General methods in proof theory for modal logic - Lecture 1

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Outline of the tutorial

- Lecture 1: An introduction to proof theory via the sequent calculus, and an introduction to normal modal logics defined via syntax and relational semantics.
- Lecture 2: Limits of the sequent framework. Case study $S5$. No cutfree sequent calculus, but a hypersequent calculus
- Lecture 3: Proof theoretic methods case study: cut-elimination methods for provability logics. The sequent calculus is not enough: other proof-theoretic formalisms (labelled, nested, display calculus) for obtaining analytic calculi for modal logics.
- Lecture 4: Non-normal logics (and their neighbourhood semantics). Ackermann's lemma/Tseitin transformation to obtain logical rules. Case study: Mimamsa Deontic Logic.

Proof theory

- Proof theory treats a proof as a formal mathematical object, facilitating its analysis, and also the study of the provability relation, by mathematical techniques.
- A proof is typically defined by first defining a proof system
- Our emphasis is on structural proof theory: the study of various proof systems for logics and their structural properties, and using the proof system to study the logic of interest.
- There are essentially two degrees of freedom here: choose the logic and then choose/construct a proof system for the logic
- To begin with, let's start with a very familiar logic: propositional classical logic **Cp**. Classical logic consists of the set of formulae with evaluate to $\top$ under the usual truth table semantics.
- Let us introduce a proof system for it. This proof system is called a Hilbert calculus...

The Hilbert calculus **hCp** for classical logic **Cp**

- Classical language: countable set of propositional variables $p_1, p_2, \ldots$ and logical connectives $\rightarrow, \neg, \wedge, \vee, \bot, \top$.
- Every propositional variable and $\bot$ and $\top$ is a formula. If A and B are formulae, then so are $A \rightarrow B$, $\neg A$, $A \wedge B$, $A \vee B$
- The Hilbert calculus **hCp** consists of the following axiom schemata (schematic variable $\mathcal{A}, \mathcal{B}, \mathcal{C}$ stand for formulae):

Ax 1: $\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{A})$

Ax 2: $(\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})) \rightarrow ((\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\mathcal{A} \rightarrow \mathcal{C}))$

Ax 3: $(\neg\mathcal{A} \rightarrow \neg\mathcal{B}) \rightarrow ((\neg\mathcal{A} \rightarrow \mathcal{B}) \rightarrow \mathcal{A})$

and other axioms for $\wedge, \vee, \top, \bot$ (omitted for brevity)

and a single rule called *modus ponens*:

$$\frac{\mathcal{A} \qquad \mathcal{A} \rightarrow \mathcal{B}}{\mathcal{B}}\ \mathrm{MP}$$

Derivation of $A \to A$

Definition (derivation)

A formal proof or derivation of B is the finite sequence $C_1, C_2, \ldots, C_n \equiv B$ of formulae where each element C_j is an axiom instance or follows from two earlier elements by modus ponens.

$$
\begin{array}{ll}
\text{Ax 1:} & \mathcal{A} \to (\mathcal{B} \to \mathcal{A}) \\
\text{Ax 2:} & (\mathcal{A} \to (\mathcal{B} \to \mathcal{C})) \to ((\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})) \\
\text{Ax 3:} & (\neg\mathcal{A} \to \neg\mathcal{B}) \to ((\neg\mathcal{A} \to \mathcal{B}) \to \mathcal{A}) \\
\text{MP:} & \mathcal{A} \quad \mathcal{A} \to \mathcal{B} / \mathcal{B}
\end{array}
$$

1	$((A \to ((A \to A) \to A)) \to ((A \to (A \to A)) \to (A \to A)))$	Ax 2
2	$(A \to ((A \to A) \to A))$	Ax 1
3	$((A \to (A \to A)) \to (A \to A))$	MP: 1 and 2
4	$(A \to (A \to A))$	Ax 1
5	$A \to A$	MP: 3 and 4

A drawback of the Hilbert calculus: derivations lack a discernible structure

- Consider the derivation of $A \to A$:

1	$((A \to ((A \to A) \to A)) \to ((A \to (A \to A)) \to (A \to A)))$	Ax 2
2	$(A \to ((A \to A) \to A))$	Ax 1
3	$((A \to (A \to A)) \to (A \to A))$	MP: 1 and 2
4	$(A \to (A \to A))$	Ax 1
5	$A \to A$	MP: 3 and 4

What is the relation of the derivation to $A \to A$? How could we construct its derivation? Is there an algorithm? and if so, what is its complexity? Is there a derivation of $(p \to p) \to \neg(p \to p)$?

There is no obvious structural relationship between $A \to A$ and its derivation (and MP is the culprit)

A new proof system: the sequent calculus **sCp**

$$\frac{}{p, X \vdash Y, p}\ \text{init} \qquad \frac{}{\bot, X \vdash Y}\ \bot\text{l} \qquad \frac{}{X \vdash Y, \top}\ \top\text{r}$$

$$\frac{X \vdash Y, A}{\neg A, X \vdash Y}\ \neg l \qquad \frac{A, X \vdash Y}{X \vdash Y, \neg A}\ \neg r$$

$$\frac{A, B, X \vdash Y}{A \wedge B, X \vdash Y}\ \wedge l \qquad \frac{X \vdash Y, A \qquad X \vdash Y, B}{X \vdash Y, A \wedge B}\ \wedge r$$

$$\frac{A, X \vdash Y \qquad B, X \vdash Y}{A \vee B, X \vdash Y}\ \vee l \qquad \frac{X \vdash Y, A, B}{X \vdash Y, A \vee B}\ \vee r$$

$$\frac{X \vdash Y, A \qquad B, X \vdash Y}{A \to B, X \vdash Y}\ \to l \qquad \frac{A, X \vdash Y, B}{X \vdash Y, A \to B}\ \to\text{r}$$

- No axioms, only rules built from sequents of the form $X \vdash Y$
- X, Y are multiset of formulae)
- X is the antecedent, Y the succedent
- Aside: original sequent calculus presented in Gentzen's (1935) highly readable work

$$\frac{}{p, X \vdash Y, p}\ \mathrm{init} \qquad \frac{}{\bot, X \vdash Y}\ \bot\mathrm{l} \qquad \frac{}{X \vdash Y, \top}\ \top\mathrm{r}$$

$$\frac{X \vdash Y, A}{\neg A, X \vdash Y}\ \neg l \qquad \frac{A, X \vdash Y}{X \vdash Y, \neg A}\ \neg r$$

$$\frac{A, B, X \vdash Y}{A \wedge B, X \vdash Y}\ \wedge l \qquad \frac{X \vdash Y, A \qquad X \vdash Y, B}{X \vdash Y, A \wedge B}\ \wedge r$$

$$\frac{A, X \vdash Y \qquad B, X \vdash Y}{A \vee B, X \vdash Y}\ \vee l \qquad \frac{X \vdash Y, A, B}{X \vdash Y, A \vee B}\ \vee r$$

$$\frac{X \vdash Y, A \qquad B, X \vdash Y}{A \to B, X \vdash Y}\ \to l \qquad \frac{A, X \vdash Y, B}{X \vdash Y, A \to B}\ \to\mathrm{r}$$

- Above the line are premises and below is the conclusion
- A 0-premise rule is called an initial sequent
- A derivation in the sequent calculus is an initial sequent or a rule applied to derivations of the premise(s).
- A derivation can be viewed a tree with vertices labelled by sequents. The root is the endsequent

$$\dfrac{}{p, X \vdash Y, p}\ \text{init} \qquad \dfrac{}{\bot, X \vdash Y}\ \bot\text{l} \qquad \dfrac{}{X \vdash Y, \top}\ \top\text{r}$$

$$\dfrac{X \vdash Y, A}{\neg A, X \vdash Y}\ \neg l \qquad \dfrac{A, X \vdash Y}{X \vdash Y, \neg A}\ \neg r$$

$$\dfrac{A, B, X \vdash Y}{A \wedge B, X \vdash Y}\ \wedge l \qquad \dfrac{X \vdash Y, A \qquad X \vdash Y, B}{X \vdash Y, A \wedge B}\ \wedge r$$

$$\dfrac{A, X \vdash Y \qquad B, X \vdash Y}{A \vee B, X \vdash Y}\ \vee l \qquad \dfrac{X \vdash Y, A, B}{X \vdash Y, A \vee B}\ \vee r$$

$$\dfrac{X \vdash Y, A \qquad B, X \vdash Y}{A \to B, X \vdash Y}\ \to l \qquad \dfrac{A, X \vdash Y, B}{X \vdash Y, A \to B}\ \to\text{r}$$

- A principal formula is the formula containing the newly introduced logical connective
- The auxiliary formula(e) are the formulae in the premises
- The multisets X and Y are the context

A derivation in **sCp**

$$
\dfrac{\dfrac{\dfrac{\dfrac{A, A \to (B \to C) \vdash C, A \qquad \dfrac{B, A \vdash C, A \qquad \dfrac{B, A \vdash C, B \qquad C, B, A \vdash C}{B \to C, B, A \vdash C}\,{\to}\mathrm{l}}{B, A, A \to (B \to C) \vdash C}\,{\to}\mathrm{l}}{A, A \to B, (A \to (B \to C)) \vdash C}\,{\to}\mathrm{l}}{A \to B, (A \to (B \to C)) \vdash A \to C}\,{\to}\mathrm{r}}{(A \to (B \to C)) \vdash (A \to B) \to (A \to C)}\,{\to}\mathrm{r}}{\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))}\,{\to}\mathrm{r}
$$

- Actually, the above is not yet a derivation. Recall that the initial sequents have the form $p, X \vdash Y, p$ *not* $A, X \vdash Y, A$.
- The height of a derivation is the maximal number of sequents on a branch in the derivation.
- The size of a formula is the number of connectives in it plus one. Another useful representation of a formula is in terms of its grammar tree.
- Note that $A, X \vdash Y, A$ is derivable: Argument by induction on the size of a formula. The base case (A is a propositional variable) is already an initial sequent!

(Height-preserving) admissibility and invertibility

- A rule r is admissible in **sCp** if the conclusion of the rule is derivable whenever the premise(s) are derivable.
- If the height of the derivation of the conclusion is no greater than the height of the premise(s), then r is height-preserving admissible in **sCp**
- A rule r of **sCp** is invertible: if a sequent instantiating conclusion is derivable, then the corresponding sequents instantiating premise(s) are derivable. If the latter have height no greater than the former then it is height-preserving
- The weakening rules lw and rw are height-preserving admissible

  $$\frac{X \vdash Y}{A, X \vdash Y}\,\mathrm{lw} \quad \frac{X \vdash Y}{X \vdash Y, A}\,\mathrm{rw}$$

  Suppose we are given a derivation d of $X \vdash Y$. Induction on the height of d. Consider the last rule r. Insert A into premise of r via IH, and hence obtain A in conclusion.
- The induction argument is simply the method of proving result. Picture the transformation of d.

(Height-preserving) admissibility and invertibility

- Every rule in **sCp** is height-preserving invertible. Induction on the height of d
- Once again: the induction argument is simply the method of proving result. Picture the transformation of d.
- The contraction rules lc and rc are height-preserving admissible

$$\frac{A, A, X \vdash Y}{A, X \vdash Y}\,\text{lc} \qquad \frac{X \vdash Y, A, A}{X \vdash Y, A}\,\text{rc}$$

  Prove both claims simultaneously (why?). I.e. Let d be a derivation. If d is derives $A, A, X \vdash Y$ then $A, X \vdash Y$ is derivable, and if d derives $X \vdash Y, A, A$ then $X \vdash Y$ is derivable. Induction on the height of d. Use hp invertibility.
- Once again: the induction argument is simply the method of proving result. Picture the transformation of d.

Relating **sCp** to classical logic

- Let **Cp** denote the set of formulae that are derivable in **hCp**.

- Since **hCp** is a Hilbert calculus for classical logic, **Cp** is the set of theorems of classical logic.
- Equivalently, **Cp** consists of those formulae that evaluate to $\top$ under the truth table semantics.

Theorem

For every formula A: $\vdash A$ is derivable in **sCp** $\Leftrightarrow A \in$ **Cp**.

- To prove this, following Gentzen, introduce a sequent calculus version of MP called the cut rule. Formula A is the cutformula.

$$\frac{A \qquad A \to B}{B}\ \text{MP} \qquad\qquad \frac{X \vdash Y, A \qquad A, U \vdash V}{X, U \vdash Y, V}\ cut$$

- We will prove the theorem by showing the following:

1. $\Gamma \vdash \Delta$ is derivable in $\mathbf{sCp} + cut \Leftrightarrow \wedge\Gamma \to \vee\Delta \in \mathbf{Cp}$ (notation)
2. $\Gamma \vdash \Delta$ is derivable in $\mathbf{sCp} + cut$ iff $\Gamma \vdash \Delta$ is derivable in **sCp**

$\vdash A$ is derivable in $\mathbf{sCp} \overset{2.}{\Leftrightarrow} \vdash A$ is derivable in $\mathbf{sCp} + cut \overset{1.}{\Leftrightarrow} A \in \mathbf{Cp}$

1a: $\Gamma \vdash \Delta$ is derivable in **sCp** + *cut* $\Rightarrow \wedge\Gamma \rightarrow \vee\Delta \in$ **Cp**

- This direction is soundness. We want to show that what the calculus derives can be translated to a theorem of classical logic.
- Use semantics or **hCp** to establish this direction.
- Argue by induction on the height of derivation of $\Gamma \vdash \Delta$.
- Translations of the initial sequents are theorems of **Cp**

$$p, X \rightarrow Y, p \qquad \text{show that } p \wedge (\wedge X) \rightarrow (\vee Y) \vee p \in \mathbf{Cp}$$

$$\bot \wedge X \rightarrow Y \qquad \text{show that } \bot \wedge (\wedge X) \rightarrow (\vee Y) \in \mathbf{Cp}$$

- Inductive step. Show for each remaining rule ρ: if the translation of every premise is a theorem of **Cp** then so is the translation of the conclusion.

$$\frac{A, X \vdash B}{X \vdash A \rightarrow B} \qquad \text{need to show:} \qquad \frac{(A \wedge (\wedge X)) \rightarrow B}{(\wedge X) \rightarrow (A \rightarrow B)}$$

1b: $\wedge\Gamma \to \vee\Delta \in \mathbf{Cp} \Rightarrow \Gamma \vdash \Delta$ is derivable in $\mathbf{sCp} + \mathit{cut}$

- Observe: $\vdash \wedge\Gamma \to \vee\Delta$ derivable in $\mathbf{sCp} + \mathit{cut}$ iff $\Gamma \vdash \Delta$ derivable $\mathbf{sCp} + \mathit{cut}$
- Show that $\vdash$ Ax is derivable in $\mathbf{sCp} + \mathit{cut}$ for every axiom Ax in **hCp**. E.g.

$$
\dfrac{\dfrac{\dfrac{\dfrac{A, A \to (B \to C) \vdash C, A \qquad \dfrac{B, A \vdash C, A \qquad \dfrac{B, A \vdash C, B \qquad C, B, A \vdash C}{B \to C, B, A \vdash C}\,{\to}\mathrm{l}}{B, A, A \to (B \to C) \vdash C}\,{\to}\mathrm{l}}{A, A \to B, (A \to (B \to C)) \vdash C}\,{\to}\mathrm{l}}{A \to B, (A \to (B \to C)) \vdash A \to C}\,{\to}\mathrm{r}}{(A \to (B \to C)) \vdash (A \to B) \to (A \to C)}\,{\to}\mathrm{r}}{\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))}\,{\to}\mathrm{r}
$$

1b: $\wedge\Gamma \to \vee\Delta \in \mathbf{Cp} \Rightarrow \Gamma \vdash \Delta$ is derivable in $\mathbf{sCp} + cut$ (ctd)

- Now let us simulate MP in the sense: if $\vdash A$ and $\vdash A \to B$ is derivable, then $\vdash B$ is derivable:

$$\dfrac{\vdash A \qquad \dfrac{\vdash A \to B \qquad \dfrac{A \vdash A \qquad B \vdash B}{A \to B, A \vdash B}\to\mathrm{l}}{A \vdash B}\,\text{cut}}{\vdash B}\,cut$$

- In this way we have that if A is derivable in $\mathbf{hCp}$ then $\vdash A$ is derivable in $\mathbf{sCp} + cut$
- It follows that

$$\begin{array}{rcr} \wedge\Gamma \to \vee\Delta \in \mathbf{Cp} & \Rightarrow & \vdash \wedge\Gamma \to \vee\Delta \text{ derivable in } \mathbf{sCp} + cut \\ & \Rightarrow & \Gamma \vdash \Delta \text{ derivable in } \mathbf{sCp} + cut \end{array}$$

2. $\Gamma \vdash \Delta$ derivable in **sCp** + *cut* iff $\Gamma \vdash \Delta$ derivable in **sCp**

- Right-to-left direction is trivial. Left-to-right is the cut-elimination theorem

Theorem (Gentzen cut-elimination theorem)

Suppose that δ is a derivation of $X \vdash Y$ in **sCp** + *cut. Then there is a transformation to eliminate instances of the cut-rule from δ to obtain a derivation δ' of $X \vdash Y$ in* **sCp**.

- First argue how to get rid of a single cut in δ
- Suppose that we are given a derivation δ of $X \vdash Y$ containing a single occurrence of the cutrule as the last rule of the derivation. Argue by principal induction on the size of the cutformula and secondary induction on cutheight (sum of the premise derivation heights) that there is a cutfree derivation of $X \vdash Y$.
- Again: induction is method of proving; picture transformation
- If δ multiple cuts, repeat the argument, always choosing a topmost cut (i.e. a cut that has no cut above it in the derivation)

Proof of Gentzen's *Hauptsatz*

Consider a derivation concluding with the cut-rule:

$$\frac{X \vdash Y, A \qquad A, U \vdash V}{X, U \vdash Y, V}\ cut$$

- (Base case) A derivation of minimal height concluding in a cutrule must have the left and right premise as initial sequents.

  $$\frac{p, X \vdash Y, p \qquad q, U \vdash V, q}{\text{depends on whether cut-formula is } p \text{ or } q \text{ or something else}}\ \text{cut}$$

  In every case the conclusion is already an initial sequent so we don't need the cut!
- Argument when either initial sequent is ($\bot$l) or ($\top$r) is similar
- (Inductive case) Consider the following possibilities
  1. cut-formula A is not principal in one of the premises
  2. cut-formula A is principal in both premises

Proof of Gentzen's *Hauptsatz* II

A is not principal in one of the premises of the cutrule e.g.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ X' \vdash^{k} Y', A\end{array}}{X \vdash^{k+1} Y, C \vee D, A}\,r \qquad \begin{array}{c}\vdots\\ A, U \vdash^{l} V, C \vee D\end{array}}{X, U \vdash Y, V, C \vee D}\,\text{cut}$$

Superscript indicates height. Cutheight is $k + l + 1$. Lift the cut upwards...

$$\dfrac{\begin{array}{c}\vdots\\ X' \vdash^{k} Y', A\end{array} \qquad \begin{array}{c}\vdots\\ A, U \vdash^{l} V, C \vee D\end{array}}{X', U \vdash Y', V, C \vee D}\,\text{cut}$$

Derivation has reduced cutheight $k + l$ $(< k + l + 1)$ so apply induction hypothesis to get cutfree derivation $X', U \vdash Y', V, C \vee D$.

Apply rule r to $X', U \vdash Y', V, C \vee D$ to get cutfree derivation of $X, U \vdash Y, V, C \vee D$. Cutfree derivation has greater height!

Proof of Gentzen's *Hauptsatz* III

- The cutformula A is principal in both premises e.g.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A, X \vdash^k Y, B\end{array}}{X \vdash^{k+1} Y, A \to B}\,{\to}\mathrm{r} \qquad \dfrac{\begin{array}{c}\vdots\\ U \vdash^l V, A\end{array} \qquad \begin{array}{c}\vdots\\ B, U \vdash^m V\end{array}}{A \to B, U \vdash^{1+\max l,m} V}\,{\to}\mathrm{l}}{X, U \vdash Y, V}\,\mathrm{cut}$$

Lift the cut upwards. . .

$$\dfrac{\begin{array}{c}\vdots\\ A, X \vdash Y, B\end{array} \qquad \begin{array}{c}\vdots\\ B, U \vdash V\end{array}}{A, X, U \vdash Y, V}\,\mathrm{cut}$$

Since size $|B|$ of the cutformula smaller than before $(A \to B)$ apply the induction hypothesis to get cutfree derivation of $A, X, U \vdash Y, V$.

Proof of Gentzen's *Hauptsatz* IV

From above: apply the induction hypothesis to obtain a cutfree derivation of $A, X, U \vdash Y, V$. Now proceed:

$$\dfrac{\begin{array}{c}\vdots \\ U \vdash V, A\end{array} \qquad \begin{array}{c}\vdots \\ A, X, U \vdash Y, V\end{array}}{X, U, U \vdash Y, V, V}$$

Since the size $|A|$ of the cutformula is smaller than before $(A \to B)$ apply the induction hypothesis to obtain a cutfree derivation of $X, U, U \vdash Y, V, V$ (the duplicates are because we applied cut twice)

By admissibility of lc and rc we get $X, U \vdash Y, V$ as required.

- cutfree proof is typically much longer than proof with cuts
- Cut-elimination: eliminating lemmata from a math. proof
- Computational interpretations

Hilbert calculus **hCp** and sequent calculus **sCp** compared

$$\frac{}{p, X \vdash Y, p}\ \text{init} \qquad \frac{}{\bot, X \vdash Y}\ \bot\text{l} \qquad \frac{}{X \vdash Y, \top}\ \top\text{r}$$

$$\frac{X \vdash Y, A}{\neg A, X \vdash Y}\ \neg l \qquad \frac{A, X \vdash Y}{X \vdash Y, \neg A}\ \neg r$$

$$\frac{A, B, X \vdash Y}{A \wedge B, X \vdash Y}\ \wedge l \qquad \frac{X \vdash Y, A \qquad X \vdash Y, B}{X \vdash Y, A \wedge B}\ \wedge r$$

$$\frac{A, X \vdash Y \qquad B, X \vdash Y}{A \vee B, X \vdash Y}\ \vee l \qquad \frac{X \vdash Y, A, B}{X \vdash Y, A \vee B}\ \vee r$$

$$\frac{X \vdash Y, A \qquad B, X \vdash Y}{A \to B, X \vdash Y}\ \to l \qquad \frac{A, X \vdash Y, B}{X \vdash Y, A \to B}\ \to\text{r}$$

- We have traded many axioms and few rules in **hCp** for no axioms and many rules in **sCp**. So what's the point?
- The aim was to remove MP to obtain the subformula property: every formula in the premise(s) is a subformula of a formula in the conclusion
- To do this we first introduced a more general version of MP (the cut rule) and showed how it could be eliminated

sCp has the Subformula property, **hCp** does not

- Subformula property: every formula in the premise(s) is a subformula of a formula in the conclusion
- If all the rules of the calculus satisfy this property, the calculus is analytic
- Analyticity is crucial to using the calculus (for consistency, decidability. . .) as we shall see
- Unlike in the Hilbert calculus, the proof has a nice structure!
- To be precise: there are properties weaker than the subformula property which can be useful (e.g. analytic cut). The point is to meaningfully relate the premises to the conclusion.

Applications: Consistency of classical logic

Consistency of classical logic is the statement that $A \wedge \neg A \notin \mathbf{Cp}$.

Theorem

Classical logic is consistent.

Proof by contradiction. Suppose that $A \wedge \neg A \in \mathbf{Cp}$. Then $A \wedge \neg A$ is derivable in **sCp** (completeness). Let us try to derive it (read upwards from $\vdash A \wedge \neg A$):

$$\frac{\vdash A \qquad \dfrac{A \vdash}{\vdash \neg A}}{\vdash A \wedge \neg A}$$

So $\vdash A$ and $A \vdash$ are derivable. Thus $\vdash$ must be derivable in $\mathbf{sCp} + \mathit{cut}$ (use cut) and hence in **sCp** (by cut-elimination). This is impossible (why?) QED.

Applications: Decidability of classical logic

Theorem

Decidability of **Cp**.

- Starting from a given formula A, repeatedly apply the rules backwards (choosing some formula as principal).
- Since each rule reduces the complexity of the sequent (a logical connective is deleted), the backward proof search terminates under any choice of principal formulae
- There are only finitely many backward proof searches. If one is a derivation, then $A \in \mathbf{Cp}$ otherwise it is not.
- Note: argument (as above) fails in $\mathbf{sCp} + lc + rc$. Suppose your favourite calculus obliges the inclusion of contraction in some way (e.g. most calculi for intuitionistic logic). Then other arguments may be available.
- Substructural logics side comment: deleting weakening from **Ip** leads to $\mathbf{FL}_{ec}$ (proved decidable by Kripke, 1959).
- Deleting weakening and exchange leads to $\mathbf{FL}_{c}$ proved undecidable (Chvalovsky and Horcik, 2016)

Modal Logics

"Modal languages are simple yet expressive languages for talking about relational structures"
Modal Logic (Blackburn, Venema and de Rijke)

- Augment the usual boolean connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$, $\bot$, $\top$) with modal operators like (but not limited to) $\Diamond$ and $\Box$.
- No variable binding, so the language is simpler than first-order.
- A relational structure is a set with a collection of relations on the set.

- Relational structures appear everywhere.
- E.g. to describe mathematical structures, theoretical computer science (model program execution as a set of states, where the binary relations model the behaviour of the program), knowledge representation, economics, computational linguistics
- We could already imagine that first-order and second-order languages are well-equipped to talk about relational structures
- The point is that modal languages are very simple languages to describe relational structures

Modal language

- Let $\mathcal{V}$ be a set of variables. The formulae of modal logic are:

$$\mathcal{F} ::= \mathcal{V} \mid \mathcal{F} \wedge \mathcal{F} \mid \mathcal{F} \vee \mathcal{F} \mid \mathcal{F} \rightarrow \mathcal{F} \mid \neg \mathcal{F} \mid \Box \mathcal{F}$$

  with $\Diamond A$ abbreviating the formula $\neg \Box \neg A$
- Equivalently $\Box A$ abbreviating $\neg \Diamond \neg A$.

- Alternatively we could include both $\Diamond$ and $\Box$ in the signature
- So $\Diamond A$ and $\Box A$ are said to be duals of each other
- Recall $\forall A = \neg \exists \neg A$.

Some standard interpretations of the modal operators

1. $\Diamond A$ as 'it is possibly the case that A'. So $\Box A$ reads 'it is not possible that not A' or simply 'it is necessarily the case that A'.

   So what can we say about statement like $A \to \Diamond A$ and $\Diamond A \to \Box\Diamond A$? Do these follow as a logical consequence?

2. Epistemic logic. Read $\Box A$ as 'the agent knows A'. Or have lots of modal operators and read $\Box_i A$ as 'the i^{th} agent knows A.

   Since we use the word knowledge, we would expect $\Box A \to A$ ('if the agent knows A then A'—contrast with belief) . But is it the case that $A \to \Box A$ ('if A, then the agent knows it')? What about $\Box A \to \Box\Box A$?

Some standard interpretations (cont.)

1. Provability. Read $\Box A$ as 'it is provable in Peano arithmetic that A'. It may be shown that $\Box(\Box A \to A) \to \Box A$ (Löb formula) holds.

2. Temporal language. Read $\Diamond A$ as 'A holds in some future time' and $\blacklozenge A$ as 'A held at some past time'.
(what is $\Box A$ and $\blacksquare A$?)

3. Propositional dynamic logic. $\langle\pi\rangle A$ as 'some terminating execution of program π from the present state leads to a state bearing information A' . So $[\pi]A$ is 'every execution of program π from the present state leads to a state bearing information A'

Talking about relational structures via the modal language

- A frame consists of a nonempty set W of worlds and a binary relation $R \subseteq W \times W$.

- A model is a pair (F, V) where $F = (W, R)$ is a frame and V is a function mapping each propositional variable to a subset $V(p) \subseteq W$ 'valuation'.
- Truth (satisfaction) at a world w in a model M is defined via:

$$
\begin{aligned}
M, w \models p &\quad \text{iff} \quad w \in V(p) \\
M, w \models A \wedge B &\quad \text{iff} \quad M, w \models A \text{ and } M, w \models B \\
M, w \models A \vee B &\quad \text{iff} \quad M, w \models A \text{ or } M, w \models B \\
M, w \models A \to B &\quad \text{iff} \quad M, w \not\models A \text{ or } M, w \models B \\
M, w \models \neg A &\quad \text{iff} \quad M, w \not\models A \\
M, w \models \Box A &\quad \text{iff} \quad \forall v \in W.(Rwv \Rightarrow M, v \models A) \\
M, w \models \Diamond A &\quad \text{iff} \quad \exists v \in W.(Rwv \;\&\; M, v \models A)
\end{aligned}
$$

- If $M, w \models A$ then A is satisfied in M at w.

Validity I

- A frame is a formalisation of the phenomenon we wish to capture (time as a linearly ordered set).
- A model 'dresses up' the frame with information (the program executes at $t = 4$).
- Since logic is concerned with reasoning (invariant under local information), we need to consider those things that hold under all possible models.
- A formula is valid at a world w of a frame $F = (W, R)$ if it is satisfied at w in every model (F, V)
- A formula is valid if it is valid on all frames at every world
- Classical theorems (i.e. $A \in \mathbf{Cp}$) are valid

Validity II

Definition

Formula A is valid at a world w in a frame F ($F, w \models A$) if *for all valuations* V it is the case that $(F, V), w \models A$.

Formula A is valid on the frame F if it is valid at every world in F.

Formula A is valid on a class $\mathcal{F}$ of frames if A is valid on every frame in $\mathcal{F}$.

- Given a class $\mathcal{F}$ of frames, the set $\Lambda_{\mathcal{F}}$ of formulae valid on $\mathcal{F}$ is called the logic of $\mathcal{F}$.
- The definition of validity utilises second-order quantification: 'over all valuations V' (over all subsets of W).

The logics of various frame classes

- The logic of all frames
- The logic of transitive frames i.e.

  $$\{ A \,|\, F \models A \text{ for every frame } F \text{ s.t. } F \models \forall xyz.(Rxy \wedge Ryz \rightarrow Rxz)\}$$

- The logic of reflexive frames

  $$\{ A \,|\, F \models A \text{ for every frame } F \text{ s.t. } F \models \forall x.Rxx\}$$

- The logic of finite (irreflexive) transitive trees (cannot be described by a first-order formula!)

Syntactic definition of modal logics

- The semantic definition we have seen is in terms of the structures the modal language intends to talk about i.e. relational structures.
- The valid formulae then represent the properties that are invariant under local information
- When we are concerned solely about such valid formulae, it makes sense to abstract away the details of the relational structure.
- Recall we have seen this before! Instead of talking about the theorems of classical logic as those that are valued $\top$ under all truth table valuations, we generated the set of theorems by consideration of the provability relation
- In other words, we want nice syntactic mechanisms for generating $\Lambda_{\mathcal{F}}$ for a given class $\mathcal{F}$ of frames

A Hilbert calculus **hK** for the normal modal logic **K**

- Define the Hilbert calculus **hK** to be the extension of the Hilbert calculus **hCp** for classical propositional logic with the following axioms and rule:

$$\Box(A \to B) \to (\Box A \to \Box B) \qquad\qquad \Box A \leftrightarrow \neg\Diamond\neg B$$

$$\frac{A}{\Box A}\ \text{necessitation}$$

- Axiom top left is called the normality axiom.
- Axiomatic extensions of **hK** are called normal modal logics.
- Non-normal modal logics are also interesting, they will be discussed in Lecture 4
- Syntactically speaking, the normal axiom permits modus ponens under $\Box$; necessitation allows us to add boxes.

Soundness and completeness of **hK** wrt semantics

- The claim is that **K** is the logic of all frames i.e. $\mathbf{K} = \Lambda_{\mathcal{F}}$ where $\mathcal{F}$ is the class of all frames.
- What is derivable in **hK** is valid on all frames (soundness)
- A formula valid on all frames is derivable (completeness)
- Soundness of the axioms. Let M be an arbitrary model and w some world in M. Show that each axiom holds on M at w.
- Next show soundness of the rules. Supposing that the premises are valid show that the conclusion is also valid
- Completeness entails showing that if A is valid on all frames, then A is a theorem of the Hilbert calculus. We omit the argument here since we can obtain the result using the sequent calculus introduced later.

Some axiomatic extensions of **hK**

- Consider the following axioms

$$
\begin{array}{rll}
4: & \Box p \to \Box\Box p & \text{(or perhaps more clearly } \Diamond\Diamond p \to \Diamond p\text{)} \\
T: & \Box p \to p & \text{(or perhaps more clearly } p \to \Diamond p\text{)} \\
L: & \Box(\Box p \to p) \to \Box p & \text{(Löb axiom)}
\end{array}
$$

- We claim that the addition of these axioms to **hK** yield the following logics:

K4	the logic of transitive frames
KT	the logic of reflexive frames
KL	the logic of finite (irreflexive) transitive trees

- For historical reasons, axiom T is reflexivity (and not transitivity!)
- Check soundness. Completeness is non-trivial.

Obtaining a sequent calculus for **K**

- Let's try to derive the normality axiom $\Box(A \to B) \to (\Box A \to \Box B)$ in **sCp**:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \to B, A \vdash B}\,\to l}{\vdots}}{\Box(A \to B), \Box A \vdash \Box B}}{\dfrac{\Box(A \to B) \vdash (\Box A \to \Box B)}{\vdash \Box(A \to B) \to (\Box A \to \Box B)}\,\to r}\,\to r$$

- How to fill in the $\ldots$?
- We might 'guess' the following

$$\dfrac{X \vdash A}{\Box X \vdash \Box A}\,\Box K$$

- Here $\Box X$ is notation

$$X = \{A_1, \ldots, A_n\} \qquad \Box X = \{\Box A_1, \ldots, \Box A_n\}$$

A sequent calculus **sK** for the modal logic K

- Add the $\Box$K rule to the sequent calculus for classical logic.

$$\frac{X \vdash A}{\Box X \vdash \Box A}\,\Box K$$

- We claim that **sK** is sound and complete for **K**
- Soundness. In the case of **sCp** we argued soundness from premise to conclusion. For the $\Box K$ rule, it is easier to argue contrapositively. Suppose that $\wedge \Box X \rightarrow \Box A$ is not valid. We need to show that $\wedge X \rightarrow A$ is not valid.
- Completeness: Show that sK derives all the axioms of hK and simulates all the rules.
- The $\Box K$ rule simulates necessitation. Add the cut-rule to simulate MP
- Since we ultimately want a calculus with the subformula property, we need to show (surprise. . .) cut-elimination.

Cut-elimination for *sK*

- Recall the Gentzen-style cut-elimination (primary induction on size of cutformula, secondary induction on cutheight)
  1. Base case. Consider when the cutheight is minimal.
  2. Inductive case. Either the cutformula is principal in both premises or it is not principal in at least one premise.
- Let us consider the case of principal cuts (i.e. cutformula is principal in both premises)

$$\dfrac{\dfrac{X \vdash A}{\Box X \vdash \Box A}\,\Box\mathrm{K} \qquad \dfrac{A, Y \vdash C}{\Box A, \Box Y \vdash \Box C}\,\Box\mathrm{K}}{\Box X, \Box Y \vdash \Box C}\,\mathrm{cut}$$

Lift cut, then apply induction hypothesis, finally reapply $\Box$K

$$\dfrac{X \vdash A \qquad A, Y \vdash C}{X, Y \vdash C}\,\mathrm{cut}$$

induction hypothesis yields cutfree:

$$\dfrac{X, Y \vdash C}{\Box X, \Box Y \vdash \Box C}\,\Box\mathrm{K}$$

A sequent calculus **sK**4 for **K**4

- Recall: **K**4 is the logic of transitive frames (T is for reflexive, remember?)
- Here is the rule encountered in the literature.

$$\frac{\Box X, X \vdash A}{\Box X \vdash \Box A}\,\Box 4$$

- Soundness and completeness of **sK**4 wrt **K**4
- Check soundness of $\Box 4$ and derive the 4 axiom.
- Simulating modus ponens leads us to introduce the cutrule...
- ...subformula property considerations motivate us to eliminate the cutrule...
- ...blah blah...

The modal provability logic GL

- $GL = \mathbf{K} + \Box(\Box p \supset p) \supset \Box p$ (Löb's axiom)
- characterised by the class $\mathcal{F}_{GL}$ of Kripke frames satisfying transitivity and no ∞-R-chains (finite transitive trees)
- I.e. for every formula A: $A \in GL$ iff $\mathcal{F}_{GL} \models A$
- proof omitted
- Interpreting $\Box A$ as "$\bar{A}$ is provable in Peano arithmetic" (frequently written $Bew(\bar{A})$) GL is sound and complete wrt formal provability interpretation in Peano arithmetic (Solovay, 1976).
- Hence the name provability logic
- The logic is decidable (a benefit of studying a fragment of Peano arithmetic)

A sequent calculus for *GL*

- **K**:

$$\frac{X \Rightarrow A}{\Box X \Rightarrow \Box A}\ \Box\mathrm{K}$$

- **K**4 (the 4 axiom is $\Box A \supset \Box\Box A$ and corresponds to transitivity)

$$\frac{X, \Box X \Rightarrow A}{\Box X \Rightarrow \Box A}\ \Box 4$$

- **GL** (axiomatised by addition of $\Box(\Box A \supset A) \supset \Box A$ to **K**)

$$\frac{\Box X, X, \Box A \Rightarrow A}{\Box X \Rightarrow \Box A}\ GLR$$

(Sambin and Valentini, 1982).
$\Box A$ is called the diagonal formula. Motivated from $\Box 4$ rule.

The sequent calculus **sGL** for **GL**

Initial sequents: $A \Rightarrow A$ for each formula A

Logical rules:

$$\frac{X \Rightarrow Y, A}{X, \neg A \Rightarrow Y}\ L\neg \qquad \frac{A, X \Rightarrow Y}{X \Rightarrow Y, \neg A}\ R\neg$$

$$\frac{A_i, X \Rightarrow Y}{A_1 \wedge A_2, X \Rightarrow Y}\ L\wedge \qquad \frac{X \Rightarrow Y, A_1 \qquad X \Rightarrow Y, A_2}{X \Rightarrow Y, A_1 \wedge A_2}\ R\wedge$$

$$\frac{A_1, X \Rightarrow Y \qquad A_2, X \Rightarrow Y}{A_1 \vee A_2, X \Rightarrow Y}\ L\vee \qquad \frac{X \Rightarrow Y, A_i}{X \Rightarrow Y, A_1 \vee A_2}\ R\vee$$

$$\frac{X \Rightarrow Y, A \qquad B, U \Rightarrow W}{A \supset B, X, U \Rightarrow Y, W} \rightarrow\!\mathrm{L} \qquad \frac{A, X \Rightarrow Y, B}{X \Rightarrow Y, A \supset B} \rightarrow\! R$$

Modal rule:

$$\frac{\Box X, X, \Box A \Rightarrow A}{\Box X \Rightarrow \Box A}\ GLR$$

Structural rules:

$$\frac{X \Rightarrow Y}{A, X \Rightarrow Y}\ LW \qquad \frac{X \Rightarrow Y}{X \Rightarrow Y, A}\ RW$$

Soundness of **sGL** wrt **K***L*

- As before soundness can be verified by taking the contrapositive of each rule and falsifying on a finite transitive irreflexive trees.
- Let us consider the rule *GLR*
- Omitting the context for simplicity, suppose that the conclusion of *GLR* is falsifiable so there is a model M s.t. $M, w_0 \not\models \Box A$. Then there exists w_1 s.t. $M, w_1 \models \neg A$. If $M, w_1 \models \Box A$ then the premise of *GLR* is falsified.
- If $M, w_1 \not\models \Box A$ then there exists w_2 s.t. $M, w_2 \models \neg A$. If $M, w_2 \models \Box A$ then the premise of *GLR* is falsified.
- ... and so on...
- We cannot continue this indefinitely because the trees are finite!
- To see why transitivity is required, consider the contexts too.

Completeness of **sGL** wrt **K**L

- Completeness: simulate *modus ponens* with cut; eliminate cut to obtain subformula property
- An alternative semantic proof of completeness: since $\mathcal{F}_{\mathbf{GL}} \models A$ implies **sGL** derives $\vdash A$, taking the contrapositive it suffices to prove:

  if there is no derivation of $\vdash A$ in **sGL** then $\mathcal{F}_{\mathbf{GL}} \not\models A$

- Idea. Suppose that there is no derivation of $\vdash A$. Use this to build a finite tree that falsifies A at the root.
- Nonetheless, the proof of cut-elimination is interesting so let us sketch the proof.

Syntactic cut-elimination for *GL* - a brief history

- Leivant (1981) suggests a syntactic proof, counter-example by Valentini (1982)
- new proof of syntactic CE for GLS_{set} proposed by Valentini (1983) — induction on $degree \cdot \omega^2 + width \cdot \omega + cutheight$
- Subsequently Borga (1983) and Sasaki (2001) present new proofs
- Moen (2001) claimed that Valentini's proof has a gap when contractions are made explicit
- Many other proofs were subsequently presented as an alternative (e.g. Mints, Negri)
- Goré and R. (2008) show Moen's claim is incorrect, Valentini's argument is sound, and introduce new transformations to deal with contraction
- Dawson and Goré (2010) verify this argument in Isabelle/HOL

Sambin Normal Form

The interesting case is the Sambin Normal Form (SNF) where both Π and Ω are cutfree

$$
\dfrac{\dfrac{\begin{array}{c}\Pi \\ \Box X, X, \Box B \overset{k}{\Rightarrow} B\end{array}}{\Box X \overset{k+1}{\Rightarrow} \Box B}\, GLR \qquad \dfrac{\begin{array}{c}\Omega \\ \Box B, B, \Box U, U, \Box D \overset{l}{\Rightarrow} D\end{array}}{\Box B, \Box U \overset{l+1}{\Rightarrow} \Box D}\, GLR}{\Box X, \Box U \Rightarrow \Box D}\, cut(\Box B)
$$

cut-height is $(k+1) + (l+1)$. degree of cut-formula is $d(\Box B)$.

The principal case — a derivation in SNF

A derivation is in Sambin Normal Form when:

- the last rule is the cut rule with cutfree premises
- the cut-formula is principal by *GLR* in both premises

A naive transformation to eliminate cut:

$$
\dfrac{
\dfrac{\dfrac{\Pi}{\Box X, X, \Box B \overset{k}{\Rightarrow} B}}{\Box X \overset{k+1}{\Rightarrow} \Box B}\,GLR
\qquad
\dfrac{\dfrac{\Pi}{X, \Box X, \Box B \overset{k}{\Rightarrow} B} \qquad \dfrac{\Omega}{\Box B, B, \Box U, U, \Box D \overset{l}{\Rightarrow} D}}{X, \Box X, \Box B, \Box U, U, \Box D \overset{[k,l]+1}{\Rightarrow} D}\,cut_1
}{
\dfrac{\dfrac{X, \Box X, \Box X, \Box U, U, \Box D \Rightarrow D}{X, \Box X, \Box U, U, \Box D \Rightarrow D}\,LC^*(\Box X)}{\Box X, \Box U \Rightarrow \Box D}\,GL
}\,cut_2
$$

Cut-height is $k + l$ (cut_1) and $(k+1) + ([k,l]+1)$ (cut_2)
Problem with cut_2 !

A successful transformation for SNF

Transform derivation in SNF to:

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\Sigma\\ \Box X, X \Rightarrow B\end{array} \quad \dfrac{\dfrac{\begin{array}{c}\Pi\\ \Box X, X, \Box B \overset{k}{\Rightarrow} B\end{array}}{\Box X \overset{k+1}{\Rightarrow} \Box B}\;GLR \quad \begin{array}{c}\Omega\\ \Box B, B, \Box U, U, \Box D \overset{l}{\Rightarrow} D\end{array}}{\Box X, B, \Box U, U, \Box D \Rightarrow D}\;cut_1}{\Box X, \Box X, X, \Box U, U, \Box D \Rightarrow D}\;cut_2}{\Box X, X, \Box U, U, \Box D \Rightarrow D}\;LC^*(\Box X)}{\Box X, \Box U \Rightarrow \Box D}\;GLR
$$

where Σ is some cut-free derivation.

- cut_1 has cut-height $(k+1)+l$
- cut_2 has smaller degree of cut-formula

New task: obtain a cut-free derivation of $\Box X, X \Rightarrow B$ from a derivation of $\Box X, X, \Box B \Rightarrow B$

A sketch of the proof of $\Box X, X \vdash B$ from $\Box X, X, \Box B \vdash B$

The width is the number n of occurrences of the following schema, where no GLR rule occurrences appear between GLR_1 and GLR_2

$$\cfrac{\cfrac{\cfrac{\cfrac{\Box G, G, \Box B, B, \Box C \Rightarrow C}{\Box G, \Box B \Rightarrow \Box C}\, GLR_2}{\vdots}}{\Box X, X, \Box B \Rightarrow B}}{\Box X \vdash \Box B}\, GLR_1$$

If $n = 0$ then the $\Box B$ in $\Box X, X, \Box B \Rightarrow B$ has either been introduced by

1. $LW(\Box B)$. In this case delete the $LW(\Box B)$ rule. Or,
2. the initial sequent $\Box B \Rightarrow \Box B$. Replace with $\Box X \Rightarrow \Box B$.

In this way we obtain a derivation of $\Box X, X \vdash B$.

The width is the number n of occurrences of the following schema, where no GLR rule occurrences appear between GLR_1 and GLR_2

$$\cfrac{\cfrac{\cfrac{\Box G, G, \Box B, B, \Box C \Rightarrow C}{\Box G, \Box B \Rightarrow \Box C}\,GLR_2}{\vdots}}{\cfrac{\Box X, X, \Box B \Rightarrow B}{\Box X \vdash \Box B}\,GLR_1}$$

If $n = k + 1$, each occurrence of the above schema is deleted as follows. Replace below left by below right.

$$\frac{\Box G, G, \Box B, B, \Box C \Rightarrow C}{\Box G, \Box B \Rightarrow \Box C}\,GLR_2 \qquad \frac{\Box C \Rightarrow \Box C}{\Box G, \Box B, \Box C \Rightarrow \Box C}\,\mathsf{lw}$$

Continuing downwards we obtain a derivation of $\Box X, \Box C \vdash \Box B$ with smaller width.

Now proceed:

$$\dfrac{\Box X, \Box C \vdash \Box B \qquad \dfrac{\Box X, X, \Box B \vdash B \qquad \Box G, G, \Box B, B, \Box C \Rightarrow C}{\Box X, X, \Box G, G, \Box B, \Box C \vdash C}\text{ cut}}{\Box X, \Box X, X, \Box G, G, \Box C, \Box C \vdash C}\text{ cut}$$

The second cut has lesser width than before! So we obtain a cutfree derivation of $\Box X, X, \Box G, G, \Box C \vdash C$.

Now replace below left in original derivation with below right.

$$\dfrac{\Box G, G, \Box B, B, \Box C \Rightarrow C}{\Box G, \Box B \Rightarrow \Box C}\ GLR_2 \qquad \dfrac{\dfrac{\Box X, X, \Box G, G, \Box C \vdash C}{\Box X, \Box G \vdash \Box C}\text{ GLR}}{\Box X, \Box G, \Box B \vdash \Box C}\text{ lw}$$

We thus obtain a derivation of the following of lesser width.

$$\dfrac{\Box X, X, \Box B \Rightarrow B}{\Box X \vdash \Box B}\ GLR_1$$

GL, **Grz** and **Go**

$$L: \quad \Box(\Box p \supset p) \supset \Box p \quad \text{(Löb's axiom)}$$

$$Grz: \quad \Box(\Box(p \supset \Box p) \supset p) \supset p$$

$$Go: \quad \Box(\Box(p \supset \Box p) \supset p) \supset \Box p$$

$$\mathbf{GL}{=}\mathbf{K} + L \quad \mathbf{Go}{=}\mathbf{K} + Go \quad \mathbf{Grz}{=}\mathbf{K} + Grz$$

A sequent calculus for **Grz** is obtained by adding the rules below left and center. For **Go** add rule below right.

$$\frac{B, X \Rightarrow Y}{\Box B, X \Rightarrow Y} \qquad \frac{\Box X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B} \qquad \frac{\Box X, X, \Box(B \supset \Box B) \Rightarrow B}{\Box X \Rightarrow \Box B}\ GoR$$

- **sGrz** has cut-elimination (Borga and Gentilini, 1986). Reflexivity rule above left simplifies argument.
- Cut-elimination for **sGo** (Goré and R., 2013).
- The proof requires a deeper study of the derivation (not just the GoR_2 rule instance). Extends Valentini's argument for **sGL** and uses a quaternary induction measure

Extending the sequent calculus to present more logics

- The sequent calculus is simple to work with
- However, it is hard to extend the proofs of cut-elimination for axiomatic extensions...
- The addition of a new rule typically breaks cut-elimination
- This motivates the extension of the sequent calculus to yield modular extensions (see next page!)

Labelled Sequents

A very general method for constructing sequent calculi from frame conditions was developed e.g. in (Viganò, 2000), (Negri, 2005 and 2011)

Main idea: Explicitly include the Kripke semantics in the calculus

Definition

Let $u, v, w, \ldots$ be a countably infinite set of labels.

- A labelled modal formula has the form $w : A$ for a label w and a modal formula A.
- A relational term has the form wRv for labels w, v.
- A labelled sequent is a sequent consisting of labelled modal formulae and relational terms.

The calculus G3K

The modal rules of the labelled sequent calculus G3K for modal logic **K** are

$$\frac{\Gamma, wRv \vdash \Delta, v : A}{\Gamma \vdash \Delta, w : \Box A}\ R\Box \qquad \frac{\Gamma, v : A, w : \Box A, wRv \vdash \Delta}{\Gamma, w : \Box A, wRv \vdash \Delta}\ L\Box$$

(v does not occur in Γ, Δ)

Intuition behind the rules:

- $R\Box$ is equivalent to the condition

$$\forall v. \, (wRv \implies v : A) \implies w : \Box A$$

- $L\Box$ is equivalent to the condition

$$w : \Box A \text{ and } wRv \implies v : A$$

The calculus G3K - propositional part

The propositional rules of G3K are essentially the standard ones extended with labels:

$$\frac{}{\Gamma, w : \bot \vdash \Delta}\ L\bot$$

$$\frac{}{\Gamma, w : p \vdash w : p, \Delta} \qquad \frac{}{\Gamma, wRv \vdash wRv, \Delta}$$

$$\frac{\Gamma, w : A, w : B \vdash \Delta}{\Gamma, w : A \wedge B \vdash \Delta}\ L\wedge \qquad \frac{\Gamma \vdash w : A, \Delta \quad \Gamma \vdash w : B, \Delta}{\Gamma \vdash w : A \wedge B, \Delta}\ R\wedge$$

$$\frac{\Gamma, w : A \vdash \Delta \quad \Gamma, w : B \vdash \Delta}{\Gamma, w : A \vee B \vdash \Delta}\ L\vee \qquad \frac{\Gamma \vdash w : A, w : B\Delta}{\Gamma \vdash w : A \vee B\Delta}\ R\vee$$

$$\frac{\Gamma, w : B \rightarrow \Delta \quad \Gamma \vdash w : A, \Delta}{\Gamma, w : A \rightarrow B \vdash \Delta}\ L\rightarrow \qquad \frac{\Gamma, w : A \rightarrow w : B, \Delta}{\Gamma \vdash w : A \rightarrow B, \Delta}\ R\rightarrow$$

The calculus G3K

Example

The axiom $\Box(p \to q) \to (\Box p \to \Box q)$ is derived as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, v : q, v : p \vdash v : q}\,\text{init} \qquad \dfrac{}{\Gamma, v : p \vdash v : p, v : q}\,\text{init}}{w : \Box(p \to q), w : \Box p, wRv, v : p \to q, v : p \vdash v : q}\,L\to}{w : \Box(p \to q), w : \Box p, wRv \vdash v : q}\,L\Box}{\dfrac{w : \Box(p \to q), w : \Box p \vdash w : \Box q}{w : \Box(p \to q) \to (\Box p \to \Box q)}\,R\to}\,R\Box
$$

The calculus G3K - useful properties

Proposition

The following properties can all be established by standard methods (mostly induction on the depth of the derivation):

- The sequent $\Gamma, w : A \vdash w : A, \Delta$ is derivable for every A
- Substitution of labels $\dfrac{\Gamma \vdash \Delta}{\Gamma(v/w) \vdash \Delta(v/w)}$ is depth-preserving admissible.
- Weakening is depth-preserving admissible.
- The labelled necessitation rule $\dfrac{\vdash w : A}{\vdash w : \Box A}$ is derivable.
- The rules of G3K are depth-preserving invertible.
- Contraction is depth-preserving admissible.

Soundness and completeness

The cut rule in the labelled sequent framework, written cut_ℓ, comes in two shapes, depending on the shape of the cut formula:

$$\frac{\Gamma \vdash \Delta, w : A \quad w : A, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi} \qquad \frac{\Gamma \vdash \Delta, wRv \quad wRv, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi}$$

Theorem

The calculus $\mathsf{G3Kcut}_\ell$ *is sound and complete for modal logic* $\mathbf{K}$*, i.e., for every formula A:*

$$A \text{ is a theorem of } \mathbf{K} \quad \textit{iff} \quad \vdash w : A \text{ is derivable in } \mathsf{G3Kcut}_\ell\ .$$

Sketch of proof.

Since the labelled necessitation rule is admissible, deriving the axioms of $\mathbf{K}$ and simulating modus ponens using cut_ℓ is enough.

Cut Elimination for G3K

The cut elimination proof is essentially the standard one, using a double induction on the size of the cut formula and the height of the cut (the sum of the depths of the derivations of its premisses).

The interesting case:

$$\dfrac{\dfrac{\Gamma, wRx \vdash \Delta, x : A}{\Gamma \vdash \Delta, w : \Box A}\, R\Box \qquad \dfrac{w : \Box A, wRv, v : A, \Sigma \vdash \Pi}{w : \Box A, wRv, \Sigma \vdash \Pi}\, L\Box}{\Gamma, wRv, \Sigma \vdash \Delta, \Pi}\, \mathrm{cut}_\ell$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\Gamma, wRx \vdash \Delta, x : A}{\Gamma, wRv \vdash \Delta, v : A}\, sb \qquad \dfrac{\dfrac{\Gamma, wRx \vdash \Delta, x : A}{\Gamma \vdash \Delta, w : \Box A}\, R\Box \quad w : \Box A, wRv, v : A, \Sigma \vdash \Pi}{\Gamma, v : A, wRv, \Sigma \vdash \Delta, \Pi}\, \mathrm{cut}_\ell}{\Gamma, wRv, \Gamma, wRv, \Sigma \vdash \Delta, \Delta, \Pi}\, \mathrm{cut}_\ell}{\Gamma, wRv, \Sigma \vdash \Delta, \Pi}\, \mathrm{Con}$$

Cut Elimination for G3K

The cut elimination proof is essentially the standard one, using a double induction on the size of the cut formula and the height of the cut (the sum of the depths of the derivations of its premisses).

Theorem

The labelled cut rule is admissible in G3K*. Hence the calculus* G3K *is cut-free complete for modal logic* **K***, i.e.:*

If A is a theorem of **K** *then* $\vdash w : A$ *is derivable in* G3K .

Converting frame conditions into rules

Definition

A geometric axiom is a formula of the form

$$\forall\vec{x}(P \to \exists\vec{y_1}M_1 \vee \cdots \vee \exists\vec{y_n}M_n)$$

where

- the M_j and P are conjunctions of relational terms
- the variables $\vec{y_j}$ are not free in P.

Examples

- $\forall x\, xRx$ for reflexivity
- $\forall x, y, z\, (xRy \wedge yRz \to xRz)$ for transitivity
- $\forall x, y\, (xRy \to yRx)$ for symmetry
- $\forall x, y, z\, (xRy \wedge xRz \to \exists w\, (yRw \wedge zRw))$ for directedness

Converting frame conditions into rules

Definition

A geometric axiom is a formula of the form

$$\forall \vec{x}(P \to \exists \vec{y_1} M_1 \vee \cdots \vee \exists \vec{y_n} M_n)$$

where

- the M_j and P are conjunctions of relational terms
- the variables $\vec{y_j}$ are not free in P.

Theorem

The geometric axiom above is equivalent to the geometric rule

$$\frac{\Gamma, \bar{P}, \bar{M}_1(z_1/y_1) \vdash \Delta \quad \ldots \quad \Gamma, \bar{P}, \bar{M}_n(z_n/y_n) \vdash \Delta}{\Gamma, \bar{P} \vdash \Delta}$$

with $\bar{M}_i$ and $\bar{P}$ the multisets of relational atoms in M_i resp. P, and $z_1, \ldots, z_n$ not in the conclusion.

Converting frame conditions into rules: Examples

- Reflexivity $\forall x\, xRx$ is converted to

$$\frac{\Gamma, yRy \vdash \Delta}{\Gamma \vdash \Delta}$$

- Transitivity $\forall x, y, z\, (xRy \wedge yRz \to xRz)$ is converted to

$$\frac{\Gamma, xRy, yRz, xRz \vdash \Delta}{\Gamma, xRy, yRz \vdash \Delta}$$

- Symmetry $\forall x, y\, (xRy \to yRx)$ is converted to

$$\frac{\Gamma, xRy, yRz \vdash \Delta}{\Gamma, xRy \vdash \Delta}$$

- Directedness $\forall x, y, z\, (xRy \wedge xRz \to \exists w\, (yRw \wedge zRw))$ gives

$$\frac{\Gamma, xRy, xRz, yRv, zRv \vdash \Delta}{\Gamma, xRy, xRz \vdash \Delta}\ v \text{ not in conclusion}$$

Converting frame conditions into rules: Contraction

To obtain the nice structural properties for extensions of G3K with geometric rules we need to close the rule set under contraction:

Definition

A geometric rule set satisfies the closure condition if for every rule

$$\frac{\Gamma, \bar{P}, Q, R, \bar{M}_1(z_1/y_1) \vdash \Delta \quad \dots \quad \Gamma, \bar{P}, Q, R, \bar{M}_n(z_n/y_n) \vdash \Delta}{\Gamma, \bar{P}, Q, R \vdash \Delta}$$

and injective renaming σ with $Q\sigma = R\sigma = Q$ it also includes

$$\frac{\Gamma, \bar{P}\sigma, Q, \bar{M}_1\sigma(z_1/y_1\sigma) \vdash \Delta \quad \dots \quad \Gamma, \bar{P}\sigma, Q, \bar{M}_n\sigma(z_n/y_n\sigma) \vdash \Delta}{\Gamma, \bar{P}\sigma, Q \vdash \Delta}$$

Lemma

Contraction is admissible in extensions of G3K *with geometric rules satisfying the closure condition.*

Converting frame conditions into rules: Contraction

To obtain the nice structural properties for extensions of G3K with geometric rules we need to close the rule set under contraction:

Example

For directedness

$$\frac{\Gamma, xRy, xRz, yRv, zRv \vdash \Delta}{\Gamma, xRy, xRz \vdash \Delta}\ v \text{ not in conclusion}$$

we need to add the rule which identifies y and z and contracts the two occurrences of xRy:

$$\frac{\Gamma, xRy, yRv, yRv \vdash \Delta}{\Gamma, xRy \vdash \Delta}\ v \text{ not in conclusion}$$

Remark: Closing a rule set under contraction only demands the addition of finitely many rules and thus is unproblematic!

Cut elimination for extended calculi

The so constructed geometric rules

$$\frac{\Gamma, \bar{P}, \bar{M}_1(z_1/y_1) \vdash \Delta \quad \dots \quad \Gamma, \bar{P}, \bar{M}_n(z_n/y_n) \vdash \Delta}{\Gamma, \bar{P} \vdash \Delta}$$

have nice properties: all their active parts

- occur on the left hand side only
- consist of relational terms only
- occur in the premisses if they occur in the conclusion.

Hence we can add them to G3K without harming cut elimination!

Cut elimination for extended calculi

Theorem

If G3K* *is an extension of* G3K *by finitely many geometric rules satisfying the closure condition, then* cut$_\ell$ *is admissible in* G3K.

Proof.

As for G3K, possibly renaming variables. E.g. for directedness:

$$\dfrac{\dfrac{\Gamma \vdash \Delta, v : A}{\Gamma \vdash \Delta, w : \Box A}\, R\Box \qquad \dfrac{w : \Box A, \Sigma, xRy, xRz, yRv, zRv \vdash \Pi}{w : \Box A, \Sigma, xRy, xRz \vdash \Pi}\, dir}{\Gamma, \Sigma, xRy, xRz \vdash \Delta, \Pi}\, \mathrm{cut}_\ell$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma \vdash \Delta, v : A}{\Gamma \vdash \Delta, w : \Box A}\, R\Box \qquad \dfrac{w : \Box A, \Sigma, xRy, xRz, yRv, zRv \vdash \Pi}{w : \Box A, \Sigma, xRy, xRz, yRu, zRu \vdash \Pi}\, sub}{\Gamma, \Sigma, xRy, xRz, yRu, zRu \vdash \Delta, \Pi}\, \mathrm{cut}_\ell}{\Gamma, \Sigma, xRy, xRz \vdash \Delta, \Pi}\, dir$$

where u does not occur in $\Gamma, \Sigma, xRy, xRz \vdash \Delta, \Pi$. $\Box$

Where's the catch?

So, labelled sequent calculi seem ideal to treat modal logics.

However, there are some issues:

- Decidability results need to be shown for every single logic.
- since the method is based heavily on Kripke semantics, the modification for non-normal modal logics is not immediately clear (see however (Gilbert and Maffezioli, 2015) and recent work by Negri).
- The calculi are not fully internal: there seems not to be a formula translation of a labelled sequent.

Recovering labelled sequents with a formula translation

- Following (Fitting 2012) and (Goré and R. 2012), let us see how the labelled sequents might be restricted to those which support a formula translation.
- First of all, let us treat formulae in negation normal form (pushing all negations inwards onto the propositional variables)
- This preserves equivalence because in every extension of **K**:

$$
\begin{array}{rlrl}
\neg\Box A & = \Diamond\neg A & \neg\Diamond A & = \Box\neg A \\
\neg(A \wedge B) & = \neg A \vee \neg B & \neg(A \vee B) & = \neg A \wedge \neg B \\
\neg(A \to B) & = A \wedge \neg B & &
\end{array}
$$

- In fact, while we are at it, let us eliminate $A \to B$ in favour of $\neg A \vee B$
- Only a small apology for changing notation at this (late) stage: notation is notation, choose what works best

- With these changes, G3K can be written as follows:

$$\dfrac{}{\mathcal{R}, x : p, x : \overline{p}, \Gamma}\ \text{init} \qquad \dfrac{\mathcal{R}, x : A, x : B, \Gamma}{\mathcal{R}, x : A \vee B, \Gamma}\ \vee$$

$$\dfrac{\mathcal{R}, x : A, \Gamma \qquad \mathcal{R}, x : B, \Gamma}{\mathcal{R}, x : A \wedge B, \Gamma}\ \wedge \qquad \dfrac{\mathcal{R}, Rxy, y : A, \Gamma}{\mathcal{R}, x : \Box A, \Gamma}\ \Box^{*}$$

$$\dfrac{\mathcal{R}, Rxy, y : A, x : \Diamond A, \Gamma}{\mathcal{R}, Rxy, x : \Diamond A, \Gamma}\ \Diamond$$

*eigenvariable y does not occur in conclusion

- Here $\mathcal{R}$ consists of relational terms Rxy (possibly empty)
- Interpreting each Rxy as an edge (x, y), we naturally obtain a graph from $\mathcal{R}$
- So the labelled sequent $\mathcal{R}, \Gamma$ is a labelled graph

Labelled tree sequents = nested sequents

Definition

A labelled tree sequent (or LTS) is a labelled sequent $\mathcal{R}, \Gamma$ where $\mathcal{R}$ defines a tree

- A LTS calculus is a labelled sequent calculus where every sequent is a LTS
- Since a labelled tree sequent is a labelled tree, we can define its grammar:

$$\Gamma := A_1, \ldots, A_n, [\Gamma], \ldots, [\Gamma]$$

- With the added constraints: finite and non-empty
- This object is precisely a nested sequent; these have been investigated independently since (Kashima, 1994) and independently rediscovered by (Poggiolessi, 2009) and (Brünnler, 2009).

Nested sequent calculus/LTS calculus for **K**

- Notation: $\Gamma\{\Delta\}$ refers to an occurrence of the sequent Δ inside Γ. $\Gamma\{\ \}$ is called a context

$$\frac{}{\Gamma\{p, \overline{p}\}}\ \text{init} \qquad \frac{\Gamma\{A\} \qquad \Gamma\{B\}}{\Gamma\{A \wedge B\}}\ (\wedge) \qquad \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}\ (\vee)$$

$$\frac{\Gamma\{[\Delta, A], \Diamond A\}}{\Gamma\{[\Delta], \Diamond A\}}\ (\Diamond) \qquad \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}}\ (\Box)$$

- NS calculi (equivalently LTS calculi) have been presented for many modal logics, intuitionistic modal logics and constructive modal logics.
- Note: in general we cannot use the structural rule extensions of G3K (to present axiomatic extensions of **K**) because they are not LTS rules. Non-structural rules are typically required.

- In these systems, a nested sequent Γ below left has the formula interpretation $\mathcal{I}(\Gamma)$ below right

$$A_1, \ldots, A_n, [\Gamma_1], \ldots, [\Gamma_m] \quad A_1 \vee \ldots \vee A_n \vee \Box\mathcal{I}(\Gamma_1) \vee \ldots \vee \Box\mathcal{I}(\Gamma_m)$$

- The claim that NS calculi are more 'internal'/preferred over LS calculi because they support a formula interpretation is misleading
- More accurate: NS calculi and some LS calculi (in particular LTS calculi) support a formula interpretation. Some LS calculi seem not to.
- (Fitting 2015) extended the NS formalism to indexed nested sequents in order to give cutfree proof systems for logics like $K + \Diamond\Box p \rightarrow \Box\Diamond p$. The notational variant labelled formalism is LTS with equality (R. 2016). It is not clear if it is possible to interpret the sequents as formulae.

One final extension: the display calculus for tense logic **Kt**

- The nested sequent had a single type of nesting. Following (Goré *et al.* 2011) define a display sequent with two types of nesting $\circ[]$ and $\bullet[]$:

$$\Gamma := A_1, \ldots, A_n, \circ[\Gamma], \ldots, \circ[\Gamma], \bullet[\Gamma], \ldots, \bullet[\Gamma]$$

$$\frac{}{\Gamma, p, \overline{p}}\ \text{init} \qquad \frac{\Gamma, A, B}{\Gamma, A \vee B}\ \vee \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\ \wedge$$

$$\frac{\Gamma, \Delta, \Delta}{\Gamma, \Delta}\ \text{c} \qquad \frac{\Gamma}{\Gamma, \Delta}\ \text{w} \qquad \frac{\Gamma, \circ[\Delta]}{\bullet[\Gamma], \Delta}\ \text{rf} \qquad \frac{\Gamma, \bullet[\Delta]}{\circ[\Gamma], \Delta}\ \text{rp}$$

$$\frac{\Gamma, \bullet[A]}{\Gamma, \blacksquare A}\ \blacksquare \qquad \frac{\Gamma, \circ[A]}{\Gamma, \Box A}\ \Box \qquad \frac{\Gamma, \bullet[\Delta, A], \blacklozenge A}{\Gamma, \bullet[\Delta], \blacklozenge A}\ \blacklozenge \qquad \frac{\Gamma, \circ[\Delta, A], \Diamond A}{\Gamma, \circ[\Delta], \Diamond A}\ \Diamond$$

- (Kracht 1996) uses the structural rule below for a display calculus for $\mathbf{Kt} + \Diamond^h \Box^i p \to \Box^j \Diamond^k p = \mathbf{Kt} + \blacklozenge^h \Diamond^j p \to \Diamond^i \blacklozenge^k p$.

$$\frac{\Gamma, \circ^i\{\bullet^k\{\Delta\}\}}{\Gamma, \bullet^h\{\circ^j\{\Delta\}\}}\ d(h, i, j, k)$$

- The computation of these rules from axioms has a nice algorithm! Limitative results by (Kracht 1996) for tense logics (Display Theorem I), modal logic case open.