}}\, \dot{4}$	$\dfrac{Rxy, Rxz, \mathcal{R}, X \Rightarrow Y}{Rxy, Ryz, \mathcal{R}, X \Rightarrow Y}\, 4_{BS}$
$\dfrac{\Gamma\{[\Delta, [\Sigma]]\}}{\Gamma\{[\Delta], \Sigma\}}\, \dot{B}$	$\dfrac{Rxy, Ryz, \mathcal{R}, X \Rightarrow Y}{Rxy, \mathcal{R}[x/z], X[x/z] \Rightarrow Y[x/z]}\, B_{BS}$
$\dfrac{\Gamma\{[\Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Delta]\}}\, \dot{5}$	$\dfrac{Rxy, Ryz, \mathcal{R}, \Gamma \Rightarrow \Delta}{Ruy, Ryz, \mathcal{R}, \Gamma \Rightarrow \Delta}\, 5_{BS}$
$\dfrac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}}\, \dot{D}$	$\dfrac{Rxy, \mathcal{R}, \Gamma \Rightarrow \Delta}{\mathcal{R}, \Gamma \Rightarrow \Delta}\, D_{BS}$

Table 7.6: Nested sequent structural rules from [12] and the corresponding LTS rules. The rule $\dot{5}$ has the condition that the hole in $\Gamma\{\}\{\emptyset\}$ has depth > 0 (so it does not correspond to the root). The rule 5_{BS} has the condition that u is a state outside the upward closure of y in the frame defined by $Rxy, Ryz, \mathcal{R}$. Rule D_{BS} has the condition that y does not appear in the conclusion sequent.

nested sequent structural rules from [12] are denoted, respectively, as $\dot{T}$, $\dot{4}$, $\dot{B}$, $\dot{5}$ and $\dot{D}$. These rules, along with the corresponding LTS rules T_{BS}, 4_{BS}, B_{BS}, 5_{BS} and D_{BS} (under the obvious mapping from nested sequent rules to LTS rules) are presented in Table 7.6. The reader should verify that it is indeed the case that the conclusion sequent of each ρ_{BS} rule is a LTS sequent whenever the premise sequent of ρ is a LTS sequent.

Let NSK denote the cutfree nested sequent calculus for K titled "System K" in Brünnler and Straßburger [12].

Theorem 7.33 (Brünnler and Straßburger) *For a set $X \subseteq \{T, 4, B, 5, D\}$, let $\dot{X}$ denote the corresponding subset of $\{\dot{T}, \dot{4}, \dot{B}, \dot{5}, \dot{D}\}$ Then $NSK + \dot{X}$ is sound and complete for $K \oplus X$ and the cut-rule is syntactically admissible.*

Proof. See Brünnler and Straßburger [12] Q.E.D.

We have the following observations.

(i) The rules T_H and B_H are exactly T_{BS} and B_{BS}.

(ii) Although the rules 4_H and 4_{BS} are not identical, we can show that 4_{BS} is syntactically admissible in $G3K + 4_H$ and 4_H is syntactically admissible in $G3K + 4_{BS}$. To show the former, suppose that we are given a derivation in $G3K + 4_H$ of the premise sequent $Rxy, Rxz, \mathcal{R}, X \Rightarrow Y$ of 4_{BS}. Then, for some fresh variable u we can obtain a derivation of $Rxy, Ryz, Rxu, \mathcal{R}[u/z], X[u/z] \Rightarrow Y[u/z]$. Applying 4_H to this sequent we obtain

$$Rxy, Ryz, (\mathcal{R}[u/z])[z/u], (X[u/z])[z/u] \Rightarrow (Y[u/z])[z/u]$$

This is exactly $Rxy, Ryz, \mathcal{R}, X \Rightarrow Y$ so admissibility is proved. To show that 4_H is syntactically admissible in $G3K + 4_{BS}$ we will make use of the admissibility of the following rule (*Merge*) in $G3K$:

$$\frac{Rxy, Rxz, \mathcal{R}, X \Rightarrow Y}{Rxy, \mathcal{R}[y/z], X[y/z] \Rightarrow Y[y/z]}\ (\mathit{Merge})$$

(see Poggiolesi [57] for a proof of this result for THS calculi). Then, starting with the premise sequent of 4_H, we have

$$\frac{Rxy, Ryz, Rxu, \mathcal{R}, X \Rightarrow Y}{Rxy, Ryu, Ryz, \mathcal{R}, X \Rightarrow Y}\ 4_{BS}$$

Applying the rule (*Merge*) to the above sequent we obtain

$$Rxy, Ryz, \mathcal{R}[z/u], X[z/u] \Rightarrow Y[z/u]$$

This is precisely the conclusion sequent of 4_H so admissibility is proved.

(iii) It is not clear if the rules 5_H and 5_{BS} are inter-admissible in $G3K$.

(iv) Although D is not a 3/4 Lemmon-Scott axiom, notice that the relation rule (*Ser*) obtained from Negri's scheme is exactly the rule D_{BS}.

Lemma 7.34 *For a set $X \subseteq \{T, 4, B\} \cup \{D\}$, let X' denote the corresponding subset of $\{T_H, 4_H, B_H\} \cup \{(Ser)\}$ (formula D corresponds to the (Ser) rule). Then the LTS calculus $G3K + X'$ is sound and complete for $K \oplus X$ and the cut-rule is syntactically admissible.*

Proof. Let X_{BS} denote the subset of $\{T_{BS}, 4_{BS}, B_{BS}, D_{BS}\}$ corresponding to X. Following from the admissibility of the cut-rule in the corresponding nested sequent calculi (Theorem 7.33) we have that the cut-rule is admissible in $G3K + X_{BS}$. From our observations above it is clear that $G3K + X_{BS}$ and $G3K + X'$ derive exactly the same sequents. It follows that the cut-rule is admissible in $G3K + X'$. Q.E.D.

Although Lemma 7.34 *suggests* that Theorem 7.32 can be strengthened to obtain cutfree LTS calculi (and hence nested sequent/THS calculi) for modal logics axiomatised by 3/4 Lemmon-Scott axioms, a general proof of cut-admissibility has yet to be presented. Assuming that such a general proof can be obtained, the next question would be to see how much further Theorem 7.32 can be extended from the 3/4 Lemmon-Scott axioms, to include axioms such as connectedness (3 axiom) and directedness (2 axiom), for example. Indeed, we have already seen that we can obtain a cutfree LTS calculus for $K \oplus D$ although the seriality axiom D is not a 3/4 Lemmon-Scott axiom.

Poggiolesi [55] presents cutfree THS calculi for the logics $K \oplus X$ ($X \subseteq \{T, 4, B, 5\}$). The calculi presented there contain non-structural rules called "Special logical rules", corresponding to each formula in X. In order to prove cut-admissibility, she proves admissibility of certain "Special (THS) structural rules" which are identical or similar to the rules T_{BS}, 4_{BS}, B_{BS} and 5_{BS} (under the mapping from LTS to THS). From our work here we know that the "Special logical rules" can be deleted in favour of the appropriate THS structural rules. It would be interesting to identify the syntactic relationship between such "special logical rules" and "Special structural rules" as this may yield an insight into the generation of structural rules leading to cutfree calculi.

7.5 Conclusion

We have answered a question posed by Poggiolesi regarding the connections between the THS calculus $CSGL$ for GL and the labelled sequent calculus $G3GL$ for GL. A crucial aspect of this work is showing that $\Rightarrow x : A$ is derivable in $G3GL$ iff $\Rightarrow x : A$ is derivable in the LTS calculus $\mathbb{TL}CSGL$ (Theorem 7.27).

Negri has identified a large class of modal logics that can be presented using cutfree labelled sequent calculi. Meanwhile, the Generalised Hein's scheme proposes a method for constructing LTS calculi for certain modal logics. The problem is that it is not clear how to eliminate the cut-rule from the calculi.

Lemma 7.34 shows that cut-elimination holds for some concrete instances. An extension of this work would be to develop a method of embedding suitable labelled sequent calculi inside LTS calculi (analogous to Theorem 7.27) thus enabling us to obtain cutfree LTS calculi for the general case.

Negri [53] has shown how to obtain cutfree labelled sequent calculi for a large class of superintuitionistic logics [16] that are specified by their frame conditions. It would be interesting to see if it is possible to extend the techniques in this chapter to obtain THS/nested sequent calculi for these logics.

Chapter 8

Conclusion

In this thesis, we have presented conclusions and further research in each chapter. Here we provide a broad overview of the results.

We presented syntactic proofs of cut-elimination for GL and Go. In the case of GL, this work resolves the controversy regarding Valentini's original proof and Moen's counterclaim. In particular, we show how to lift Valentini's argument to a sequent calculus for GL built from multisets and prove that the induction measure is well-founded, and we have identified a mistake in Moen's claim. The proof for Go means that syntactic cut-elimination has been shown now for the provability logics GL, Grz and Go, filling a gap in the literature. Our study indicates that the proof for Go requires the deepest level of analysis — this is perhaps related to the fact that $Go \subset GL$ and $Go \subset Grz$. A prominent feature of these logics is that the corresponding sequent calculi contain an inference rule with a diagonal formula. That is, each sequent calculus contains a rule with a formula crossing from the antecedent (of the premise sequent) to the succedent (of the conclusion sequent). Typically the difficulties for the cut-elimination proof arise from this rule. As future work, the transformations in these proofs can be abstracted in order to give a uniform account of cut-elimination for sequent calculi containing inference rules with diagonal formulae. As an application, we would like to use these techniques to obtain a syntactic proof of cut-elimination for a traditional sequent calculus for the logic $S4.3.1$, as Shimura's [67] proof for this logic requires a modification of the traditional sequent calculus.

In Part II we confirmed that Kracht's characterisation of properly displayable modal logics is incorrect and proposed a new characterisation. Although the complete characterisation rests on a conjecture that still needs to be proved[1], even

[1]M. Kracht has given a 'proof' for the conjecture, but we have shown that his proof is

without this conjecture our work significantly extends the class of modal logics that are properly displayable. Using these results we showed how to properly display superintuitionistic logics axiomatised by formulae of a certain syntactic form. Moreover, we also demonstrated how to properly display superintuitionistic logics that are specificed by suitable semantic frame conditions. Thus, our work provides a systematic method of constructing display calculi for a large class of superintuitionistic logics.

Finally, in Part III we identified a subclass of the labelled sequent calculus called labelled tree sequent calculi and proved that they are notational variants of the tree-hypersequent and nested sequent calculus. We then showed how to embed Negri's labelled sequent calculus $G3GL$ for provability logic GL in a labelled tree sequent calculus. Exploiting this embedding and the notational variant result, we obtained a mapping between derivations in Poggiolesi's [58] tree-hypersequent calculus $CSGL$ and the labelled sequent calculus $G3GL$, for derivations of formulae in GL. This completely answers a question posed as future work in Poggiolesi [58]. Using this mapping we can obtain soundness and completeness, and cut-admissibility for $CSGL$ from the existing results for $G3GL$, alleviating the need for independent proofs. Next, we proved cut-elimination for certain labelled tree sequent calculi that were constructed using a general scheme for obtaining inference rules from 3/4 Lemmon-Scott formulae. An extension of this work would be to obtain a generalised proof of cut-admissibility for logics axiomatised over K using 3/4 Lemmon-Scott formulae. Our work here indicates that this problem can be phrased in terms of importing results from suitable labelled sequent calculi into labelled tree sequent calculi.

incomplete. He completely agrees with our analysis regarding this problem and concedes that it is not clear how to obtain the result: personal correspondence by email dated 13/Dec/2010.

Appendix A

Additional results for Chapter 5

A.1 A model-theoretic proof of Lemma 5.7

Let us introduce some notation. For a string σ constructed from $\Diamond$ and $\blacklozenge$ (ϵ denotes the empty string), recursively define the first-order formulae $\mathcal{P}_\sigma(w, v)$ as follows:

$$\begin{aligned}
\mathcal{P}_\epsilon(w, v) &= (w = v) \\
\mathcal{P}_{\Diamond\sigma}(w, v) &= (\exists w' \rhd w)\mathcal{P}_\sigma(w', v) \\
\mathcal{P}_{\blacklozenge\sigma}(w, v) &= (\exists w' \lhd w)\mathcal{P}_\sigma(w', v)
\end{aligned}$$

For example, $\mathcal{P}_{\Diamond\blacklozenge}(w, s)$ is the formula $(\exists w' \rhd w)(\exists w'' \lhd w')(s = w'')$.

Lemma A.1 *Let σ be a (possibly empty) string constructed from $\Diamond$ and $\blacklozenge$. Observe that for any tense formula A, model M and state w:*

$$M, w \models \sigma A \textit{ iff there exists } v \textit{ such that } \mathcal{P}_\sigma(w, v) \textit{ and } M, v \models A$$

Proof. Induction on the length of σ. Q.E.D.

Intuitively, $\mathcal{P}_\sigma(w, v)$ is a statement specifying the path between states w and v in terms of existential quantifers.

The notation $\overset{\vee}{f}$ is defined in Section 5.1.2.

Proof.[Second proof of Lemma 5.7] It suffices to prove that for every frame F, $F \models f(p_i) \supset \overset{\vee}{f}(p_i \wedge D_i)$ iff $F \models f(p_i \wedge \neg D_i) \supset \bot$. We argue in each direction by contradiction.

Assume that there is some F such that $F \models f(p_i \wedge \neg D_i) \supset \bot$ and $F \not\models f(p_i) \supset \overset{\vee}{f}(p_i \wedge D_i)$. The latter implies that there exists some model M based on

F, and state w such that $M, w \not\models f(p_i) \supset \overset{\vee}{f}(p_i \wedge D_i)$. Therefore $M, w \models f(p_i)$ and $M, w \not\models \overset{\vee}{f}(p_i \wedge D_i)$. From $M, w \models f(p_i)$ by repeated use of Lemma A.1 it follows that there exist states $v_1, \dots, v_n$ and strings $\sigma_1, \dots, \sigma_n$ in $\Diamond, \blacklozenge$ such that $\mathcal{P}_{\sigma_i}(w, v_i)$ and $M, v_i \models p_i$. Moreover, since $M, w \not\models \overset{\vee}{f}(p_i \wedge D_i)$ it must be the case that $M, v_i \not\models p_i \wedge D_i$ and hence $M, v_i \not\models D_i$ $(1 \leq i \leq n)$. Therefore $M, v_i \models p_i \wedge \neg D_i$ for each i, so $M, w \models f(p_i \wedge \neg D_i)$. Since $F \models f(p_i \wedge \neg D_i) \supset \bot$ it follows that $M, w \models \bot$. This is impossible so we have obtained a contradiction.

Now for the other direction. Assume that there is some frame F such that $F \models f(p_i) \supset \overset{\vee}{f}(p_i \wedge D_i)$ and $F \not\models f(p_i \wedge \neg D_i) \supset \bot$. Then there exists some model $M = (F, V)$ and state w such that $M, w \not\models f(p_i \wedge \neg D_i) \supset \bot$. Thus $M, w \models f(p_i \wedge \neg D_i)$. Therefore there exist $v_1, \dots, v_n$ and strings $\sigma_1, \dots, \sigma_n$ constructed from $\Diamond, \blacklozenge$ such that $\mathcal{P}_{\sigma_i}(w, v_i)$ and $M, v_i \models p_i \wedge \neg D_i$. Therefore $M, v_i \models p_i$ for each i, and thus $M, w \models f(p_i)$. Since $F \models f(p_i) \supset \overset{\vee}{f}(p_i \wedge D_i)$ by assumption, we must have $M, w \models \overset{\vee}{f}(p_i \wedge D_i)$. Define the set $\mathbf{u}^{(i)}$ by

$$s \in \mathbf{u}^{(i)} \text{ iff } \mathcal{P}_{\sigma_i}(w, s) \text{ and } M, s \models p_i \wedge D_i$$

Since $\overset{\vee}{f}(X_0, \dots, X_n)$ is constructed from X_i using $\Diamond, \blacklozenge$ and $\vee$, using the equivalences at (5.1) we can push the disjunction outwards to write $\overset{\vee}{f}(X_0, \dots, X_n)$ as $\bigvee_i \sigma_i X_i$ where σ_i is a string constructed from $\Diamond$ and $\blacklozenge$. Then $\overset{\vee}{f}(p_i \wedge D_i)$ is equivalent to $\bigvee_i \sigma_i(p_i \wedge D_i)$. Since $M, w \models \overset{\vee}{f}(p_i \wedge D_i)$ at least one of the disjuncts must be satisfied at the state w, so at least one of the sets in $\{\mathbf{u}^{(i)}\}$ must be non-empty. If every $D_i = \bot$ then we have an immediate contradiction. Otherwise suppose that $D_i \neq \bot$ $(i \in I)$ for some set $I \subseteq \{1, \dots, n\}$. Informally speaking, the sets $\{\mathbf{u}^{(i)}\}_{i \in I}$ consist of the states that 'witness' $M, w \models \overset{\vee}{f}(p_i \wedge D_i)$. Also observe that $v_i \notin \mathbf{u}^{(i)}$ for every $i \in I$ since $M, v_i \models p_i \wedge \neg D_i$.

Let $M' = (F, V')$ be the model obtained from $M = (F, V)$ by setting

$$V'(p_i) = V(p_i) \setminus \mathbf{u}^{(i)} \text{ for each propositional variable } p_i$$

Informally speaking, the model M' is obtained from M by 'switching-off' p_i at states s satisfying $\mathcal{P}_{\sigma_i}(w, s)$ and $p_i \wedge D_i$.

Clearly $M', v_i \models p_i$ for each i, so $M', w \models f(p_i)$ and thus $M', w \models \overset{\vee}{f}(p_i \wedge D_i)$. Therefore there must be some $i^* \in I$ and state s such that $\mathcal{P}_{\sigma_{i^*}}(w, s)$ and $M', s \models p_{i^*} \wedge D_{i^*}$. Since the formula $p_{i^*} \wedge D_{i^*}$ is positive, by Lemma 4.21 it is upward monotone in all propositional variables. Since $V'(p) \subseteq V(p)$ for all p, we have $M, s \models p_{i^*} \wedge D_{i^*}$. Then it must be the case that $s \in \mathbf{u}^{(i^*)}$, which would mean that $s \notin V'(p_{i^*})$ and thus $M', s \not\models p_{i^*}$. This is impossible, however, for we have already noted that $M', s \models p_{i^*} \wedge D_{i^*}$. We have arrived at a contradiction. Q.E.D.

We observe that we can depict a 'fragment', respectively, of the models M and M' used in the above proof, for some $D_i \neq \bot$, as follows:

$$\begin{tikzcd} & v_i \quad p_i \wedge \neg D_i \\ w \arrow[ur, "\mathcal{R}_{\sigma_i}"] \arrow[dr, "\mathcal{R}_{\sigma_i}"] & \\ & s \in \mathbf{u}^{(i)} \quad p_i \wedge D_i \end{tikzcd} \qquad \begin{tikzcd} & v_i \quad p_i \wedge \neg D_i \\ w \arrow[ur, "\mathcal{R}_{\sigma_i}"] \arrow[dr, "\mathcal{R}_{\sigma_i}"] & \\ & s \in \mathbf{u}^{(i)} \quad \neg p_i \wedge D_i \end{tikzcd}$$

A.2 Computing M-formulae from primitive modal formulae

Recall that a *basic primitive formula* is a formula built from $\wedge$ and $\Diamond$ using propositional variables and $\top$. We will need the following result.

Lemma A.2 *Let $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_m)$ be sequences of basic primitive formulae. Let $(B'_1, \ldots, B'_m)$ be the sequence obtained from $(B_1, \ldots, B_m)$ by replacing each B_i that contains propositional variables not occurring in $\bigcup_i Var(A_i)$ with $\bot$. Then*

$$\mathcal{F} \models A_1 \wedge \ldots \wedge A_n \supset B_1 \vee \ldots \vee B_m \textit{ iff } \mathcal{F} \models A_1 \wedge \ldots \wedge A_n \supset B'_1 \vee \ldots \vee B'_m$$

Proof. First suppose that B_m contains a propositional variable z not occurring in $\bigcup_i Var(A_i)$. We need to show that

$$\mathcal{F} \models A_1 \wedge \ldots \wedge A_n \supset B_1 \vee \ldots \vee B_m \text{ iff } \mathcal{F} \models A_1 \wedge \ldots \wedge A_n \supset B_1 \vee \ldots \vee B_{m-1} \vee \bot$$

The right-to-left direction is straightforward. For the left-to-right direction we proceed by contradiction.

Assume that there is some frame F such that

$$F \models A_1 \wedge \ldots \wedge A_n \supset B_1 \vee \ldots \vee B_{m-1} \vee B_m \tag{A.1}$$

and $F \not\models A_1 \wedge \ldots \wedge A_n \supset B_1 \vee \ldots \vee B_{m-1} \vee \bot$. Then there exists a model $M = (F, V)$ and state w such that

$$M, w \models A_1 \wedge \ldots \wedge A_n \text{ and } M, w \not\models B_1 \vee \ldots \vee B_{m-1} \vee \bot$$

From (A.1) it follows that $M, w \not\models B_i$ $(i < m)$ and $M, w \models B_m$. Because B_m is a basic primitive formula, we can falsify B_m in the model M' obtained from M by setting $V'(z) = \emptyset$ and keeping all other valuations unchanged. Since z does not occur negatively in the B_i, we have $M, w \not\models B_i$ $(i \leq m)$. Also, because z does not occur in the A_i, we have $M, w \models A_i$ $(i \leq n)$. Thus

$$M', w \not\models A_1 \wedge \ldots \wedge A_n \supset B_1 \vee \ldots \vee B_{m-1} \vee B_m$$

contradicting (A.1) so the claim is proved.

The main result follows from repeated application of this argument. Q.E.D.

In the following we make use of the fact that primitive modal formulae belong to the class of Sahlqvist formulae, and hence define a first-order definable class of frames.

Lemma A.3 *Let $A \supset B$ be a primitive modal formulae. Then there is an effectively computable set $\{M_i\}$ of M-formulae such that $K \oplus (A \supset B) = K \oplus \{M_i\}$.*

Proof. First observe that any basic primitive formula D can be written in the form $\Diamond \ldots \Diamond(r \wedge D')$ where r is a propositional variable or $\top$ and D' is a basic primitive formula. Consider the formula $\neg D$:

$$\neg\Diamond \ldots \Diamond(r \wedge D') \approx_{Kt} \Box \ldots \Box((r \supset \bot) \vee \neg D')$$

By induction on the size of D, it follows that the negation of a basic primitive formula *containing each propositional variable at most once* is equivalent to an M-formula.

Now, from the primitive modal formula $A \supset B$, using standard equivalences we can write this formula as a conjunction of primitive modal formulae $\{s_i\}$, where each s_i has the form

$$A_1 \wedge \ldots \wedge A_n \supset B_1 \vee \ldots \vee B_m \tag{A.2}$$

where each formula in $\{A_i\}, \{B_i\}$ is a basic primitive formula. By the definition of primitive modal axiom, the antecedent of (A.2) contains each propositional variable at most once. Furthermore, by Lemma A.2 there is a formula

$$A_1 \wedge \ldots \wedge A_n \supset B'_1 \vee \ldots \vee B'_m \tag{A.3}$$

where $\{B'_i\}$ is a set of basic primitive formulae such that (A.2) and (A.3) define the same class of frames, and every propositional variable in the succedent also occurs in the antecedent. Clearly (A.3) is equivalent to

$$\neg A_1 \vee \ldots \vee A_n \vee (\top \supset B'_1) \vee \ldots \vee (\top \supset B'_m) \tag{A.4}$$

Each $\neg A_i$ term is equivalent to an M-formula, and each $\top \supset B'_i$ term is either equivalent to an M-formula or to $\bot$. Thus the above formula is an M-formula. We have shown that each primitive modal formula s_i corresponds to an M-formula M_i. Since the sets $\{s_i\}$ and $\{M_i\}$ are Sahlqvist formulae, by the Sahlqvist completeness theorem, $K \oplus \{s_i\} = K \oplus \{M_i\}$. It follows that $K \oplus (A \supset B) = K \oplus \{M_i\}$.

Q.E.D.

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