Report on 3-Valued Łukasiewicz Logic

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We check a number of properties in the 3-valued Łukasiewicz logic.
The logic contains the connectives

$$\wedge, \rightarrow, \neg, \vee, \otimes, \oplus$$

and truth values

$$\mathbf{0}, \mathbf{1/2}, \mathbf{1}.$$

The truth value $\mathbf{1}$ is designated.

1 Wajsberg's axioms for Łukasiewicz logic

Proposition 1 *The formula* $(A \rightarrow (B \rightarrow A))$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[(A \rightarrow (B \rightarrow A))^{\mathbf{1}}]$$

Derivation of $[(A \rightarrow (B \rightarrow A))^{\mathbf{1}}]$:

$$\dfrac{\dfrac{\overset{\text{axiom for } A}{[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, (B \rightarrow A)^{\mathbf{1/2}}]} \qquad \dfrac{\overset{\text{axiom for } A}{[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}]} \quad \overset{\text{axiom for } B}{[A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]}}{[A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, (B \rightarrow A)^{\mathbf{1/2}}]}}{[A^{\mathbf{0}}, (B \rightarrow A)^{\mathbf{1}}, (B \rightarrow A)^{\mathbf{1/2}}]} \qquad \dfrac{\overset{\text{axiom for } A}{[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}]} \quad \overset{\text{axiom for } A}{[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}]}}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, (B \rightarrow A)^{\mathbf{1}}]}$$

$$\overline{[(A \rightarrow (B \rightarrow A))^{\mathbf{1}}]}$$

Proposition 2 *The formula* $((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)))$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)))^{\mathbf{1}}]$$

Derivation of $[((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)))^{\mathbf{1}}]$:

$$\dfrac{\overset{\text{axiom for } B}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}]} \quad \overset{\text{axiom for } C}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}]}}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, (B \rightarrow C)^{\mathbf{1/2}}]} \qquad [A$$

$$\overline{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, (A}$$

$$\overset{\text{axiom for } C}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, ((B \rightarrow C) \rightarrow (A \rightarrow C))^{\mathbf{1/2}}]}$$

$$\overline{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, (B \rightarrow C)^{\mathbf{0}}, ((B \rightarrow C) \rightarrow (A \rightarrow C))^{\mathbf{1}}}$$

Proposition 3 *The formula* $(((A \to \neg A) \to A) \to A)$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[(((A \to \neg A) \to A) \to A)^{\mathbf{1}}]$$

Derivation of $[(((A \to \neg A) \to A) \to A)^{\mathbf{1}}]$:

$$\text{axiom for } A$$
$$[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}]$$
$$[A^{\mathbf{0}}, A^{\mathbf{1}}, \neg A^{\mathbf{1/2}}]$$
$$[A^{\mathbf{0}}, A^{\mathbf{1}}, \neg A^{\mathbf{1}}, \neg A^{\mathbf{1/2}}] \qquad \text{axio}\ldots\ [A^{\mathbf{0}}, A^{\mathbf{1}}, \ldots$$

$$\text{axiom for } A\ [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, \neg A^{\mathbf{1}}, \neg A^{\mathbf{1/2}}] \qquad \text{axiom for } A\ [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, \neg A^{\mathbf{1}}] \qquad \text{axiom for } A\ [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, (A \to \neg A)^{\mathbf{1/2}}] \qquad [A^{\mathbf{0}}, A^{\mathbf{1}}, (A \to \neg A)^{\mathbf{1}}]$$

$$\text{axiom for } A\ [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}] \qquad [A^{\mathbf{1}}, A^{\mathbf{1/2}}, (A \to \neg A)^{\mathbf{1}}] \qquad [A^{\mathbf{0}}, A^{\mathbf{1}}, ((A \to \neg A) \to A)^{\mathbf{1/2}}]$$

$$[A^{\mathbf{1}}, A^{\mathbf{1/2}}, ((A \to \neg A) \to A)^{\mathbf{0}}] \qquad [A^{\mathbf{1}}, ((A \ldots$$

$$[(((A \to \neg A) \to A) \to A)^{\mathbf{1}}]$$

Proposition 4 *The formula* $((\neg A \to \neg B) \to (B \to A))$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[((\neg A \to \neg B) \to (B \to A))^{\mathbf{1}}]$$

Derivation of $[((\neg A \to \neg B) \to (B \to A))^{\mathbf{1}}]$:

$$\text{axiom for } B\ [A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}] \qquad \text{axiom for } A\ [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}] \qquad \text{axiom for } B\ [A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}] \qquad \text{a}\ldots\ [A^{\mathbf{0}}, A\ldots$$

$$[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, \neg B^{\mathbf{0}}] \qquad [A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, \neg A^{\mathbf{1}}] \qquad [A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, \neg B^{\mathbf{0}}] \qquad [A^{\mathbf{1}}, A\ldots$$

$$[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, (\neg A \to \neg B)^{\mathbf{0}}] \qquad \text{axiom for } A\ [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, (\neg A \to \neg B)^{\mathbf{0}}] \qquad [A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, (\neg A \ldots$$

$$[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, (B \to A)^{\mathbf{1/2}}, (\neg A \to \neg B)^{\mathbf{0}}] \qquad [A^{\mathbf{1}}, \ldots$$

$$[(B \to A)^{\mathbf{1}}, (B \to A)^{\mathbf{1/2}}, (\neg A \to \neg B)^{\mathbf{0}}]$$

2 Bernays's axioms for classical logic

Proposition 5 *The formula* $(A \to (B \to A))$ *is a tautology.*

Proposition 6 *The formula* $((A \to (A \to B)) \to (A \to B))$ *is **not** a tautology.*

Proposition 7 *The formula* $((A \to (B \to C)) \to (B \to (A \to C)))$ *is a tautology.*

Proposition 8 *The formula* $((B \to C) \to ((A \to B) \to (A \to C)))$ *is a tautology.*

Proposition 9 *The formula* $((A \wedge B) \to A)$ *is a tautology.*

Proposition 10 *The formula* $((A \wedge B) \to B)$ *is a tautology.*

Proposition 11 *The formula* $((A \to B) \to ((A \to C) \to (A \to (B \wedge C))))$ *is a tautology.*

Proposition 12 *The formula* $(A \to (A \vee B))$ *is a tautology.*

Proposition 13 *The formula* $(B \to (A \vee B))$ *is a tautology.*

Proposition 14 *The formula* $((B \to A) \to ((C \to A) \to ((B \vee C) \to A)))$ *is a tautology.*

Proposition 15 *The formula* $((A \to B) \to (\neg B \to \neg A))$ *is a tautology.*

Proposition 16 *The formula* $((A \to \neg A) \to \neg A)$ *is* ***not*** *a tautology.*

Proposition 17 *The formula* $(A \to \neg\neg A)$ *is a tautology.*

Proposition 18 *The formula* $(\neg\neg A \to A)$ *is a tautology.*

3 Classical tautologies not intuitionistically valid

Proposition 19 *The formula* $(A \vee \neg A)$ *is* ***not*** *a tautology.*

The problem is equivalent to proving the following sequent:

$$[(A \vee \neg A)^{\mathbf{1}}]$$

Derivation of $[(A \vee \neg A)^{\mathbf{1}}]$:

$$\dfrac{\dfrac{\dfrac{\text{hypothesis}}{[A^{\mathbf{0}}, A^{\mathbf{1}}]}}{[A^{\mathbf{1}}, \neg A^{\mathbf{1}}]}}{[(A \vee \neg A)^{\mathbf{1}}]}$$

List of counter-examples:

$$[A^{\mathbf{1/2}}]$$

Proposition 20 *The formula* $(\neg A \vee \neg\neg A)$ *is* ***not*** *a tautology.*

The problem is equivalent to proving the following sequent:

$$[(\neg A \vee \neg\neg A)^{\mathbf{1}}]$$

Derivation of $[(\neg A \vee \neg\neg A)^{\mathbf{1}}]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{hypothesis}}{[A^{\mathbf{0}}, A^{\mathbf{1}}]}}{[A^{\mathbf{0}}, \neg A^{\mathbf{0}}]}}{[A^{\mathbf{0}}, \neg\neg A^{\mathbf{1}}]}}{[\neg A^{\mathbf{1}}, \neg\neg A^{\mathbf{1}}]}}{[(\neg A \vee \neg\neg A)^{\mathbf{1}}]}$$

List of counter-examples:

$$[A^{\mathbf{1/2}}]$$

Proposition 21 *The formula* $((A \to \neg A) \to \neg A)$ *is* ***not*** *a tautology.*

The problem is equivalent to proving the following sequent:

$$[((A \to \neg A) \to \neg A)^{\mathbf{1}}]$$

Derivation of $[((A \to \neg A) \to \neg A)^{\mathbf{1}}]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{axiom for } A\\ [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}]\end{array}}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, \neg A^{\mathbf{0}}]} \quad \begin{array}{c}\text{axiom for } A\\ [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}]\end{array}}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, (A \to \neg A)^{\mathbf{0}}]}}{[A^{\mathbf{0}}, \neg A^{\mathbf{1/2}}, (A \to \neg A)^{\mathbf{0}}]}}{[\neg A^{\mathbf{1}}, \neg A^{\mathbf{1/2}}, (A \to \neg A)^{\mathbf{0}}]} \quad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{axiom for } A\\ [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, \neg A^{\mathbf{1/2}}]\end{array} \quad \dfrac{\begin{array}{c}\text{hypothesis}\\ [A^{\mathbf{0}}, A^{\mathbf{1}}]\end{array}}{[A^{\mathbf{0}}, A^{\mathbf{1}}, \neg A^{\mathbf{0}}]}}{[A^{\mathbf{0}}, A^{\mathbf{1}}, (A \to \neg A)^{\mathbf{1/2}}]}}{[A^{\mathbf{0}}, \neg A^{\mathbf{0}}, (A \to \neg A)^{\mathbf{1/2}}]} \quad \dfrac{\begin{array}{c}\text{axiom for } A\\ [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, \neg A^{\mathbf{1/2}}]\end{array} \quad \dfrac{\begin{array}{c}\text{hypothesis}\\ [A^{\mathbf{0}}, A^{\mathbf{1}}]\end{array}}{[A^{\mathbf{0}}, A^{\mathbf{1}}, \neg A^{\mathbf{0}}]}}{[A^{\mathbf{0}}, A^{\mathbf{1}}, (A \to \neg A)^{\mathbf{1/2}}]}}{[A^{\mathbf{0}}, (A \to \neg A)^{\mathbf{0}}, (A \to \neg A)^{\mathbf{1/2}}]}}{[\neg A^{\mathbf{1}}, (A \to \neg A)^{\mathbf{0}}, (A \to \neg A)^{\mathbf{1/2}}]}}{[((A \to \neg A) \to \neg A)^{\mathbf{1}}]}$$

List of counter-examples:

$$[A^{\mathbf{1/2}}]$$

Proposition 22 *The formula* $((A \to B) \lor (B \to A))$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[((A \to B) \lor (B \to A))^{\mathbf{1}}]$$

Derivation of $[((A \to B) \lor (B \to A))^{\mathbf{1}}]$:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{axiom for } A\\ [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]\end{array} \quad \begin{array}{c}\text{axiom for } B\\ [A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]\end{array}}{[A^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, (B \to A)^{\mathbf{1}}]} \quad \dfrac{\begin{array}{c}\text{axiom for } A\\ [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}]\end{array} \quad \begin{array}{c}\text{axiom for } A\\ [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]\end{array}}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, (B \to A)^{\mathbf{1}}]}}{[(A \to B)^{\mathbf{1}}, (B \to A)^{\mathbf{1}}]}}{[((A \to B) \lor (B \to A))^{\mathbf{1}}]}$$

Proposition 23 *The formula* $((\neg A \to A) \to A)$ *is* ***not*** *a tautology.*

The problem is equivalent to proving the following sequent:

$$[((\neg A \to A) \to A)^{\mathbf{1}}]$$

Derivation of $[((\neg A \to A) \to A)^{\mathbf{1}}]$:

$$\dfrac{\dfrac{\begin{array}{c}\text{axiom for } A\\ [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}]\end{array} \quad \dfrac{\begin{array}{c}\text{axiom for } A\\ [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}]\end{array}}{[A^{\mathbf{1}}, A^{\mathbf{1/2}}, \neg A^{\mathbf{1}}]}}{[A^{\mathbf{1}}, A^{\mathbf{1/2}}, (\neg A \to A)^{\mathbf{0}}]} \quad \dfrac{\dfrac{\begin{array}{c}\text{axiom for } A\\ [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, \neg A^{\mathbf{1/2}}]\end{array} \quad \dfrac{\begin{array}{c}\text{hypothesis}\\ [A^{\mathbf{0}}, A^{\mathbf{1}}]\end{array}}{[A^{\mathbf{0}}, A^{\mathbf{1}}, \neg A^{\mathbf{1}}]}}{[A^{\mathbf{0}}, A^{\mathbf{1}}, (\neg A \to A)^{\mathbf{1/2}}]} \quad \dfrac{\dfrac{\begin{array}{c}\text{axiom for } A\\ [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, \neg A^{\mathbf{1/2}}]\end{array} \quad \dfrac{\begin{array}{c}\text{hypothesis}\\ [A^{\mathbf{0}}, A^{\mathbf{1}}]\end{array}}{[A^{\mathbf{0}}, A^{\mathbf{1}}, \neg A^{\mathbf{1}}]}}{[A^{\mathbf{0}}, A^{\mathbf{1}}, (\neg A \to A)^{\mathbf{1/2}}]}}{[A^{\mathbf{1}}, \neg A^{\mathbf{1}}, (\neg A \to A)^{\mathbf{1/2}}]}}{[A^{\mathbf{1}}, (\neg A \to A)^{\mathbf{0}}, (\neg A \to A)^{\mathbf{1/2}}]}}{[((\neg A \to A) \to A)^{\mathbf{1}}]}$$

List of counter-examples:

$$[A^{\mathbf{1/2}}]$$

Proposition 24 *The formula* $(((A \to B) \to A) \to A)$ *is* ***not*** *a tautology.*

The problem is equivalent to proving the following sequent:

$$[(((A \to B) \to A) \to A)^{\mathbf{1}}]$$

Derivation of $[(((A \to B) \to A) \to A)^{\mathbf{1}}]$:

hypothesis   axiom for A

$[A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}$

axiom for A axiom for A axiom for A

$[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, (A \to B)^{\mathbf{1/2}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1}}, (A \to B)^{\mathbf{1}}]$

axiom for A

$[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}]$ $[A^{\mathbf{1}}, A^{\mathbf{1/2}}, (A \to B)^{\mathbf{1}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1}}, ((A \to B) \to A)^{\mathbf{1/2}}]$

$[A^{\mathbf{1}}, A^{\mathbf{1/2}}, ((A \to B) \to A)^{\mathbf{0}}]$ $[A$

$[(((A \to B) \to A) \to A)^{\mathbf{1}}]$

List of counter-examples:

$$[A^{\mathbf{1/2}}, B^{\mathbf{0}}]$$

4 Some more interesting tautologies

Proposition 25 *The formula* $(A \to (A \to A))$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[(A \to (A \to A))^{\mathbf{1}}]$$

Derivation of $[(A \to (A \to A))^{\mathbf{1}}]$:

axiom for A axiom for A axiom for A axiom for A

$[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, (A \to A)^{\mathbf{1/2}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, (A \to A)^{\mathbf{1/2}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}]$

$[A^{\mathbf{0}}, (A \to A)^{\mathbf{1}}, (A \to A)^{\mathbf{1/2}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1/2}}, (A \to A)^{\mathbf{1}}]$

$[(A \to (A \to A))^{\mathbf{1}}]$

Proposition 26 *The formula* $((A \land (A \to B)) \to B)$ *is* ***not*** *a tautology.*

The problem is equivalent to proving the following sequent:

$$[((A \land (A \to B)) \to B)^{\mathbf{1}}]$$

Derivation of $[((A \land (A \to B)) \to B)^{\mathbf{1}}]$:

axiom for B axiom for

$[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B$

axiom for B hypothesis   axiom for $(A \to B)$ $[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, (A \to B)^{\mathbf{1/2}}]$

$[A^{\mathbf{0}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$ $[A^{\mathbf{0}}, B^{\mathbf{1}}, (A \to B)^{\mathbf{0}}, (A \to B)^{\mathbf{1}}, (A \to B)^{\mathbf{1/2}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, ($

$[A^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, (A \to B)^{\mathbf{0}}]$ $[A^{\mathbf{0}}, B^{\mathbf{1}}, (A \land (A \to B))^{\mathbf{1/2}},$

$[B^{\mathbf{1}}, B^{\mathbf{1/2}}, (A \land (A \to B))^{\mathbf{0}}]$ $[B^{\mathbf{1}}, (A \land (A \to B))^{\mathbf{0}}, (A \land ($

$[((A \land (A \to B)) \to B)^{\mathbf{1}}]$

List of counter-examples:

$$[A^{\mathbf{1/2}}, B^{\mathbf{0}}]$$

Proposition 27 *The formula* $((A \to B) \to ((C \to A) \to (C \to B)))$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[((A \to B) \to ((C \to A) \to (C \to B)))^{\mathbf{1}}]$$

Derivation of $[((A \to B) \to ((C \to A) \to (C \to B)))^{\mathbf{1}}]$:

axiom for B

$[A^{\mathbf{0}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, ((C \to A) \to (C \to B))^{\mathbf{1/2}}]$ $[A^{\mathbf{0}}, I$

axiom for B

$[A^{\mathbf{0}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, (C \to B)^{\mathbf{1/2}}, ((C \to A) \to (C \to B))^{\mathbf{1/2}}]$ $[A^{\mathbf{0}}, B^{\mathbf{0}}, B^{\mathbf{1}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, (C \to B)^{\mathbf{1/2}}, ((C$

$[A^{\mathbf{0}}, B^{\mathbf{0}}, (C \to B)^{\mathbf{1}}, (C \to B)^{\mathbf{1/2}}, ((C \to A) \to (C \to B))^{\mathbf{1/2}}]$

$[B$

Proposition 28 *The formula* $((A \to B) \to ((B \to C) \to (A \to C)))$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[((A \to B) \to ((B \to C) \to (A \to C)))^{\mathbf{1}}]$$

Derivation of $[((A \to B) \to ((B \to C) \to (A \to C)))^{\mathbf{1}}]$:

axiom for B — axiom for C

$[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}]$

$[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, (B \to C)^{\mathbf{1/2}}]$ $[A$

axiom for C

$[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, ((B \to C) \to (A \to C))^{\mathbf{1/2}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, (A$

$[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, (B \to C)^{\mathbf{0}}, ((B \to C) \to (A \to C))$

Proposition 29 *The formula* $((A \to (A \to B)) \to (A \to B))$ *is* ***not*** *a tautology.*

The problem is equivalent to proving the following sequent:

$$[((A \to (A \to B)) \to (A \to B))^{\mathbf{1}}]$$

Derivation of $[((A \to (A \to B)) \to (A \to B))^{1}]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{axiom for } B\\ [A^{0}, A^{1/2}, B^{0}, B^{1}, B^{1/2}]\end{array} \quad \begin{array}{c}\text{axiom for } A\\ [A^{0}, A^{1}, A^{1/2}, B^{1}, B^{1/2}]\end{array}}{[A^{0}, A^{1/2}, B^{1}, B^{1/2}, (A \to B)^{0}]} \quad \begin{array}{c}\text{axiom for } A\\ [A^{0}, A^{1}, A^{1/2}, B^{1}, B^{1/2}]\end{array}}{[A^{0}, A^{1/2}, B^{1}, B^{1/2}, (A \to (A \to B))^{0}]} \quad \begin{array}{c}\text{axiom for } B\\ [A^{0}, A^{1}, B^{0}, B^{1}, B^{1/2}, (A \to (A \to B))^{0}]\end{array}}{[A^{0}, B^{1}, B^{1/2}, (A \to B)^{1/2}, (A \to (A \to B))^{0}]} \quad \begin{array}{c}\text{ax}\ldots\\ [A^{0}, A^{1}\ldots\end{array}}{[(A \to B)^{1}, (A \to \ldots}$$

List of counter-examples:

$$[A^{1/2}, B^{0}]$$

Proposition 30 *The formula* $((A \to \neg A) \to \neg A)$ *is **not** a tautology.*

The problem is equivalent to proving the following sequent:

$$[((A \to \neg A) \to \neg A)^{1}]$$

Derivation of $[((A \to \neg A) \to \neg A)^{1}]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{axiom for } A\\ [A^{0}, A^{1}, A^{1/2}]\end{array}}{[A^{0}, A^{1/2}, \neg A^{0}]} \quad \begin{array}{c}\text{axiom for } A\\ [A^{0}, A^{1}, A^{1/2}]\end{array}}{[A^{0}, A^{1/2}, (A \to \neg A)^{0}]}}{[A^{0}, \neg A^{1/2}, (A \to \neg A)^{0}]}}{[\neg A^{1}, \neg A^{1/2}, (A \to \neg A)^{0}]} \quad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{axiom for } A\\ [A^{0}, A^{1}, A^{1/2}, \neg A^{1/2}]\end{array} \quad \dfrac{\begin{array}{c}\text{hypothesis}\\ [A^{0}, A^{1}]\end{array}}{[A^{0}, A^{1}, \neg A^{0}]}}{[A^{0}, A^{1}, (A \to \neg A)^{1/2}]}}{[A^{0}, \neg A^{0}, (A \to \neg A)^{1/2}]} \quad \dfrac{\begin{array}{c}\text{axiom for } A\\ [A^{0}, A^{1}, A^{1/2}, \neg A^{1/2}]\end{array} \quad \dfrac{\begin{array}{c}\text{hypothesis}\\ [A^{0}, A^{1}]\end{array}}{[A^{0}, A^{1}, \neg A^{0}]}}{[A^{0}, A^{1}, (A \to \neg A)^{1/2}]}}{[A^{0}, (A \to \neg A)^{0}, (A \to \neg A)^{1/2}]}}{[\neg A^{1}, (A \to \neg A)^{0}, (A \to \neg A)^{1/2}]}}{[((A \to \neg A) \to \neg A)^{1}]}$$

List of counter-examples:

$$[A^{1/2}]$$

5 Some popular consequences

Proposition 31 *The following consequence holds:*

$$A, (A \to B) \vdash B$$

The problem is equivalent to proving the following sequent:

$$[A^{0}, A^{1/2}, B^{1}, (A \to B)^{0}, (A \to B)^{1/2}]$$

Derivation of $[A^{0}, A^{1/2}, B^{1}, (A \to B)^{0}, (A \to B)^{1/2}]$:

$$\dfrac{\dfrac{\begin{array}{c}\text{axiom for } B\\ [A^{0}, A^{1/2}, B^{0}, B^{1}, B^{1/2}]\end{array} \quad \begin{array}{c}\text{axiom for } A\\ [A^{0}, A^{1}, A^{1/2}, B^{0}, B^{1}]\end{array}}{[A^{0}, A^{1/2}, B^{0}, B^{1}, (A \to B)^{1/2}]} \quad \begin{array}{c}\text{axiom for } A\\ [A^{0}, A^{1}, A^{1/2}, B^{1}, (A \to B)^{1/2}]\end{array}}{[A^{0}, A^{1/2}, B^{1}, (A \to B)^{0}, (A \to B)^{1/2}]}$$

Proposition 32 *The following consequence holds:*

$$(A \to B), \neg B \vdash \neg A$$

The problem is equivalent to proving the following sequent:

$$[\neg A^{\mathbf{1}}, \neg B^{\mathbf{0}}, \neg B^{\mathbf{1/2}}, (A \to B)^{\mathbf{0}}, (A \to B)^{\mathbf{1/2}}]$$

Derivation of $[\neg A^{\mathbf{1}}, \neg B^{\mathbf{0}}, \neg B^{\mathbf{1/2}}, (A \to B)^{\mathbf{0}}, (A \to B)^{\mathbf{1/2}}]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } B}{[A^{\mathbf{0}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, (A \to B)^{\mathbf{1/2}}]} \quad \dfrac{\overset{\text{axiom for } A}{[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]} \quad \overset{\text{axiom for } B}{[A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]}}{[A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, (A \to B)^{\mathbf{1/2}}]}}{[A^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, (A \to B)^{\mathbf{0}}, (A \to B)^{\mathbf{1/2}}]}}{[A^{\mathbf{0}}, B^{\mathbf{1}}, \neg B^{\mathbf{1/2}}, (A \to B)^{\mathbf{0}}, (A \to B)^{\mathbf{1/2}}]}}{[A^{\mathbf{0}}, \neg B^{\mathbf{0}}, \neg B^{\mathbf{1/2}}, (A \to B)^{\mathbf{0}}, (A \to B)^{\mathbf{1/2}}]}}{[\neg A^{\mathbf{1}}, \neg B^{\mathbf{0}}, \neg B^{\mathbf{1/2}}, (A \to B)^{\mathbf{0}}, (A \to B)^{\mathbf{1/2}}]}$$

Proposition 33 *The following consequence holds:*

$$(A \to B), (B \to C) \vdash (A \to C)$$

The problem is equivalent to proving the following sequent:

$$[(A \to B)^{\mathbf{0}}, (A \to B)^{\mathbf{1/2}}, (A \to C)^{\mathbf{1}}, (B \to C)^{\mathbf{0}}, (B \to C)^{\mathbf{1/2}}]$$

Derivation of $[(A \to B)^{\mathbf{0}}, (A \to B)^{\mathbf{1/2}}, (A \to C)^{\mathbf{1}}, (B \to C)^{\mathbf{0}}, (B \to C)^{\mathbf{1/2}}]$:

$$\dfrac{\dfrac{\overset{\text{axiom for } C}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, (B \to C)^{\mathbf{1/2}}]} \quad \overset{\text{axiom for } B}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, (B \to C)^{\mathbf{1/2}}]}}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, (B \to C)^{\mathbf{0}}, (B \to C)^{\mathbf{1/2}}]} \quad \dfrac{\overset{\text{axiom for } C}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/}} \ldots} \quad \ldots}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/}} \ldots}}{[A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, (A \to C)^{\mathbf{1}}, (B \to C)^{\mathbf{0}}, (B \to C)^{\mathbf{1/2}}] \quad \ldots}$$

Proposition 34 *The following consequence holds:*

$$(A \vee B), \neg A \vdash B$$

The problem is equivalent to proving the following sequent:

$$[B^{\mathbf{1}}, \neg A^{\mathbf{0}}, \neg A^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}, (A \vee B)^{\mathbf{1/2}}]$$

Derivation of $[B^{\mathbf{1}}, \neg A^{\mathbf{0}}, \neg A^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}, (A \vee B)^{\mathbf{1/2}}]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } B}{[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]} \quad \overset{\text{axiom for } B}{[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]} \quad \overset{\text{axiom for } A}{[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}]}}{[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]} \quad \overset{\text{axiom for } A}{[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]}}{[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, (A \vee B)^{\mathbf{0}}, (A \vee B)^{\mathbf{1/2}}]}}{[A^{\mathbf{1}}, B^{\mathbf{1}}, \neg A^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}, (A \vee B)^{\mathbf{1/2}}]}}{[B^{\mathbf{1}}, \neg A^{\mathbf{0}}, \neg A^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}, (A \vee B)^{\mathbf{1/2}}]}$$

Proposition 35 *The following consequence holds:*

$$(\neg C \vee \neg D), (A \to C), (B \to D) \vdash (\neg A \vee \neg B)$$

The problem is equivalent to proving the following sequent:

$$[(A \to C)^{\mathbf{0}}, (A \to C)^{\mathbf{1/2}}, (B \to D)^{\mathbf{0}}, (B \to D)^{\mathbf{1/2}}, (\neg A \vee \neg B)^{\mathbf{1}}, (\neg C \vee \neg D)^{\mathbf{0}}, (\neg C \vee \neg D)^{\mathbf{1/2}}]$$

Derivation of $[(A \to C)^{\mathbf{0}}, (A \to C)^{\mathbf{1/2}}, (B \to D)^{\mathbf{0}}, (B \to D)^{\mathbf{1/2}}, (\neg A \vee \neg B)^{\mathbf{1}}, (\neg C \vee \neg D)^{\mathbf{0}}, (\neg C \vee \neg D)^{\mathbf{1/2}}]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{axiom for } D}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, D^{\mathbf{0}}, D^{\mathbf{1}}, D^{\mathbf{1/2}}, (\neg C \vee \neg D)^{\mathbf{1/2}}]}}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, D^{\mathbf{0}}, D^{\mathbf{1/2}}, \neg D^{\mathbf{0}}, (\neg C \vee \neg D)^{\mathbf{1/2}}]} \quad \dfrac{\dfrac{\text{axiom for } C}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, D^{\mathbf{0}}, D^{\mathbf{1/2}}, (\neg C \vee \neg D)^{\mathbf{1/2}}]}}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, D^{\mathbf{0}}, D^{\mathbf{1/2}}, \neg C^{\mathbf{0}}, (\neg C \vee \neg D)^{\mathbf{1/2}}]}}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, D^{\mathbf{0}}, D^{\mathbf{1/2}}, (\neg C \vee \neg D)^{\mathbf{0}}, (\neg C \vee \neg D)^{\mathbf{1/2}}]}}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, D^{\mathbf{0}}, D^{\mathbf{1/2}}, \neg B^{\mathbf{1}}, (\neg C \vee \neg D)^{\mathbf{0}}, (\neg C \vee \neg D)^{\mathbf{1/2}}]}}{[A^{\mathbf{1/2}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, D^{\mathbf{0}}, D^{\mathbf{1/2}}, \neg A^{\mathbf{1}}, \neg B^{\mathbf{1}}, (\neg C \vee \neg D)^{\mathbf{0}}, (\neg C \vee \neg D)^{\mathbf{1/2}}]}}{[A^{\mathbf{1/2}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, D^{\mathbf{0}}, D^{\mathbf{1/2}}, (\neg A \vee \neg B)^{\mathbf{1}}, (\neg C \vee \neg D)^{\mathbf{0}}, (\neg C \vee \neg D)^{\mathbf{1/2}}]}$$

Proposition 36 *The following consequence holds:*

$$(A \vee B), (A \to C), (B \to D) \vdash (C \vee D)$$

The problem is equivalent to proving the following sequent:

$$[(A \to C)^{\mathbf{0}}, (A \to C)^{\mathbf{1/2}}, (B \to D)^{\mathbf{0}}, (B \to D)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}, (A \vee B)^{\mathbf{1/2}}, (C \vee D)^{\mathbf{1}}]$$

Derivation of $[(A \to C)^{\mathbf{0}}, (A \to C)^{\mathbf{1/2}}, (B \to D)^{\mathbf{0}}, (B \to D)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}, (A \vee B)^{\mathbf{1/2}}, (C \vee D)^{\mathbf{1}}]$:

$$\dfrac{\dfrac{\dfrac{\text{axiom for } C}{[A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, D^{\mathbf{0}}, D^{\mathbf{1}}, D^{\mathbf{1/2}}]}}{[A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, D^{\mathbf{0}}, D^{\mathbf{1/2}}, (C \vee D)^{\mathbf{1}}]} \quad \dfrac{\dfrac{\text{axiom for } C}{[A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, D^{\mathbf{0}}, D^{\mathbf{1}}, D^{\mathbf{1/2}}]}}{[A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, D^{\mathbf{0}}, D^{\mathbf{1/2}}, (C \vee D)^{\mathbf{1}}]}}{[A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, D^{\mathbf{0}}, D^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1/2}}, (C \vee D)^{\mathbf{1}}]}$$

Proposition 37 *The following consequence* ***does not*** *hold:*

$$(A \to (B \to C)) \vdash ((A \wedge B) \to C)$$

The problem is equivalent to proving the following sequent:

$$[(A \to (B \to C))^{\mathbf{0}}, (A \to (B \to C))^{\mathbf{1/2}}, ((A \wedge B) \to C)^{\mathbf{1}}]$$

Derivation of $[(A \to (B \to C))^{\mathbf{0}}, (A \to (B \to C))^{\mathbf{1/2}}, ((A \wedge B) \to C)^{\mathbf{1}}]$:

$[A^{\mathbf{0}}, \ldots$

axiom for C
$[A^{\mathbf{1/2}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, (A \wedge B)^{\mathbf{0}}]$

axiom for C
$[A^{\mathbf{1/2}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, (A \wedge B)^{\mathbf{0}}, (A \wedge B)^{\mathbf{1/2}}]$

axiom for C
$[A^{\mathbf{1/2}}, B^{\mathbf{1}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, (A \wedge B)^{\mathbf{0}}]$

$[A^{\mathbf{1/2}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, ((A \wedge B) \to C)^{\mathbf{1}}]$ $\quad [A^{\mathbf{1/2}} \ldots$

$[A^{\mathbf{1/2}}, C^{\mathbf{0}}, (B \to C)^{\mathbf{1/2}}, ((A \wedge B) \to C)^{\mathbf{1}}]$

List of counter-examples:

$$[A^{\mathbf{1/2}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}]$$

Proposition 38 *The following consequence holds:*

$$((A \wedge B) \to C) \vdash (A \to (B \to C))$$

The problem is equivalent to proving the following sequent:

$$[(A \to (B \to C))^{\mathbf{1}}, ((A \wedge B) \to C)^{\mathbf{0}}, ((A \wedge B) \to C)^{\mathbf{1/2}}]$$

Derivation of $[(A \to (B \to C))^{\mathbf{1}}, ((A \wedge B) \to C)^{\mathbf{0}}, ((A \wedge B) \to C)^{\mathbf{1/2}}]$:

axiom for B
$[A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, C^{\mathbf{1}}, C^{\mathbf{1/}} \ldots$

axiom for B
$[A^{\mathbf{0}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, ((A \wedge B) \to C)^{\mathbf{1/2}}]$

axiom for C
$[A^{\mathbf{0}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, ((A \wedge B) \to C)^{\mathbf{1/2}}]$ $\quad [A^{\mathbf{0}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, (\ldots$

$[A^{\mathbf{0}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, ((A \wedge B) \to C)^{\mathbf{0}}, ((A \wedge B) \to C)^{\mathbf{1/2}}]$

Proposition 39 *The following consequence holds:*

$$(A \to B) \vdash (\neg B \to \neg A)$$

The problem is equivalent to proving the following sequent:

$$[(A \to B)^{\mathbf{0}}, (A \to B)^{\mathbf{1/2}}, (\neg B \to \neg A)^{\mathbf{1}}]$$

Derivation of $[(A \to B)^{\mathbf{0}}, (A \to B)^{1/2}, (\neg B \to \neg A)^{\mathbf{1}}]$:

axiom for B $\qquad\qquad\qquad\qquad\qquad\qquad$ axiom for B

$$[A^{\mathbf{0}}, A^{1/2}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{1/2}] \qquad\qquad [A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{1/2}]$$

axiom for B $\qquad\qquad$ axiom for A

$$[A^{\mathbf{0}}, A^{1/2}, B^{\mathbf{0}}, B^{1/2}, \neg B^{\mathbf{0}}] \quad [A^{\mathbf{0}}, A^{1/2}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{1/2}, \neg B^{1/2}] \quad [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{1/2}, B^{\mathbf{0}}, \neg B^{\mathbf{0}}] \quad [A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, \neg B^{1/2}]$$

$$[A^{\mathbf{0}}, A^{1/2}, B^{\mathbf{0}}, B^{1/2}, \neg A^{1/2}, \neg B^{\mathbf{0}}] \quad [A^{\mathbf{0}}, A^{1/2}, B^{\mathbf{0}}, B^{1/2}, \neg B^{\mathbf{0}}, \neg B^{1/2}] \quad [A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{0}}, \neg A^{1/2}, \neg B^{\mathbf{0}}] \quad [A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{0}}, \neg B^{\mathbf{0}}, \neg B^{1/2}]$$

$$[A^{1/2}, B^{\mathbf{0}}, B^{1/2}, \neg A^{\mathbf{1}}, \neg A^{1/2}, \neg B^{\mathbf{0}}] \quad [A^{1/2}, B^{\mathbf{0}}, B^{1/2}, \neg A^{\mathbf{1}}, \neg B^{\mathbf{0}}, \neg B^{1/2}] \quad [A^{\mathbf{1}}, B^{\mathbf{0}}, \neg A^{\mathbf{1}}, \neg A^{1/2}, \neg B^{\mathbf{0}}] \quad [A^{\mathbf{1}}, B^{\mathbf{0}}, \neg A^{\mathbf{1}}, \neg B^{\mathbf{0}}, \neg B^{1/2}]$$

$$[A^{1/2}, B^{\mathbf{0}}, B^{1/2}, (\neg B \to \neg A)^{\mathbf{1}}] \qquad [A^{\mathbf{1}}, B^{\mathbf{0}}, (\neg B \to \neg A)^{\mathbf{1}}]$$

$$[B^{\mathbf{0}}, (A \to B)^{1/2}, (\neg B \to \neg A)^{\mathbf{1}}]$$

$$[(A \to B)^{\mathbf{0}}, (A -$$

Proposition 40 *The following consequence holds:*

$$(\neg A \to \neg B) \vdash (B \to A)$$

The problem is equivalent to proving the following sequent:

$$[(B \to A)^{\mathbf{1}}, (\neg A \to \neg B)^{\mathbf{0}}, (\neg A \to \neg B)^{1/2}]$$

Derivation of $[(B \to A)^{\mathbf{1}}, (\neg A \to \neg B)^{\mathbf{0}}, (\neg A \to \neg B)^{1/2}]$:

axiom for B

$$[A^{\mathbf{1}}, A^{1/2}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{1/2}]$$

axiom for A

$$[A^{\mathbf{1}}, A^{1/2}, B^{\mathbf{0}}, B^{\mathbf{1}}, \neg B^{1/2}] \qquad [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{1/2}, B^{\mathbf{0}}, B^{\mathbf{1}}, \neg B^{\mathbf{0}}]$$

$$[A^{\mathbf{1}}, A^{1/2}, B^{\mathbf{0}}, B^{\mathbf{1}}, \neg A^{1/2}, \neg B^{1/2}] \qquad [A^{\mathbf{1}}, A^{1/2}, B^{\mathbf{0}}, B^{\mathbf{1}}, \neg A^{\mathbf{1}}, \neg B^{\mathbf{0}}]$$

axiom for A $\qquad\qquad$ axiom for B

$$[A^{\mathbf{1}}, A^{1/2}, B^{\mathbf{0}}, B^{\mathbf{1}}, (\neg A \to \neg B)^{1/2}] \qquad [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{1/2}, B^{\mathbf{0}}, (\neg A \to \neg B)^{1/2}] \qquad [A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{1/2}, (\neg A$$

$$[A^{\mathbf{1}}, A^{1/2}, B^{\mathbf{0}}, \neg B^{\mathbf{0}}, (\neg A \to \neg B)^{1/2}] \qquad [A^{\mathbf{1}}, A^{1/2}, B^{\mathbf{0}}, \neg A^{\mathbf{1}}, (\neg A \to \neg B)^{1/2}] \qquad [A^{\mathbf{1}}, B^{\mathbf{0}}, B^{1/2}, \neg B^{\mathbf{0}}, (\neg A$$

$$[A^{\mathbf{1}}, A^{1/2}, B^{\mathbf{0}}, (\neg A \to \neg B)^{\mathbf{0}}, (\neg A \to \neg B)^{1/2}] \qquad [A$$

$$[(B \to A)^{\mathbf{1}}, (\neg A \to \neg B)^{\mathbf{0}}, (\neg A \to \neg B)^{1/2}]$$

Proposition 41 *The following consequence holds:*

$$(A \to B), (A \to C) \vdash (A \to (B \wedge C))$$

The problem is equivalent to proving the following sequent:

$$[(A \to B)^{\mathbf{0}}, (A \to B)^{1/2}, (A \to C)^{\mathbf{0}}, (A \to C)^{1/2}, (A \to (B \wedge C))^{\mathbf{1}}]$$

Derivation of $[(A \to B)^{\mathbf{0}}, (A \to B)^{1/2}, (A \to C)^{\mathbf{0}}, (A \to C)^{1/2}, (A \to (B \wedge C))^{\mathbf{1}}]$:

axiom for C $\qquad\qquad$ axiom for B $\qquad\qquad$ axiom for C

$$[A^{\mathbf{0}}, A^{1/2}, B^{\mathbf{0}}, B^{1/2}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{1/2}, (B \wedge C)^{1/2}] \quad [A^{\mathbf{0}}, A^{1/2}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{1/2}, C^{\mathbf{0}}, C^{1/2}, (B \wedge C)^{1/2}] \quad [A^{\mathbf{0}}, A^{1/2}, B^{\mathbf{0}}, B^{1/2}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{1/2}]$$

$$[A^{\mathbf{0}}, A^{1/2}, B^{\mathbf{0}}, B^{1/2}, C^{\mathbf{0}}, C^{1/2}, (B \wedge C)^{\mathbf{1}}, (B \wedge C)^{1/2}] \qquad [A^{\mathbf{0}}, A^{1/2}, B^{\mathbf{0}}, B^{1/2},$$

$$[A^{1/2}, B^{\mathbf{0}}, B^{1/2}, C^{\mathbf{0}}, C^{1/2}, (A \to (B \wedge C))^{\mathbf{1}}]$$

$$[A^{1/2},$$

Proposition 42 *The following consequence holds:*

$$((A \vee B) \to C) \vdash ((A \to C) \wedge (B \to C))$$

The problem is equivalent to proving the following sequent:

$$[((A \to C) \wedge (B \to C))^{\mathbf{1}}, ((A \vee B) \to C)^{\mathbf{0}}, ((A \vee B) \to C)^{\mathbf{1/2}}]$$

Derivation of $[((A \to C) \wedge (B \to C))^{\mathbf{1}}, ((A \vee B) \to C)^{\mathbf{0}}, ((A \vee B) \to C)^{\mathbf{1/2}}]$:

axiom for B $\quad$ axiom for B $\quad$ axiom for A

$[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}]$ $\quad$ $[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}]$ $\quad$ $[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, C$

$[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1/2}}]$

$[B^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]$

axiom for C

$[B^{\mathbf{0}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, ((A \vee B) \to C)^{\mathbf{1/2}}]$ $\quad$ $[B^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, ((A \vee B) \to C)^{\mathbf{1/}}$

$[B^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}, ((A \vee B) \to C)^{\mathbf{0}}, ((A \vee B) \to C)^{\mathbf{1/2}}]$

6 Some popular equivalences

Proposition 43 *The formulas* $((B \vee C) \wedge A)$ *and* $((B \wedge A) \vee (C \wedge A))$ *are equivalent.*

The problem is equivalent to proving the following sequents:

$$[((B \vee C) \wedge A)^{\mathbf{0}}, ((B \vee C) \wedge A)^{\mathbf{1/2}}, ((B \wedge A) \vee (C \wedge A))^{\mathbf{1}}]$$
$$[((B \vee C) \wedge A)^{\mathbf{1}}, ((B \wedge A) \vee (C \wedge A))^{\mathbf{0}}, ((B \wedge A) \vee (C \wedge A))^{\mathbf{1/2}}]$$

Derivation of $[((B \vee C) \wedge A)^{\mathbf{0}}, ((B \vee C) \wedge A)^{\mathbf{1/2}}, ((B \wedge A) \vee (C \wedge A))^{\mathbf{1}}]$:

axiom for A $\quad$ axiom

$[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}]$ $\quad$ $[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}$

axiom for A

$[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, (C \wedge A)^{\mathbf{1}}]$ $\quad$ $[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, (C \wedge A$

$[A^{\mathbf{0}}, A^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, (B \wedge A)^{\mathbf{1}}, (C \wedge A)^{\mathbf{1}}]$

$[A^{\mathbf{0}}, A^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, ((B \wedge A) \vee (C \wedge A))^{\mathbf{1}}]$

axiom for A

$[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, (B \vee C)^{\mathbf{0}}, ((B \wedge A) \vee (C \wedge A))^{\mathbf{1}}]$

Derivation of $[((B \vee C) \wedge A)^{\mathbf{1}}, ((B \wedge A) \vee (C \wedge A))^{\mathbf{0}}, ((B \wedge A) \vee (C \wedge A))^{\mathbf{1/2}}]$:

axiom for A $\quad$ axiom for A $\quad$ axiom for C

$[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, C^{\mathbf{0}}]$ $\quad$ $[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}]$ $\quad$ $[A^{\mathbf{0}}, A^{\mathbf{1}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}]$

$[A^{\mathbf{0}}, A^{\mathbf{1}}, C^{\mathbf{0}}, (C \wedge A)^{\mathbf{1/2}}]$

$[A^{\mathbf{0}}, A^{\mathbf{1}}, C^{\mathbf{0}}, (C \wedge A)^{\mathbf{0}}, (C \wedge A)^{\mathbf{1/2}}]$

axiom for A $\quad$ axiom for A

$[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, C^{\mathbf{0}}, (C \wedge A)^{\mathbf{1/2}}]$ $\quad$ $[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, (C$

$[A^{\mathbf{0}}, A^{\mathbf{1}},$

Proposition 44 *The formulas* $(A \wedge (B \vee C))$ *and* $((A \wedge B) \vee (A \wedge C))$ *are equivalent.*

The problem is equivalent to proving the following sequents:

$$[(A \wedge (B \vee C))^{0}, (A \wedge (B \vee C))^{1/2}, ((A \wedge B) \vee (A \wedge C))^{1}]$$
$$[(A \wedge (B \vee C))^{1}, ((A \wedge B) \vee (A \wedge C))^{0}, ((A \wedge B) \vee (A \wedge C))^{1/2}]$$

Derivation of $[(A \wedge (B \vee C))^{0}, (A \wedge (B \vee C))^{1/2}, ((A \wedge B) \vee (A \wedge C))^{1}]$:

axiom for C — $[A^{0}, A^{1/2}, B^{1}, C^{0}, C^{1}, C^{1/2}]$ axiom for A — $[A^{0}, A^{1}, A^{1/2}, B^{1}, C^{0}, C^{1/2}]$

$[A^{0}, A^{1/2}, B^{1}, C^{0}, C^{1/2}, (A \wedge C)^{1}]$ $[A^{0}, A$

$[A^{0}, A^{1/2}, C^{0}, C^{1/2}, (A \wedge B)^{1}, (A$

$[A^{0}, A^{1/2}, C^{0}, C^{1/2}, ((A \wedge B) \vee (A$

axiom for $(B \vee C)$ — $[A^{0}, (B \vee C)^{0}, (B \vee C)^{1}, (B \vee C)^{1/2}, ((A \wedge B) \vee (A \wedge C))^{1}]$

Derivation of $[(A \wedge (B \vee C))^{1}, ((A \wedge B) \vee (A \wedge C))^{0}, ((A \wedge B) \vee (A \wedge C))^{1/2}]$:

axiom for C — $[A^{0}, B^{1}, C^{0}, C^{1}, C^{1/2}]$ axiom for C — $[A^{0}, A^{1/2}, B^{1}, C^{0}, C^{1}, C^{1/2}]$ axiom for A — $[A^{0}, A^{1}, A^{1/2}, B^{1}, C^{0}, C^{1}]$ axiom for C — $[A^{0}, B^{1}, B^{1/2}, C^{0}, C^{1}, C^{1/2}]$ axiom — $[A^{0}, A^{1/2}, B^{1}, B$

$[A^{0}, B^{1}, C^{0}, C^{1}, (A \wedge C)^{1/2}]$ $[A^{0}, B^{1}, B^{1/2},$

$[A^{0}, B^{1}, C^{0}, C^{1}, (A \wedge C)^{0}, (A \wedge C)^{1/2}]$

Proposition 45 *The formulas* $((B \wedge C) \vee A)$ *and* $((B \vee A) \wedge (C \vee A))$ *are equivalent.*

The problem is equivalent to proving the following sequents:

$$[((B \vee A) \wedge (C \vee A))^{1}, ((B \wedge C) \vee A)^{0}, ((B \wedge C) \vee A)^{1/2}]$$
$$[((B \vee A) \wedge (C \vee A))^{0}, ((B \vee A) \wedge (C \vee A))^{1/2}, ((B \wedge C) \vee A)^{1}]$$

Derivation of $[((B \vee A) \wedge (C \vee A))^{1}, ((B \wedge C) \vee A)^{0}, ((B \wedge C) \vee A)^{1/2}]$:

axiom for C — $[A^{0}, A^{1}, B^{0}, C^{0}, C^{1}, C^{1/2}]$ axiom for C — $[A^{0}, A^{1}, B^{0}, B^{1/2}, C^{0}, C^{1}, C^{1/2}]$ axiom — $[A^{0}, A^{1}, B^{0}, B$

$[A^{0}, A^{1}, B^{0}, C^{0}, C^{1}, (B \wedge C)^{1/2}]$

axiom for A — $[A^{0}, A^{1}, A^{1/2}, C^{1}]$ axiom for A — $[A^{0}, A^{1}, A^{1/2}, C^{1}, (B \wedge C)^{1/2}]$ $[A^{0}, A^{1}, C^{1}, (B \wedge C)^{0}, (B \wedge C)^{1/2}]$

$[A^{0}, A^{1}, C^{1}, ((B \wedge C) \vee A)^{1/2}]$

Derivation of $[((B \vee A) \wedge (C \vee A))^{\mathbf{0}}, ((B \vee A) \wedge (C \vee A))^{\mathbf{1/2}}, ((B \wedge C) \vee A)^{\mathbf{1}}]$:

axiom for A | axiom for A

$$[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, (B \wedge C)^{\mathbf{1}}] \qquad [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, C^{\mathbf{1/2}}, (B \wedge C)^{\mathbf{1}}]$$

$$[A^{\mathbf{0}}, A^{\mathbf{1/2}}, ((B \wedge C) \vee A)^{\mathbf{1}}] \qquad [A^{\mathbf{0}}, A^{\mathbf{1/2}}, C^{\mathbf{1/2}}, ((B \wedge C) \vee A)^{\mathbf{1}}]$$

$$[A^{\mathbf{0}}, A^{\mathbf{1/2}}, (C \vee A)^{\mathbf{1/2}}, ((B \wedge C) \vee A$$

axiom for $(C \vee A)$

$$[(B \vee A)^{\mathbf{0}}, (C \vee A)^{\mathbf{0}}, (C \vee A)^{\mathbf{1}}, (C \vee A)^{\mathbf{1/2}}, ((B \wedge C) \vee A)^{\mathbf{1}}]$$

Proposition 46 *The formulas* $(A \vee (B \wedge C))$ *and* $((A \vee B) \wedge (A \vee C))$ *are equivalent.*

The problem is equivalent to proving the following sequents:

$$[((A \vee B) \wedge (A \vee C))^{\mathbf{1}}, (A \vee (B \wedge C))^{\mathbf{0}}, (A \vee (B \wedge C))^{\mathbf{1/2}}]$$
$$[((A \vee B) \wedge (A \vee C))^{\mathbf{0}}, ((A \vee B) \wedge (A \vee C))^{\mathbf{1/2}}, (A \vee (B \wedge C))^{\mathbf{1}}]$$

Derivation of $[((A \vee B) \wedge (A \vee C))^{\mathbf{1}}, (A \vee (B \wedge C))^{\mathbf{0}}, (A \vee (B \wedge C))^{\mathbf{1/2}}]$:

axiom for C | axiom for C | axiom for B

$$[A^{\mathbf{1}}, B^{\mathbf{0}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}] \quad [A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}] \quad [A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1}}]$$

$$[A^{\mathbf{1}}, B^{\mathbf{0}}, C^{\mathbf{0}}, C^{\mathbf{1}}, (B \wedge C)^{\mathbf{1/2}}]$$

$$[A^{\mathbf{1}}, B^{\mathbf{0}}, C^{\mathbf{0}}, C^{\mathbf{1}}, (B \wedge C)^{\mathbf{0}}, (B \wedge C)^{\mathbf{1/2}}]$$

axiom for C

$$[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}]$$

$$[A^{\mathbf{1}}, B^{\mathbf{0}}, C^{\mathbf{0}}, C^{\mathbf{1}}, (A \vee (B \wedge C)$$

$$[A^{\mathbf{1}}, C^{\mathbf{1}}, (B \wedge C)^{\mathbf{0}}, (A \vee (B \wedge C$$

Derivation of $[((A \vee B) \wedge (A \vee C))^{\mathbf{0}}, ((A \vee B) \wedge (A \vee C))^{\mathbf{1/2}}, (A \vee (B \wedge C))^{\mathbf{1}}]$:

axiom for C | axiom for B

$$[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1}}, C^{\mathbf{1/2}}] \qquad [A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}]$$

$$[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, (B \wedge C)^{\mathbf{1}}]$$

$$[B^{\mathbf{0}}, B^{\mathbf{1/2}}, C^{\mathbf{0}}, C^{\mathbf{1/2}}, (A \vee (B \wedge C))^{\mathbf{1}}]$$

axiom for $(A \vee C)$

$$[(A \vee B)^{\mathbf{0}}, (A \vee C)^{\mathbf{0}}, (A \vee C)^{\mathbf{1}}, (A \vee C)^{\mathbf{1/2}}, (A \vee (B \wedge C))^{\mathbf{1}}]$$

7 Interdefinability of connectives

Proposition 47 *The equality* $(A \to B) = (\neg A \vee B)$ *does* ***not*** *hold.*

The problem is equivalent to proving the following sequents:

$$[(A \to B)^{\mathbf{0}}, (\neg A \vee B)^{\mathbf{1}}, (\neg A \vee B)^{\mathbf{1/2}}]$$
$$[(A \to B)^{\mathbf{1}}, (\neg A \vee B)^{\mathbf{0}}, (\neg A \vee B)^{\mathbf{1/2}}]$$
$$[(A \to B)^{\mathbf{1/2}}, (\neg A \vee B)^{\mathbf{0}}, (\neg A \vee B)^{\mathbf{1}}]$$

Derivation of $[(A \to B)^{\mathbf{0}}, (\neg A \vee B)^{\mathbf{1}}, (\neg A \vee B)^{\mathbf{1/2}}]$:

$$\text{axiom for } A \;\; [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}]$$
$$[A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, \neg A^{\mathbf{1/2}}] \qquad \text{axiom for } A \;\; [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$$
$$\text{axiom for } B \;\; [A^{\mathbf{0}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}] \qquad \text{axiom for } B \;\; [A^{\mathbf{0}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, \neg A^{\mathbf{1/2}}] \qquad [A^{\mathbf{0}}, B^{\mathbf{0}}, B^{\mathbf{1}}, \neg A^{\mathbf{0}}, \neg A^{\mathbf{1/2}}] \qquad \text{axiom for } B \;\; [A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}] \qquad [A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, \neg A^{\mathbf{1/2}}]$$
$$[A^{\mathbf{0}}, B^{\mathbf{0}}, B^{\mathbf{1}}, (\neg A \vee B)^{\mathbf{1/2}}] \qquad [A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{1}}, (\neg A \vee B)^{\mathbf{1/2}}]$$
$$[B^{\mathbf{0}}, B^{\mathbf{1}}, \neg A^{\mathbf{1}}, (\neg A \vee B)^{\mathbf{1/2}}] \qquad [A^{\mathbf{1}}, B^{\mathbf{1}}, \neg A^{\mathbf{1}}, (\neg A \vee B)^{\mathbf{1/2}}]$$
$$[B^{\mathbf{0}}, (\neg A \vee B)^{\mathbf{1}}, (\neg A \vee B)^{\mathbf{1/2}}] \qquad [A^{\mathbf{1}}, (\neg A \vee B)^{\mathbf{1}}, (\neg A \vee B)^{\mathbf{1/2}}]$$
$$[(A \to B)^{\mathbf{0}}, (\neg A \vee B)^{\mathbf{1}}, (\neg A \vee B)^{\mathbf{1/2}}]$$

Derivation of $[(A \to B)^{\mathbf{1}}, (\neg A \vee B)^{\mathbf{0}}, (\neg A \vee B)^{\mathbf{1/2}}]$:

$$\text{axiom for } A \;\; [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$$
$$\text{axiom for } A \;\; [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}] \qquad [A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, \neg A^{\mathbf{1/2}}]$$
$$\text{axiom for } B \;\; [A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}] \qquad [A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, \neg A^{\mathbf{1/2}}] \qquad [A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, \neg A^{\mathbf{0}}, \neg A^{\mathbf{1/2}}]$$
$$[A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, (\neg A \vee B)^{\mathbf{1/2}}] \qquad \text{axio} \;\; [A^{\mathbf{0}}, A^{\mathbf{1/2}},$$
$$\text{axiom for } B \;\; [A^{\mathbf{0}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, (\neg A \vee B)^{\mathbf{1/2}}] \qquad [A^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, \neg A^{\mathbf{0}}, (\neg A \vee B)^{\mathbf{1/2}}]$$
$$[A^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, (\neg A \vee B)^{\mathbf{0}}, (\neg A \vee B)^{\mathbf{1/2}}]$$
$$[(A \to B)^{\mathbf{1}}, (\neg A \vee B)^{\mathbf{0}}, (\neg$$

Derivation of $[(A \to B)^{\mathbf{1/2}}, (\neg A \vee B)^{\mathbf{0}}, (\neg A \vee B)^{\mathbf{1}}]$:

$$\text{axiom for } A \;\; [A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}] \qquad \text{hypothesis} \;\; [A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}]$$
$$[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, \neg A^{\mathbf{1}}] \qquad \text{hypothesis} \;\; [A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}] \qquad [A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, \neg A^{\mathbf{1}}]$$
$$\text{axiom for } B \;\; [A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, \neg A^{\mathbf{1}}] \qquad [A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1/2}}, (\neg A \vee B)^{\mathbf{1}}] \qquad [A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, \neg A^{\mathbf{1}}] \qquad [A^{\mathbf{1}}, B^{\mathbf{0}}, (\neg A \vee B)^{\mathbf{1}}]$$
$$[A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, (\neg A \vee B)^{\mathbf{1}}] \qquad [A^{\mathbf{1/2}}, B^{\mathbf{1/2}}, \neg A^{\mathbf{0}}, (\neg A \vee B)^{\mathbf{1}}] \qquad [A^{\mathbf{1}}, B^{\mathbf{0}}, (\neg A \vee B)^{\mathbf{1}}] \qquad [A^{\mathbf{1}}, B^{\mathbf{0}}, \neg A^{\mathbf{0}}, (\neg A \vee B)^{\mathbf{1}}]$$
$$[A^{\mathbf{1/2}}, B^{\mathbf{1/2}}, (\neg A \vee B)^{\mathbf{0}}, (\neg A \vee B)^{\mathbf{1}}] \qquad [A^{\mathbf{1}}, B^{\mathbf{0}}, (\neg A \vee B)^{\mathbf{0}}, (\neg A \vee B)^{\mathbf{1}}]$$
$$[(A \to B)^{\mathbf{1/2}}, (\neg A \vee B)^{\mathbf{0}}, (\neg A \vee B)^{\mathbf{1}}]$$

List of counter-examples:

$$[A^{\mathbf{1/2}}, B^{\mathbf{1/2}}]$$

Proposition 48 *The equality $(A \to B) = \neg(A \wedge \neg B)$ does **not** hold.*

The problem is equivalent to proving the following sequents:

$$[\neg(A \wedge \neg B)^{\mathbf{1}}, \neg(A \wedge \neg B)^{\mathbf{1/2}}, (A \to B)^{\mathbf{0}}]$$
$$[\neg(A \wedge \neg B)^{\mathbf{0}}, \neg(A \wedge \neg B)^{\mathbf{1/2}}, (A \to B)^{\mathbf{1}}]$$
$$[\neg(A \wedge \neg B)^{\mathbf{0}}, \neg(A \wedge \neg B)^{\mathbf{1}}, (A \to B)^{\mathbf{1/2}}]$$

Derivation of $[\neg(A\wedge\neg B)^{\mathbf{1}},\neg(A\wedge\neg B)^{\mathbf{1/2}},(A\to B)^{\mathbf{0}}]$:

$$
\dfrac{
\dfrac{\dfrac{\overset{\text{axiom for } B}{[A^{\mathbf{0}},B^{\mathbf{0}},B^{\mathbf{1}},B^{\mathbf{1/2}},(A\to B)^{\mathbf{0}}]}}{[A^{\mathbf{0}},B^{\mathbf{0}},B^{\mathbf{1}},\neg B^{\mathbf{1/2}},(A\to B)^{\mathbf{0}}]}}{[A^{\mathbf{0}},B^{\mathbf{1}},\neg B^{\mathbf{1}},\neg B^{\mathbf{1/2}},(A\to B)^{\mathbf{0}}]}
\quad
\dfrac{\dfrac{\overset{\text{axiom for } B}{[A^{\mathbf{0}},A^{\mathbf{1/2}},B^{\mathbf{0}},B^{\mathbf{1}},B^{\mathbf{1/2}}]}\quad\overset{\text{axiom for } A}{[A^{\mathbf{0}},A^{\mathbf{1}},A^{\mathbf{1/2}},B^{\mathbf{1}},B^{\mathbf{1/2}}]}}{[A^{\mathbf{0}},A^{\mathbf{1/2}},B^{\mathbf{1}},B^{\mathbf{1/2}},(A\to B)^{\mathbf{0}}]}}{[A^{\mathbf{0}},A^{\mathbf{1/2}},B^{\mathbf{1}},\neg B^{\mathbf{1/2}},(A\to B)^{\mathbf{0}}]}
\quad
\overset{\text{axiom for } A}{[A^{\mathbf{0}},A^{\mathbf{1}},A^{\mathbf{1/2}},B^{\mathbf{1}},(A\to B)^{\mathbf{0}}]}
}{
\dfrac{\dfrac{\dfrac{\dfrac{[A^{\mathbf{0}},B^{\mathbf{1}},(A\wedge\neg B)^{\mathbf{1/2}},(A\to B)^{\mathbf{0}}]}{[A^{\mathbf{0}},\neg B^{\mathbf{0}},(A\wedge\neg B)^{\mathbf{1/2}},(A\to B)^{\mathbf{0}}]}}{[(A\wedge\neg B)^{\mathbf{0}},(A\wedge\neg B)^{\mathbf{1/2}},(A\to B)^{\mathbf{0}}]}}{[\neg(A\wedge\neg B)^{\mathbf{1/2}},(A\wedge\neg B)^{\mathbf{0}},(A\to B)^{\mathbf{0}}]}}{[\neg(A\wedge\neg B)^{\mathbf{1}},\neg(A\wedge\neg B)^{\mathbf{1/2}},(A\to B)^{\mathbf{0}}]}
}
$$

Derivation of $[\neg(A\wedge\neg B)^{\mathbf{0}},\neg(A\wedge\neg B)^{\mathbf{1/2}},(A\to B)^{\mathbf{1}}]$:

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } B}{[A^{\mathbf{0}},B^{\mathbf{0}},B^{\mathbf{1}},B^{\mathbf{1/2}}]}\quad\overset{\text{axiom for } B}{[A^{\mathbf{0}},A^{\mathbf{1/2}},B^{\mathbf{0}},B^{\mathbf{1}},B^{\mathbf{1/2}}]}}{[B^{\mathbf{0}},B^{\mathbf{1/2}},(A\to B)^{\mathbf{1}}]}}{[B^{\mathbf{0}},\neg B^{\mathbf{1/2}},(A\to B)^{\mathbf{1}}]}}{[B^{\mathbf{0}},\neg B^{\mathbf{1}},\neg B^{\mathbf{1/2}},(A\to B)^{\mathbf{1}}]}
\quad
\dfrac{\dfrac{\overset{\text{axiom for } B}{[A^{\mathbf{0}},A^{\mathbf{1/2}},B^{\mathbf{0}},B^{\mathbf{1}},B^{\mathbf{1/2}}]}\quad\overset{\text{axiom for } B}{[A^{\mathbf{0}},A^{\mathbf{1/2}},B^{\mathbf{0}},B^{\mathbf{1}},B^{\mathbf{1/2}}]}}{[A^{\mathbf{1/2}},B^{\mathbf{0}},B^{\mathbf{1/2}},(A\to B)^{\mathbf{1}}]}}{[A^{\mathbf{1/2}},B^{\mathbf{0}},\neg B^{\mathbf{1/2}},(A\to B)^{\mathbf{1}}]}
\quad
\dfrac{\overset{\text{axiom for } A}{[A^{\mathbf{0}},A^{\mathbf{1}},A^{\mathbf{1/2}},B^{\mathbf{0}},B^{\mathbf{1}},B^{\mathbf{1/2}}]}\quad [}{[A^{\mathbf{1}},A^{\mathbf{1/2}},B^{\mathbf{0}},(A}
}{
\dfrac{[B^{\mathbf{0}},(A\wedge\neg B)^{\mathbf{1/2}},(A\to B)^{\mathbf{1}}]}{[\neg B^{\mathbf{1}},(A\wedge\neg B)^{\mathbf{1/2}},(A\to B)^{\mathbf{1}}]}
}
$$

Derivation of $[\neg(A\wedge\neg B)^{\mathbf{0}},\neg(A\wedge\neg B)^{\mathbf{1}},(A\to B)^{\mathbf{1/2}}]$:

$$
\dfrac{
\dfrac{\dfrac{\overset{\text{axiom for } B}{[A^{\mathbf{0}},A^{\mathbf{1/2}},B^{\mathbf{0}},B^{\mathbf{1}},B^{\mathbf{1/2}}]}\quad\overset{\text{hypothesis}}{[A^{\mathbf{0}},A^{\mathbf{1}},B^{\mathbf{0}},B^{\mathbf{1}}]}}{[A^{\mathbf{0}},B^{\mathbf{0}},B^{\mathbf{1}},(A\to B)^{\mathbf{1/2}}]}}{[A^{\mathbf{0}},B^{\mathbf{1}},\neg B^{\mathbf{1}},(A\to B)^{\mathbf{1/2}}]}
\quad
\dfrac{\overset{\text{axiom for } A}{[A^{\mathbf{0}},A^{\mathbf{1}},A^{\mathbf{1/2}},B^{\mathbf{1}},B^{\mathbf{1/2}}]}\quad\overset{\text{hypothesis}}{[A^{\mathbf{0}},A^{\mathbf{1}},B^{\mathbf{0}},B^{\mathbf{1}}]}}{[A^{\mathbf{0}},A^{\mathbf{1}},B^{\mathbf{1}},(A\to B)^{\mathbf{1/2}}]}
}{
\dfrac{\dfrac{\dfrac{\dfrac{[A^{\mathbf{0}},B^{\mathbf{1}},(A\wedge\neg B)^{\mathbf{1}},(A\to B)^{\mathbf{1/2}}]}{[A^{\mathbf{0}},\neg B^{\mathbf{0}},(A\wedge\neg B)^{\mathbf{1}},(A\to B)^{\mathbf{1/2}}]}}{[(A\wedge\neg B)^{\mathbf{0}},(A\wedge\neg B)^{\mathbf{1}},(A\to B)^{\mathbf{1/2}}]}}{[\neg(A\wedge\neg B)^{\mathbf{1}},(A\wedge\neg B)^{\mathbf{1}},(A\to B)^{\mathbf{1/2}}]}}{[\neg(A\wedge\neg B)^{\mathbf{0}},\neg(A\wedge\neg B)^{\mathbf{1}},(A\to B)^{\mathbf{1/2}}]}
}
$$

List of counter-examples:

$$[A^{\mathbf{1/2}},B^{\mathbf{1/2}}]$$

Proposition 49 *The equality* $(A\vee B)=((A\to B)\to B)$ *holds.*

The problem is equivalent to proving the following sequents:

$$
\begin{array}{c}
[((A\to B)\to B)^{\mathbf{1}},((A\to B)\to B)^{\mathbf{1/2}},(A\vee B)^{\mathbf{0}}]\\
[((A\to B)\to B)^{\mathbf{0}},((A\to B)\to B)^{\mathbf{1/2}},(A\vee B)^{\mathbf{1}}]\\
[((A\to B)\to B)^{\mathbf{0}},((A\to B)\to B)^{\mathbf{1}},(A\vee B)^{\mathbf{1/2}}]
\end{array}
$$

Derivation of $[((A \to B) \to B)^{\mathbf{1}}, ((A \to B) \to B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$:

axiom for B axiom for A

$[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$

axiom for B

$[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$ $[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$

$[A^{\mathbf{1}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, (A \to B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$

axiom for B

$[B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, ((A \to B) \to B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$ $[A^{\mathbf{1}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, ((A \to B) \to$ …

$[B^{\mathbf{1}}, B^{\mathbf{1/2}}, (A \to B)^{\mathbf{0}}, ((A \to B) \to B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$

Derivation of $[((A \to B) \to B)^{\mathbf{0}}, ((A \to B) \to B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$:

axiom for B axiom for B axiom for A axi…

$[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$ $[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$ axiom for B $[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1}},$ …

$[A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$ $[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$ $[A^{\mathbf{0}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, (A \vee B)^{\mathbf{1}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B$ …

$[B^{\mathbf{0}}, B^{\mathbf{1/2}}, (A \to B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$ $[B^{\mathbf{0}}, (A \to B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1}}]$

$[B^{\mathbf{0}}, ((A \to B) \to B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$

$[((A \to$ …

Derivation of $[((A \to B) \to B)^{\mathbf{0}}, ((A \to B) \to B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]$:

axiom for B axiom for B axio…

$[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$ $[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$ $[A^{\mathbf{0}}, A^{\mathbf{1}},$ …

axiom for B

$[A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1/2}}]$ $[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]$

$[B^{\mathbf{0}}, B^{\mathbf{1}}, (A \to B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1/2}}]$

axiom for B

$[B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, (A \to B)^{\mathbf{0}}, (A \vee B)^{\mathbf{1/2}}]$ $[B^{\mathbf{0}}, B^{\mathbf{1}}, (A \to B$ …

$[B^{\mathbf{0}}, ((A \to B) \to B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]$

Proposition 50 *The equality* $(A \vee B) = \neg(\neg A \wedge \neg B)$ *holds.*

The problem is equivalent to proving the following sequents:

$$[\neg(\neg A \wedge \neg B)^{\mathbf{1}}, \neg(\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$$
$$[\neg(\neg A \wedge \neg B)^{\mathbf{0}}, \neg(\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$$
$$[\neg(\neg A \wedge \neg B)^{\mathbf{0}}, \neg(\neg A \wedge \neg B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]$$

Derivation of $[\neg(\neg A \wedge \neg B)^{\mathbf{1}}, \neg(\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$:

axiom for B

$$[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$$
$$[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$$
$$[A^{\mathbf{1}}, B^{\mathbf{1}}, \neg B^{\mathbf{1}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$$

axiom for B $[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$ axiom for A $[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$

$$[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$$
$$[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$$
$$[A^{\mathbf{1}}, B^{\mathbf{1}}, \neg A^{\mathbf{1/2}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$$

axiom for A

$$[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, (A \vee B)^{\mathbf{0}}]$$
$$[A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{1}}, \neg A^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$$
$$[A^{\mathbf{1}}, B^{\mathbf{1}}, \neg A^{\mathbf{1}}, \neg A^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$$

$$[A^{\mathbf{1}}, B^{\mathbf{1}}, (\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$$
$$[A^{\mathbf{1}}, \neg B^{\mathbf{0}}, (\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$$
$$[\neg A^{\mathbf{0}}, \neg B^{\mathbf{0}}, (\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$$
$$[(\neg A \wedge \neg B)^{\mathbf{0}}, (\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$$
$$[\neg(\neg A \wedge \neg B)^{\mathbf{1/2}}, (\neg A \wedge \neg B)^{\mathbf{0}}, (A \vee B)^{\mathbf{0}}]$$
$$[\neg(\neg A \wedge \neg B)^{\mathbf{1}}, \neg(\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$$

Derivation of $[\neg(\neg A \wedge \neg B)^{\mathbf{0}}, \neg(\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$:

axiom for B

$$[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$$
$$[B^{\mathbf{0}}, B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$$
$$[B^{\mathbf{0}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$$
$$[B^{\mathbf{0}}, \neg B^{\mathbf{1}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$$

axiom for B

$$[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$$
$$[A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$$
$$[A^{\mathbf{1/2}}, B^{\mathbf{0}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$$
$$[B^{\mathbf{0}}, \neg A^{\mathbf{1/2}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$$

axiom for A

$$[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}]$$
$$[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, (A \vee B)^{\mathbf{1}}]$$
$$[A^{\mathbf{0}}, B^{\mathbf{0}}, \neg A^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$$
$$[B^{\mathbf{0}}, \neg A^{\mathbf{1}}, \neg A^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$$

$$[B^{\mathbf{0}}, (\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$$
$$[\neg B^{\mathbf{1}}, (\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$$

axiom for B

$$[A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$$
$$[A^{\mathbf{0}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$$
$$[A^{\mathbf{0}}, B^{\mathbf{0}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$$
$$[A^{\mathbf{0}}, \neg B^{\mathbf{1}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$$

$[A^{\mathbf{0}},$ …

$[A^{\mathbf{0}}, A$ …

$[A^{\mathbf{0}}, A$ …

$[A^{\mathbf{0}}, \neg A$ …

$[A^{\mathbf{0}}, (\neg A$ …

$[\neg A^{\mathbf{1}}, (\neg$ …

$$[(\neg A \wedge \neg B)^{\mathbf{1}}, (\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$$
$$[\neg(\neg A \wedge \neg B)^{\mathbf{1/2}}, (\neg A \wedge \neg B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1}}]$$
$$[\neg(\neg A \wedge \neg B)^{\mathbf{0}}, \neg(\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$$

Derivation of $[\neg(\neg A \wedge \neg B)^{\mathbf{0}}, \neg(\neg A \wedge \neg B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]$:

axiom for B $[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$ axiom for B $[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$ axiom for A $[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}]$

$$[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]$$
$$[A^{\mathbf{1}}, B^{\mathbf{1}}, \neg B^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]$$

axiom for B $[A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$ axiom for A $[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]$ axio… $[A^{\mathbf{0}}, A^{\mathbf{1}}$ …

$$[A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]$$
$$[A^{\mathbf{1}}, B^{\mathbf{1}}, \neg A^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]$$

$$[A^{\mathbf{1}}, B^{\mathbf{1}}, (\neg A \wedge \neg B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]$$
$$[A^{\mathbf{1}}, \neg B^{\mathbf{0}}, (\neg A \wedge \neg B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]$$
$$[\neg A^{\mathbf{0}}, \neg B^{\mathbf{0}}, (\neg A \wedge \neg B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]$$
$$[(\neg A \wedge \neg B)^{\mathbf{0}}, (\neg A \wedge \neg B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]$$
$$[\neg(\neg A \wedge \neg B)^{\mathbf{1}}, (\neg A \wedge \neg B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]$$
$$[\neg(\neg A \wedge \neg B)^{\mathbf{0}}, \neg(\neg A \wedge \neg B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]$$

Proposition 51 *The equality $(A \wedge B) = \neg(A \rightarrow \neg B)$ does **not** hold.*

The problem is equivalent to proving the following sequents:

$$[\neg(A \rightarrow \neg B)^{\mathbf{1}}, \neg(A \rightarrow \neg B)^{\mathbf{1/2}}, (A \wedge B)^{\mathbf{0}}]$$
$$[\neg(A \rightarrow \neg B)^{\mathbf{0}}, \neg(A \rightarrow \neg B)^{\mathbf{1/2}}, (A \wedge B)^{\mathbf{1}}]$$
$$[\neg(A \rightarrow \neg B)^{\mathbf{0}}, \neg(A \rightarrow \neg B)^{\mathbf{1}}, (A \wedge B)^{\mathbf{1/2}}]$$

Derivation of $[\neg(A\to\neg B)^1, \neg(A\to\neg B)^{1/2}, (A\wedge B)^0]$:

axiom for B — hypothesis

$[A^0, A^{1/2}, B^0, B^1, B^{1/2}]$ — $[A^0, A^1, B^0, B^1]$ — hypothesis

$[A^0, A^{1/2}, B^0, B^1, \neg B^{1/2}]$ — $[A^0, A^1, B^0, B^1, \neg B^0]$ — axiom for A — $[A^0, A^1, B^0, B^1]$

$[A^0, B^0, B^1, (A\to\neg B)^{1/2}]$ — $[A^0, A^1, A^{1/2}, B^0, \neg B^{1/2}]$ — $[A^0, A^1, B^0, \neg B^0]$

$[A^0, B^0, \neg B^0, (A\to\neg B)^{1/2}]$ — $[A^0, A^1, B^0, (A\to\neg B)^{1/2}]$

$$[A^0, B^0, (A\to\neg B)^0, (A\to\neg B)^{1/2}]$$
$$[(A\wedge B)^0, (A\to\neg B)^0, (A\to\neg B)^{1/2}]$$
$$[\neg(A\to\neg B)^{1/2}, (A\wedge B)^0, (A\to\neg B)^0]$$
$$[\neg(A\to\neg B)^1, \neg(A\to\neg B)^{1/2}, (A\wedge B)^0]$$

Derivation of $[\neg(A\to\neg B)^0, \neg(A\to\neg B)^{1/2}, (A\wedge B)^1]$:

axiom for B — axiom for A

axiom for B — $[A^0, A^{1/2}, B^0, B^1, B^{1/2}]$ — axiom for A — $[A^0, A^1, A^{1/2}, B^0, B^{1/2}, \neg B^{1/2}]$

$[A^0, B^0, B^1, B^{1/2}, (A\to\neg B)^{1/2}]$ — $[A^0, A^{1/2}, B^0, B^1, \neg B^{1/2}]$ — $[A^0, A^1, A^{1/2}, B^0, B^1, \neg B^0]$ — $[A^0, A^1, B^0, B^{1/2}, (A$

$[A^0, B^0, B^1, \neg B^{1/2}, (A\to\neg B)^{1/2}]$ — $[A^0, A^{1/2}, B^0, B^1, (A\to\neg B)^{1/2}]$ — $[A^0, A^1, B^0, \neg B^{1/2}, ($

$[A^0, B^1, \neg B^1, \neg B^{1/2}, (A\to\neg B)^{1/2}]$ — $[A^0, A^{1/2}, B^1, \neg B^1, (A\to\neg B)^{1/2}]$ — $[A^0, A^1, \neg B^1, \neg B^{1/2}, ($

$[B^1, (A\to\neg B)^1, (A\to\neg B)^{1/2}]$

$$[(A\wedge B)^1, (A\to\neg B)^1, (A\to\neg B)^{1/2}]$$
$$[\neg(A\to\neg B)^{1/2}, (A\wedge B)^1, (A\to\neg B)^1]$$
$$[\neg(A\to\neg B)^0, \neg(A\to\neg B)^{1/2}, (A\wedge B)^1]$$

Derivation of $[\neg(A\to\neg B)^0, \neg(A\to\neg B)^1, (A\wedge B)^{1/2}]$:

axiom for B — axiom for B — $[A^0, A$

$[A^0, B^0, B^1, B^{1/2}, \neg B^{1/2}]$ — $[A^0, A^{1/2}, B^0, B^1, B^{1/2}]$ — axiom for B — $[A^0, A$

$[A^0, B^1, B^{1/2}, \neg B^1, \neg B^{1/2}]$ — $[A^0, A^{1/2}, B^1, B^{1/2}, \neg B^1]$ — $[A^0, A^1, B^0, B^1, B^{1/2}, \neg B^{1/2}]$ — axiom for A

$[B^1, B^{1/2}, (A\to\neg B)^1]$ — $[A^0, A^1, B^1, B^{1/2}, \neg B^1, \neg B^{1/2}]$ — $[A^0, A^1, A^{1/2}, B^1, B^{1/2}, \neg B^1]$

$[B^1, B^{1/2}, \neg B^0, (A\to\neg B)^1]$ — $[A^1, B^1, B^{1/2}, (A\to\neg B)^1]$

$[B^1, B^{1/2}, (A\to\neg B)^0, (A\to\neg B)^1]$

List of counter-examples:

$$[A^{1/2}, B^{1/2}]$$

Proposition 52 *The equality* $(A\vee B) = \neg(\neg A\wedge\neg B)$ *holds.*

The problem is equivalent to proving the following sequents:

$$[\neg(\neg A\wedge\neg B)^1, \neg(\neg A\wedge\neg B)^{1/2}, (A\vee B)^0]$$
$$[\neg(\neg A\wedge\neg B)^0, \neg(\neg A\wedge\neg B)^{1/2}, (A\vee B)^1]$$
$$[\neg(\neg A\wedge\neg B)^0, \neg(\neg A\wedge\neg B)^1, (A\vee B)^{1/2}]$$

Derivation of $[\neg(\neg A \wedge \neg B)^{\mathbf{1}}, \neg(\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]$:

$$
\dfrac{
\dfrac{\dfrac{\text{axiom for } B}{[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]}}{\dfrac{[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]}{[A^{\mathbf{1}}, B^{\mathbf{1}}, \neg B^{\mathbf{1}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]}}
\quad
\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } B}{[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]} \quad \overset{\text{axiom for } A}{[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]}}{[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]}}{[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]}}{[A^{\mathbf{1}}, B^{\mathbf{1}}, \neg A^{\mathbf{1/2}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]}
\quad
\dfrac{\dfrac{\overset{\text{axiom for } A}{[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, (A \vee B)^{\mathbf{0}}]}}{[A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{1}}, \neg A^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]}}{[A^{\mathbf{1}}, B^{\mathbf{1}}, \neg A^{\mathbf{1}}, \neg A^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]}
}{
\dfrac{[A^{\mathbf{1}}, B^{\mathbf{1}}, (\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]}{\dfrac{[A^{\mathbf{1}}, \neg B^{\mathbf{0}}, (\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]}{\dfrac{[\neg A^{\mathbf{0}}, \neg B^{\mathbf{0}}, (\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]}{\dfrac{[(\neg A \wedge \neg B)^{\mathbf{0}}, (\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]}{\dfrac{[\neg(\neg A \wedge \neg B)^{\mathbf{1/2}}, (\neg A \wedge \neg B)^{\mathbf{0}}, (A \vee B)^{\mathbf{0}}]}{[\neg(\neg A \wedge \neg B)^{\mathbf{1}}, \neg(\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{0}}]}}}}}
}
$$

Derivation of $[\neg(\neg A \wedge \neg B)^{\mathbf{0}}, \neg(\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]$:

$$
\dfrac{
\dfrac{\dfrac{\overset{\text{axiom for } B}{[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]}}{\dfrac{[B^{\mathbf{0}}, B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]}{[B^{\mathbf{0}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]}}}{[B^{\mathbf{0}}, \neg B^{\mathbf{1}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]}
\quad
\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } B}{[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]}}{[A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]}}{[A^{\mathbf{1/2}}, B^{\mathbf{0}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]}}{[B^{\mathbf{0}}, \neg A^{\mathbf{1/2}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]}
\quad
\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } A}{[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}]}}{[A^{\mathbf{0}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, (A \vee B)^{\mathbf{1}}]}}{[A^{\mathbf{0}}, B^{\mathbf{0}}, \neg A^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]}}{[B^{\mathbf{0}}, \neg A^{\mathbf{1}}, \neg A^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]}
}{
\dfrac{[B^{\mathbf{0}}, (\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]}{[\neg B^{\mathbf{1}}, (\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]}
}
\quad
\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } B}{[A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]}}{[A^{\mathbf{0}}, B^{\mathbf{0}}, B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]}}{[A^{\mathbf{0}}, B^{\mathbf{0}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]}}{[A^{\mathbf{0}}, \neg B^{\mathbf{1}}, \neg B^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]}
\quad \cdots
$$

$$
\dfrac{\cdots}{\dfrac{[(\neg A \wedge \neg B)^{\mathbf{1}}, (\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]}{\dfrac{[\neg(\neg A \wedge \neg B)^{\mathbf{1/2}}, (\neg A \wedge \neg B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1}}]}{[\neg(\neg A \wedge \neg B)^{\mathbf{0}}, \neg(\neg A \wedge \neg B)^{\mathbf{1/2}}, (A \vee B)^{\mathbf{1}}]}}}
$$

Derivation of $[\neg(\neg A \wedge \neg B)^{\mathbf{0}}, \neg(\neg A \wedge \neg B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]$:

$$
\dfrac{
\overset{\text{axiom for } B}{[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]}
\quad
\dfrac{\dfrac{\overset{\text{axiom for } B}{[A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]} \quad \overset{\text{axiom for } A}{[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{0}}, B^{\mathbf{1}}]}}{[A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]}}{[A^{\mathbf{1}}, B^{\mathbf{1}}, \neg B^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]}
\quad
\dfrac{\dfrac{\overset{\text{axiom for } B}{[A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{0}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]} \quad \overset{\text{axiom for } A}{[A^{\mathbf{0}}, A^{\mathbf{1}}, A^{\mathbf{1/2}}, B^{\mathbf{1}}, B^{\mathbf{1/2}}]} \quad \cdots}{[A^{\mathbf{0}}, A^{\mathbf{1}}, B^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]}}{[A^{\mathbf{1}}, B^{\mathbf{1}}, \neg A^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]}
}{
\dfrac{[A^{\mathbf{1}}, B^{\mathbf{1}}, (\neg A \wedge \neg B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]}{\dfrac{[A^{\mathbf{1}}, \neg B^{\mathbf{0}}, (\neg A \wedge \neg B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]}{\dfrac{[\neg A^{\mathbf{0}}, \neg B^{\mathbf{0}}, (\neg A \wedge \neg B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]}{\dfrac{[(\neg A \wedge \neg B)^{\mathbf{0}}, (\neg A \wedge \neg B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]}{\dfrac{[\neg(\neg A \wedge \neg B)^{\mathbf{1}}, (\neg A \wedge \neg B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]}{[\neg(\neg A \wedge \neg B)^{\mathbf{0}}, \neg(\neg A \wedge \neg B)^{\mathbf{1}}, (A \vee B)^{\mathbf{1/2}}]}}}}}
}
$$

8 Metaconsequences

Proposition 53 *The following meta-consequence* ***does not*** *hold:*

$$P, Q \vdash R \quad / \quad P \vdash (Q \to R)$$

The problem is equivalent to proving the following sequents:

$$[P^{\mathbf{0}}, P^{1/2}, Q^{\mathbf{1}}, (Q \to R)^{\mathbf{1}}]$$
$$[P^{\mathbf{0}}, P^{1/2}, R^{\mathbf{0}}, R^{1/2}, (Q \to R)^{\mathbf{1}}]$$

Derivation of $[P^{\mathbf{0}}, P^{1/2}, Q^{\mathbf{1}}, (Q \to R)^{\mathbf{1}}]$:

$$\frac{\overset{\text{hypothesis}}{[P^{\mathbf{0}}, P^{1/2}, Q^{\mathbf{0}}, Q^{\mathbf{1}}, R^{1}, R^{1/2}]} \quad \overset{\text{axiom for } Q}{[P^{\mathbf{0}}, P^{1/2}, Q^{\mathbf{0}}, Q^{\mathbf{1}}, Q^{1/2}, R^{\mathbf{1}}]}}{[P^{\mathbf{0}}, P^{1/2}, Q^{\mathbf{1}}, (Q \to R)^{\mathbf{1}}]}$$

Derivation of $[P^{\mathbf{0}}, P^{1/2}, R^{\mathbf{0}}, R^{1/2}, (Q \to R)^{\mathbf{1}}]$:

$$\frac{\overset{\text{axiom for } R}{[P^{\mathbf{0}}, P^{1/2}, Q^{\mathbf{0}}, R^{\mathbf{0}}, R^{\mathbf{1}}, R^{1/2}]} \quad \overset{\text{axiom for } R}{[P^{\mathbf{0}}, P^{1/2}, Q^{\mathbf{0}}, Q^{1/2}, R^{\mathbf{0}}, R^{\mathbf{1}}, R^{1/2}]}}{[P^{\mathbf{0}}, P^{1/2}, R^{\mathbf{0}}, R^{1/2}, (Q \to R)^{\mathbf{1}}]}$$

List of counter-examples:

$$[P^{\mathbf{1}}, Q^{1/2}, R^{\mathbf{0}}]$$

Proposition 54 *The following meta-consequence* ***does not*** *hold:*

$$(P \wedge Q) \vdash R \quad / \quad P \vdash (Q \to R)$$

The problem is equivalent to proving the following sequents:

$$[P^{\mathbf{0}}, P^{1/2}, R^{\mathbf{0}}, R^{1/2}, (Q \to R)^{\mathbf{1}}]$$
$$[P^{\mathbf{0}}, P^{1/2}, (P \wedge Q)^{\mathbf{1}}, (Q \to R)^{\mathbf{1}}]$$

Derivation of $[P^{\mathbf{0}}, P^{1/2}, R^{\mathbf{0}}, R^{1/2}, (Q \to R)^{\mathbf{1}}]$:

$$\frac{\overset{\text{axiom for } R}{[P^{\mathbf{0}}, P^{1/2}, Q^{\mathbf{0}}, R^{\mathbf{0}}, R^{\mathbf{1}}, R^{1/2}]} \quad \overset{\text{axiom for } R}{[P^{\mathbf{0}}, P^{1/2}, Q^{\mathbf{0}}, Q^{1/2}, R^{\mathbf{0}}, R^{\mathbf{1}}, R^{1/2}]}}{[P^{\mathbf{0}}, P^{1/2}, R^{\mathbf{0}}, R^{1/2}, (Q \to R)^{\mathbf{1}}]}$$

Derivation of $[P^{\mathbf{0}}, P^{1/2}, (P \wedge Q)^{\mathbf{1}}, (Q \to R)^{\mathbf{1}}]$:

$$\frac{\dfrac{\overset{\text{hypothesis}}{[P^{\mathbf{0}}, P^{1/2}, Q^{\mathbf{0}}, Q^{\mathbf{1}}, R^{\mathbf{1}}, R^{1/2}]} \quad \overset{\text{axiom for } Q}{[P^{\mathbf{0}}, P^{1/2}, Q^{\mathbf{0}}, Q^{\mathbf{1}}, Q^{1/2}, R^{\mathbf{1}}]}}{[P^{\mathbf{0}}, P^{1/2}, Q^{\mathbf{1}}, (Q \to R)^{\mathbf{1}}]} \quad \overset{\text{axiom for } P}{[P^{\mathbf{0}}, P^{\mathbf{1}}, P^{1/2}, (Q \to R)^{\mathbf{1}}]}}{[P^{\mathbf{0}}, P^{1/2}, (P \wedge Q)^{\mathbf{1}}, (Q \to R)^{\mathbf{1}}]}$$

List of counter-examples:

$$[P^{\mathbf{1}}, Q^{1/2}, R^{\mathbf{0}}]$$

9 Program listing: `ex_lukasiewicz1.pl`

```
% Test file to check things in Lukasiewicz logic

% make sure MUltseq is loaded
:- ensure_loaded('../multseq/multseq').

% load sample properties
:- [properties].
```

```
% load the rules
:- load_logic('lukasiewicz.msq').

% define standard Omap
:- setOmap([(neg)/(-),imp/(>),and/(/\),or/(\/),equiv/(=)]).

% check all properties and write report to out.tex

:- set_option(tex_output(verbose)).

:- start_logging(ex_lukasiewicz1,'.tex').

:- print_tex(tex_title("Report␣on␣3-Valued␣\\L␣ukasiewicz␣Logic")).

:- print_tex(tex_paragraph(["We␣check␣a␣number␣of␣properties␣in␣the␣3-valued␣\\L␣ukasiewicz␣
    logic."])).

:- print_tex(tex_logic).

:- print_tex(tex_section(["Wajsberg's␣axioms␣for␣\\L␣ukasiewicz␣logic"])).

:- (member(X,[wajsberg1,wajsberg2,wajsberg3,wajsberg4]), chkProp(X), fail; true).

:- print_tex(tex_section(["Bernays's␣axioms␣for␣classical␣logic"])).

% leaving out bernays11-13 as these involve equvalence

:- set_option(tex_output(terse)).

:- (member(X,[bernays1,bernays2,bernays3,bernays4,bernays5,bernays6,bernays7,bernays8,
    bernays9,bernays10,bernays14,bernays15,bernays16,bernays17]), chkProp(X), fail; true).

:- set_option(tex_output(verbose)).

:- print_tex(tex_section(["Classical␣tautologies␣not␣intuitionistically␣valid"])).

:- (member(X,[lem,weaklem,bernays15,prelinearity,mirabilis,peirce]), chkProp(X), fail; true)
    .

:- print_tex(tex_section(["Some␣more␣interesting␣tautologies"])).

:- (member(X,[mingle,pseudomp,prefix,suffix,contraction,reductio]), chkProp(X), fail; true).


:- print_tex(tex_section(["Some␣popular␣consequences"])).

:- (member(X,[modusponens,modustollens,hyposyllogism,disjsyllogism,destrdilemma,
    constrdilemma,importation,exportation,contrapos1,contrapos2,agglomeration,sda]), chkProp
    (X), fail; true).


:- print_tex(tex_section(["Some␣popular␣equivalences"])).

:- (member(X,[ldistrright,ldistrleft]), chkProp(X), fail; true).
% Here we switch and and or
:- (member(X,[ldistrright,ldistrleft]), chkProp([or/(/\),and/(\/)],X), fail; true).


:- print_tex(tex_section(["Interdefinability␣of␣connectives"])).

:- (member(X,[defimpor,defimpand,deforimp,deforand,defandimp,deforand]), chkProp(X), fail;
    true).
```

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:- print_tex(tex_section(["Metaconsequences"])).

:- (member(X,[deductionthm,residuation]), chkProp(X), fail; true).

:- print_tex(tex_listing("ex_lukasiewicz1.pl")).

:- stop_logging.
```