Report on Classical Logic

M. Ultseq

July 2, 2024

The logic contains the connectives

$$\wedge, \leftrightarrow, \not\rightarrow, \rightarrow, ite, |, \neg, \downarrow, \vee, \oplus$$

and truth values

$$\mathbf{f}, \mathbf{t}.$$

The truth value $\mathbf{t}$ is designated.

We verify that all classical logic satisfies some well-known properties (involving only $\wedge$, $\vee$, $\rightarrow$, $\neg$, and $\leftrightarrow$). We output proofs in "multidimensional" format, which for classical logic means just two sides to a sequent, as usual.

1 Bernays's axioms for classical logic

Proposition 1 *The formula* $(A \rightarrow (B \rightarrow A))$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid (A \rightarrow (B \rightarrow A))]$$

Derivation of $[\emptyset \mid (A \rightarrow (B \rightarrow A))]$:

$$\dfrac{\dfrac{\overset{\text{axiom for } A}{[A, B \mid A]}}{[A \mid (B \rightarrow A)]}\ \rightarrow_{\mathbf{t}}}{[\emptyset \mid (A \rightarrow (B \rightarrow A))]}\ \rightarrow_{\mathbf{t}}$$

Proposition 2 *The formula* $((A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B))$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid ((A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B))]$$

Derivation of $[\emptyset \mid ((A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B))]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } B}{[A, B \mid B]} \quad \overset{\text{axiom for } A}{[A \mid A, B]}}{[A, (A \rightarrow B) \mid B]}\ \rightarrow_{\mathbf{f}} \quad \overset{\text{axiom for } A}{[A \mid A, B]}}{[A, (A \rightarrow (A \rightarrow B)) \mid B]}\ \rightarrow_{\mathbf{f}}}{[(A \rightarrow (A \rightarrow B)) \mid (A \rightarrow B)]}\ \rightarrow_{\mathbf{t}}}{[\emptyset \mid ((A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B))]}\ \rightarrow_{\mathbf{t}}$$

Proposition 3 *The formula* $((A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)))$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid ((A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)))]$$

Derivation of $[\emptyset \mid ((A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)))]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } C}{[A,B,C \mid C]}}{[B,C \mid (A \rightarrow C)]}\rightarrow_{\mathbf{t}}}{[C \mid (B \rightarrow (A \rightarrow C))]}\rightarrow_{\mathbf{t}} \quad \dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } B}{[B \mid B,(A \rightarrow C)]}}{[\emptyset \mid B,(B \rightarrow (A \rightarrow C))]}\rightarrow_{\mathbf{t}}}{[(B \rightarrow C) \mid (B \rightarrow (A \rightarrow C))]}\rightarrow_{\mathbf{f}} \quad \dfrac{\dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } A}{[A,B \mid A,C]}}{[B \mid A,(A \rightarrow C)]}\rightarrow_{\mathbf{t}}}{[\emptyset \mid A,(B \rightarrow (A \rightarrow C))]}\rightarrow_{\mathbf{t}}}{[(A \rightarrow (B \rightarrow C)) \mid (B \rightarrow (A \rightarrow C))]}\rightarrow_{\mathbf{f}}}{[\emptyset \mid ((A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)))]}\rightarrow_{\mathbf{t}}$$

Proposition 4 *The formula* $((B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid ((B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))]$$

Derivation of $[\emptyset \mid ((B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } C}{[A,B,C \mid C]}}{[B,C \mid (A \rightarrow C)]}\rightarrow_{\mathbf{t}} \quad \dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } A}{[A,C \mid A,C]}}{[C \mid A,(A \rightarrow C)]}\rightarrow_{\mathbf{t}}}{[C,(A \rightarrow B) \mid (A \rightarrow C)]}\rightarrow_{\mathbf{f}}}{[C \mid ((A \rightarrow B) \rightarrow (A \rightarrow C))]}\rightarrow_{\mathbf{t}} \quad \dfrac{\dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } B}{[B \mid B,(A \rightarrow C)]} \quad \dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } A}{[A \mid A,B,C]}}{[\emptyset \mid A,B,(A \rightarrow C)]}\rightarrow_{\mathbf{t}}}{[(A \rightarrow B) \mid B,(A \rightarrow C)]}\rightarrow_{\mathbf{f}}}{[\emptyset \mid B,((A \rightarrow B) \rightarrow (A \rightarrow C))]}\rightarrow_{\mathbf{t}}}{[(B \rightarrow C) \mid ((A \rightarrow B) \rightarrow (A \rightarrow C))]}\rightarrow_{\mathbf{f}}}{[\emptyset \mid ((B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))]}\rightarrow_{\mathbf{t}}$$

Proposition 5 *The formula* $((A \wedge B) \rightarrow A)$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid ((A \wedge B) \rightarrow A)]$$

Derivation of $[\emptyset \mid ((A \wedge B) \rightarrow A)]$:

$$\dfrac{\dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } A}{[A,B \mid A]}}{[(A \wedge B) \mid A]}\wedge_{\mathbf{f}}}{[\emptyset \mid ((A \wedge B) \rightarrow A)]}\rightarrow_{\mathbf{t}}$$

Proposition 6 *The formula* $((A \wedge B) \rightarrow B)$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid ((A \wedge B) \rightarrow B)]$$

Derivation of $[\emptyset \mid ((A \wedge B) \rightarrow B)]$:

$$\dfrac{\dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } B}{[A,B \mid B]}}{[(A \wedge B) \mid B]}\wedge_{\mathbf{f}}}{[\emptyset \mid ((A \wedge B) \rightarrow B)]}\rightarrow_{\mathbf{t}}$$

Proposition 7 *The formula* $((A \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow (B \wedge C))))$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid ((A \to B) \to ((A \to C) \to (A \to (B \wedge C))))]$$

Derivation of $[\emptyset \mid ((A \to B) \to ((A \to C) \to (A \to (B \wedge C))))]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{axiom for } C\\ [A,B,C \mid C]\end{array} \quad \begin{array}{c}\text{axiom for } B\\ [A,B,C \mid B]\end{array}}{[A,B,C \mid (B \wedge C)]}\wedge_{\mathbf{t}}}{[B,C \mid (A \to (B \wedge C))]}\to_{\mathbf{t}} \quad \dfrac{\begin{array}{c}\text{axiom for } A\\ [A,B \mid A,(B \wedge C)]\end{array}}{[B \mid A,(A \to (B \wedge C))]}\to_{\mathbf{t}}}{[B,(A \to C) \mid (A \to (B \wedge C))]}\to_{\mathbf{f}}}{[B \mid ((A \to C) \to (A \to (B \wedge C)))]}\to_{\mathbf{t}} \quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\text{axiom for } A\\ [A,C \mid A,(B \wedge C)]\end{array}}{[C \mid A,(A \to (B \wedge C))]}\to_{\mathbf{t}} \quad \dfrac{\begin{array}{c}\text{axiom for } A\\ [A \mid A,(B \wedge C)]\end{array}}{[\emptyset \mid A,(A \to (B \wedge C))]}\to_{\mathbf{t}}}{[(A \to C) \mid A,(A \to (B \wedge C))]}\to_{\mathbf{f}}}{[\emptyset \mid A,((A \to C) \to (A \to (B \wedge C)))]}\to_{\mathbf{t}}}{[(A \to B) \mid ((A \to C) \to (A \to (B \wedge C)))]}\to_{\mathbf{f}}}{[\emptyset \mid ((A \to B) \to ((A \to C) \to (A \to (B \wedge C))))]}\to_{\mathbf{t}}$$

Proposition 8 *The formula* $(A \to (A \vee B))$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid (A \to (A \vee B))]$$

Derivation of $[\emptyset \mid (A \to (A \vee B))]$:

$$\dfrac{\dfrac{\begin{array}{c}\text{axiom for } A\\ [A \mid A,B]\end{array}}{[A \mid (A \vee B)]}\vee_{\mathbf{t}}}{[\emptyset \mid (A \to (A \vee B))]}\to_{\mathbf{t}}$$

Proposition 9 *The formula* $(B \to (A \vee B))$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid (B \to (A \vee B))]$$

Derivation of $[\emptyset \mid (B \to (A \vee B))]$:

$$\dfrac{\dfrac{\begin{array}{c}\text{axiom for } B\\ [B \mid A,B]\end{array}}{[B \mid (A \vee B)]}\vee_{\mathbf{t}}}{[\emptyset \mid (B \to (A \vee B))]}\to_{\mathbf{t}}$$

Proposition 10 *The formula* $((B \to A) \to ((C \to A) \to ((B \vee C) \to A)))$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid ((B \to A) \to ((C \to A) \to ((B \vee C) \to A)))]$$

Derivation of $[\emptyset \mid ((B \to A) \to ((C \to A) \to ((B \vee C) \to A)))]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{axiom for } A\\ [A,(B \vee C) \mid A]\end{array}}{[A \mid ((B \vee C) \to A)]}\to_{\mathbf{t}} \quad \dfrac{\begin{array}{c}\text{axiom for } A\\ [A,(B \vee C) \mid A,C]\end{array}}{[A \mid C,((B \vee C) \to A)]}\to_{\mathbf{t}}}{[A,(C \to A) \mid ((B \vee C) \to A)]}\to_{\mathbf{f}}}{[A \mid ((C \to A) \to ((B \vee C) \to A))]}\to_{\mathbf{t}} \quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\text{axiom for } A\\ [A,(B \vee C) \mid A,B]\end{array}}{[A \mid B,((B \vee C) \to A)]}\to_{\mathbf{t}} \quad \dfrac{\dfrac{\begin{array}{c}\text{axiom for } C\\ [C \mid A,B,C]\end{array} \quad \begin{array}{c}\text{axiom for } B\\ [B \mid A,B,C]\end{array}}{[(B \vee C) \mid A,B,C]}\vee_{\mathbf{f}}}{[\emptyset \mid B,C,((B \vee C) \to A)]}\to_{\mathbf{t}}}{[(C \to A) \mid B,((B \vee C) \to A)]}\to_{\mathbf{f}}}{[\emptyset \mid B,((C \to A) \to ((B \vee C) \to A))]}\to_{\mathbf{t}}}{[(B \to A) \mid ((C \to A) \to ((B \vee C) \to A))]}\to_{\mathbf{f}}}{[\emptyset \mid ((B \to A) \to ((C \to A) \to ((B \vee C) \to A)))]}\to_{\mathbf{t}}$$

Proposition 11 *The formula $((A \leftrightarrow B) \rightarrow (A \rightarrow B))$ is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid ((A \leftrightarrow B) \rightarrow (A \rightarrow B))]$$

Derivation of $[\emptyset \mid ((A \leftrightarrow B) \rightarrow (A \rightarrow B))]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{axiom for } B}{[A, B \mid B]}}{[A, B \mid (A \rightarrow B)]}\rightarrow_{\mathbf{t}} \quad \dfrac{\dfrac{\text{axiom for } A}{[A \mid A, B]}}{[\emptyset \mid A, B, (A \rightarrow B)]}\rightarrow_{\mathbf{t}}}{[(A \leftrightarrow B) \mid (A \rightarrow B)]}\leftrightarrow_{\mathbf{f}}}{[\emptyset \mid ((A \leftrightarrow B) \rightarrow (A \rightarrow B))]}\rightarrow_{\mathbf{t}}$$

Proposition 12 *The formula $((A \leftrightarrow B) \rightarrow (B \rightarrow A))$ is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid ((A \leftrightarrow B) \rightarrow (B \rightarrow A))]$$

Derivation of $[\emptyset \mid ((A \leftrightarrow B) \rightarrow (B \rightarrow A))]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{axiom for } A}{[A, B \mid A]}}{[A, B \mid (B \rightarrow A)]}\rightarrow_{\mathbf{t}} \quad \dfrac{\dfrac{\text{axiom for } B}{[B \mid A, B]}}{[\emptyset \mid A, B, (B \rightarrow A)]}\rightarrow_{\mathbf{t}}}{[(A \leftrightarrow B) \mid (B \rightarrow A)]}\leftrightarrow_{\mathbf{f}}}{[\emptyset \mid ((A \leftrightarrow B) \rightarrow (B \rightarrow A))]}\rightarrow_{\mathbf{t}}$$

Proposition 13 *The formula $((A \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow (A \leftrightarrow B)))$ is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid ((A \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow (A \leftrightarrow B)))]$$

Derivation of $[\emptyset \mid ((A \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow (A \leftrightarrow B)))]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{axiom for } B}{[A, B, (B \rightarrow A) \mid B]} \quad \dfrac{\dfrac{\text{axiom for } A}{[A, B \mid A]} \quad \dfrac{\text{axiom for } B}{[B \mid A, B]}}{[B, (B \rightarrow A) \mid A]}\rightarrow_{\mathbf{f}}}{[B, (B \rightarrow A) \mid (A \leftrightarrow B)]}\leftrightarrow_{\mathbf{t}}}{[B \mid ((B \rightarrow A) \rightarrow (A \leftrightarrow B))]}\rightarrow_{\mathbf{t}} \quad \dfrac{\dfrac{\dfrac{\text{axiom for } A}{[A, (B \rightarrow A) \mid A, B]} \quad \dfrac{\dfrac{\text{axiom for } A}{[A, B \mid A]} \quad \dfrac{\text{axiom for } B}{[B \mid A, B]}}{[B, (B \rightarrow A) \mid A]}\rightarrow_{\mathbf{f}}}{[(B \rightarrow A) \mid A, (A \leftrightarrow B)]}\leftrightarrow_{\mathbf{t}}}{[\emptyset \mid A, ((B \rightarrow A) \rightarrow (A \leftrightarrow B))]}\rightarrow_{\mathbf{t}}}{[(A \rightarrow B) \mid ((B \rightarrow A) \rightarrow (A \leftrightarrow B))]}\rightarrow_{\mathbf{f}}}{[\emptyset \mid ((A \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow (A \leftrightarrow B)))]}\rightarrow_{\mathbf{t}}$$

Proposition 14 *The formula $((A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A))$ is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid ((A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A))]$$

Derivation of $[\emptyset \mid ((A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A))]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{axiom for } B}{[A, B \mid B]}}{[A, B, \neg B \mid \emptyset]}\neg_{\mathbf{f}}}{[B, \neg B \mid \neg A]}\neg_{\mathbf{t}}}{[B \mid (\neg B \rightarrow \neg A)]}\rightarrow_{\mathbf{t}} \quad \dfrac{\dfrac{\dfrac{\text{axiom for } A}{[A, \neg B \mid A]}}{[\neg B \mid A, \neg A]}\neg_{\mathbf{t}}}{[\emptyset \mid A, (\neg B \rightarrow \neg A)]}\rightarrow_{\mathbf{t}}}{[(A \rightarrow B) \mid (\neg B \rightarrow \neg A)]}\rightarrow_{\mathbf{f}}}{[\emptyset \mid ((A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A))]}\rightarrow_{\mathbf{t}}$$

Proposition 15 *The formula* $((A \to \neg A) \to \neg A)$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid ((A \to \neg A) \to \neg A)]$$

Derivation of $[\emptyset \mid ((A \to \neg A) \to \neg A)]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } A}{[A \mid A]}}{[A, \neg A \mid \emptyset]}\,\neg_{\mathbf{f}} \qquad \overset{\text{axiom for } A}{[A \mid A]}}{[A, (A \to \neg A) \mid \emptyset]}\,\to_{\mathbf{f}}}{[(A \to \neg A) \mid \neg A]}\,\neg_{\mathbf{t}}}{[\emptyset \mid ((A \to \neg A) \to \neg A)]}\,\to_{\mathbf{t}}$$

Proposition 16 *The formula* $(A \to \neg\neg A)$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid (A \to \neg\neg A)]$$

Derivation of $[\emptyset \mid (A \to \neg\neg A)]$:

$$\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } A}{[A \mid A]}}{[A, \neg A \mid \emptyset]}\,\neg_{\mathbf{f}}}{[A \mid \neg\neg A]}\,\neg_{\mathbf{t}}}{[\emptyset \mid (A \to \neg\neg A)]}\,\to_{\mathbf{t}}$$

Proposition 17 *The formula* $(\neg\neg A \to A)$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid (\neg\neg A \to A)]$$

Derivation of $[\emptyset \mid (\neg\neg A \to A)]$:

$$\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } A}{[A \mid A]}}{[\emptyset \mid A, \neg A]}\,\neg_{\mathbf{t}}}{[\neg\neg A \mid A]}\,\neg_{\mathbf{f}}}{[\emptyset \mid (\neg\neg A \to A)]}\,\to_{\mathbf{t}}$$

2 Classical tautologies not intuitionistically valid

Proposition 18 *The formula* $(A \vee \neg A)$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid (A \vee \neg A)]$$

Derivation of $[\emptyset \mid (A \vee \neg A)]$:

$$\dfrac{\dfrac{\overset{\text{axiom for } A}{[A \mid A]}}{[\emptyset \mid A, \neg A]}\,\neg_{\mathbf{t}}}{[\emptyset \mid (A \vee \neg A)]}\,\vee_{\mathbf{t}}$$

Proposition 19 *The formula* $(\neg A \vee \neg\neg A)$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid (\neg A \vee \neg\neg A)]$$

Derivation of $[\emptyset \mid (\neg A \vee \neg\neg A)]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } A}{[A \mid A]}}{[A, \neg A \mid \emptyset]}\ \neg_{\mathbf{f}}}{[A \mid \neg\neg A]}\ \neg_{\mathbf{t}}}{[\emptyset \mid \neg A, \neg\neg A]}\ \neg_{\mathbf{t}}}{[\emptyset \mid (\neg A \vee \neg\neg A)]}\ \vee_{\mathbf{t}}$$

Proposition 20 *The formula* $((A \to B) \vee (B \to A))$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid ((A \to B) \vee (B \to A))]$$

Derivation of $[\emptyset \mid ((A \to B) \vee (B \to A))]$:

$$\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } A}{[A, B \mid A, B]}}{[A \mid B, (B \to A)]}\ \to_{\mathbf{t}}}{[\emptyset \mid (A \to B), (B \to A)]}\ \to_{\mathbf{t}}}{[\emptyset \mid ((A \to B) \vee (B \to A))]}\ \vee_{\mathbf{t}}$$

Proposition 21 *The formula* $(((A \to B) \to A) \to A)$ *is a tautology.*

The problem is equivalent to proving the following sequent:

$$[\emptyset \mid (((A \to B) \to A) \to A)]$$

Derivation of $[\emptyset \mid (((A \to B) \to A) \to A)]$:

$$\dfrac{\dfrac{\overset{\text{axiom for } A}{[A \mid A]} \qquad \dfrac{\overset{\text{axiom for } A}{[A \mid A, B]}}{[\emptyset \mid A, (A \to B)]}\ \to_{\mathbf{t}}}{[((A \to B) \to A) \mid A]}\ \to_{\mathbf{f}}}{[\emptyset \mid (((A \to B) \to A) \to A)]}\ \to_{\mathbf{t}}$$

3 Some popular consequences

Proposition 22 *The following consequence holds:*

$$A, (A \to B) \vdash B$$

The problem is equivalent to proving the following sequent:

$$[A, (A \to B) \mid B]$$

Derivation of $[A, (A \to B) \mid B]$:

$$\dfrac{\overset{\text{axiom for } B}{[A, B \mid B]} \qquad \overset{\text{axiom for } A}{[A \mid A, B]}}{[A, (A \to B) \mid B]}\ \to_{\mathbf{f}}$$

Proposition 23 *The following consequence holds:*

$$(A \to B), \neg B \vdash \neg A$$

The problem is equivalent to proving the following sequent:

$$[\neg B, (A \to B) \mid \neg A]$$

Derivation of $[\neg B, (A \to B) \mid \neg A]$:

$$\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } B}{[A, B \mid B]} \qquad \overset{\text{axiom for } A}{[A \mid A, B]}}{[A, (A \to B) \mid B]} \to_{\mathbf{f}}}{[A, \neg B, (A \to B) \mid \emptyset]} \neg_{\mathbf{f}}}{[\neg B, (A \to B) \mid \neg A]} \neg_{\mathbf{t}}$$

Proposition 24 *The following consequence holds:*

$$(A \to B), (B \to C) \vdash (A \to C)$$

The problem is equivalent to proving the following sequent:

$$[(A \to B), (B \to C) \mid (A \to C)]$$

Derivation of $[(A \to B), (B \to C) \mid (A \to C)]$:

$$\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } C}{[A, B, C \mid C]} \quad \overset{\text{axiom for } B}{[A, B \mid B, C]}}{[A, B, (B \to C) \mid C]} \to_{\mathbf{f}}}{[B, (B \to C) \mid (A \to C)]} \to_{\mathbf{t}} \qquad \dfrac{\overset{\text{axiom for } A}{[A, (B \to C) \mid A, C]}}{[(B \to C) \mid A, (A \to C)]} \to_{\mathbf{t}}}{[(A \to B), (B \to C) \mid (A \to C)]} \to_{\mathbf{f}}$$

Proposition 25 *The following consequence holds:*

$$(A \vee B), \neg A \vdash B$$

The problem is equivalent to proving the following sequent:

$$[\neg A, (A \vee B) \mid B]$$

Derivation of $[\neg A, (A \vee B) \mid B]$:

$$\dfrac{\dfrac{\overset{\text{axiom for } B}{[B \mid A, B]} \qquad \overset{\text{axiom for } A}{[A \mid A, B]}}{[(A \vee B) \mid A, B]} \vee_{\mathbf{f}}}{[\neg A, (A \vee B) \mid B]} \neg_{\mathbf{f}}$$

Proposition 26 *The following consequence holds:*

$$(\neg C \vee \neg D), (A \to C), (B \to D) \vdash (\neg A \vee \neg B)$$

The problem is equivalent to proving the following sequent:

$$[(A \to C), (B \to D), (\neg C \vee \neg D) \mid (\neg A \vee \neg B)]$$

Derivation of $[(A \to C), (B \to D), (\neg C \vee \neg D) \mid (\neg A \vee \neg B)]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{axiom for } D}{[A, B, C, D \mid D]}}{[A, B, C, D, \neg D \mid \emptyset]}\neg_{\mathbf{f}} \quad \dfrac{\dfrac{\text{axiom for } C}{[A, B, C, D \mid C]}}{[A, B, C, D, \neg C \mid \emptyset]}\neg_{\mathbf{f}}}{[A, B, C, D, (\neg C \vee \neg D) \mid \emptyset]}\vee_{\mathbf{f}}}{[A, C, D, (\neg C \vee \neg D) \mid \neg B]}\neg_{\mathbf{t}}}{[C, D, (\neg C \vee \neg D) \mid \neg A, \neg B]}\neg_{\mathbf{t}}}{[C, D, (\neg C \vee \neg D) \mid (\neg A \vee \neg B)]}\vee_{\mathbf{t}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\text{axiom for } B}{[A, B, C, (\neg C \vee \neg D) \mid B]}}{[A, C, (\neg C \vee \neg D) \mid B, \neg B]}\neg_{\mathbf{t}}}{[C, (\neg C \vee \neg D) \mid B, \neg A, \neg B]}\neg_{\mathbf{t}}}{[C, (\neg C \vee \neg D) \mid B, (\neg A \vee \neg B)]}\vee_{\mathbf{t}}}{[C, (B \to D), (\neg C \vee \neg D) \mid (\neg A \vee \neg B)]}\to_{\mathbf{f}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\text{axiom for } A}{[A, D, (\neg C \vee \neg D) \mid A, \neg B]}}{[D, (\neg C \vee \neg D) \mid A, \neg A, \neg B]}\neg_{\mathbf{t}}}{[D, (\neg C \vee \neg D) \mid A, (\neg A \vee \neg B)]}\vee_{\mathbf{t}} \quad \cdots}{[(B \to D), (\neg C \vee \neg D)}$$

$$[(A \to C), (B \to D), (\neg C \vee \neg D) \mid (\neg A \vee \neg B)]$$

Proposition 27 *The following consequence holds:*

$$(A \vee B), (A \to C), (B \to D) \vdash (C \vee D)$$

The problem is equivalent to proving the following sequent:

$$[(A \to C), (B \to D), (A \vee B) \mid (C \vee D)]$$

Derivation of $[(A \to C), (B \to D), (A \vee B) \mid (C \vee D)]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{axiom for } C}{[B, C, D \mid C, D]}}{[B, C, D \mid (C \vee D)]}\vee_{\mathbf{t}} \quad \dfrac{\dfrac{\text{axiom for } C}{[A, C, D \mid C, D]}}{[A, C, D \mid (C \vee D)]}\vee_{\mathbf{t}}}{[C, D, (A \vee B) \mid (C \vee D)]}\vee_{\mathbf{f}} \quad \dfrac{\dfrac{\text{axiom for } B}{[B, C \mid B, (C \vee D)]} \quad \dfrac{\dfrac{\text{axiom for } C}{[A, C \mid B, C, D]}}{[A, C \mid B, (C \vee D)]}\vee_{\mathbf{t}}}{[C, (A \vee B) \mid B, (C \vee D)]}\vee_{\mathbf{f}}}{[C, (B \to D), (A \vee B) \mid (C \vee D)]}\to_{\mathbf{f}} \quad \dfrac{\dfrac{\dfrac{\text{axiom for } D}{[B, D \mid A, C, D]}}{[B, D \mid A, (C \vee D)]}\vee_{\mathbf{t}} \quad \dfrac{\text{a}}{[A, D}}{[D, (A \vee B) \mid A, (C \vee}$$

$$[(A \to C), (B \to D), (A \vee B) \mid (C \vee D)]$$

Proposition 28 *The following consequence holds:*

$$(A \to (B \to C)) \vdash ((A \wedge B) \to C)$$

The problem is equivalent to proving the following sequent:

$$[(A \to (B \to C)) \mid ((A \wedge B) \to C)]$$

Derivation of $[(A \to (B \to C)) \mid ((A \wedge B) \to C)]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{axiom for } C}{[C, (A \wedge B) \mid C]}}{[C \mid ((A \wedge B) \to C)]}\to_{\mathbf{t}} \quad \dfrac{\dfrac{\dfrac{\text{axiom for } B}{[A, B \mid B, C]}}{[(A \wedge B) \mid B, C]}\wedge_{\mathbf{f}}}{[\emptyset \mid B, ((A \wedge B) \to C)]}\to_{\mathbf{t}}}{[(B \to C) \mid ((A \wedge B) \to C)]}\to_{\mathbf{f}} \quad \dfrac{\dfrac{\dfrac{\text{axiom for } A}{[A, B \mid A, C]}}{[(A \wedge B) \mid A, C]}\wedge_{\mathbf{f}}}{[\emptyset \mid A, ((A \wedge B) \to C)]}\to_{\mathbf{t}}}{[(A \to (B \to C)) \mid ((A \wedge B) \to C)]}\to_{\mathbf{f}}$$

Proposition 29 *The following consequence holds:*

$$((A \wedge B) \to C) \vdash (A \to (B \to C))$$

The problem is equivalent to proving the following sequent:

$$[((A \wedge B) \to C) \mid (A \to (B \to C))]$$

Derivation of $[((A \wedge B) \to C) \mid (A \to (B \to C))]$:

$$\dfrac{\dfrac{\dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } C}{[A,B,C \mid C]} \quad \dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } B}{[A,B \mid B,C]} \quad \genfrac{}{}{0pt}{}{\text{axiom for } A}{[A,B \mid A,C]}}{[A,B \mid C,(A \wedge B)]}\,\wedge_{\mathbf{t}}}{[A,B,((A \wedge B) \to C) \mid C]}\,\to_{\mathbf{f}}}{[A,((A \wedge B) \to C) \mid (B \to C)]}\,\to_{\mathbf{t}}}{[((A \wedge B) \to C) \mid (A \to (B \to C))]}\,\to_{\mathbf{t}}$$

4 Some popular equivalences

Proposition 30 *The formulas* $((B \vee C) \wedge A)$ *and* $((B \wedge A) \vee (C \wedge A))$ *are equivalent.*

The problem is equivalent to proving the following sequents:

$$[((B \vee C) \wedge A) \mid ((B \wedge A) \vee (C \wedge A))]$$
$$[((B \wedge A) \vee (C \wedge A)) \mid ((B \vee C) \wedge A)]$$

Derivation of $[((B \vee C) \wedge A) \mid ((B \wedge A) \vee (C \wedge A))]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } A}{[A,C \mid A,(C \wedge A)]} \quad \dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } A}{[A,C \mid A,B]} \quad \genfrac{}{}{0pt}{}{\text{axiom for } C}{[A,C \mid B,C]}}{[A,C \mid B,(C \wedge A)]}\,\wedge_{\mathbf{t}}}{[A,C \mid (B \wedge A),(C \wedge A)]}\,\wedge_{\mathbf{t}}}{[A,C \mid ((B \wedge A) \vee (C \wedge A))]}\,\vee_{\mathbf{t}} \quad \dfrac{\dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } A}{[A,B \mid A,(C \wedge A)]} \quad \genfrac{}{}{0pt}{}{\text{axiom for } B}{[A,B \mid B,(C \wedge A)]}}{[A,B \mid (B \wedge A),(C \wedge A)]}\,\wedge_{\mathbf{t}}}{[A,B \mid ((B \wedge A) \vee (C \wedge A))]}\,\vee_{\mathbf{t}}}{[A,(B \vee C) \mid ((B \wedge A) \vee (C \wedge A))]}\,\vee_{\mathbf{f}}}{[((B \vee C) \wedge A) \mid ((B \wedge A) \vee (C \wedge A))]}\,\wedge_{\mathbf{f}}$$

Derivation of $[((B \wedge A) \vee (C \wedge A)) \mid ((B \vee C) \wedge A)]$:

$$\dfrac{\dfrac{\dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } A}{[A,C \mid A]}}{[(C \wedge A) \mid A]}\,\wedge_{\mathbf{f}} \quad \dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } A}{[A,B \mid A]}}{[(B \wedge A) \mid A]}\,\wedge_{\mathbf{f}}}{[((B \wedge A) \vee (C \wedge A)) \mid A]}\,\vee_{\mathbf{f}} \quad \dfrac{\dfrac{\dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } C}{[A,C \mid B,C]}}{[(C \wedge A) \mid B,C]}\,\wedge_{\mathbf{f}} \quad \dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } B}{[A,B \mid B,C]}}{[(B \wedge A) \mid B,C]}\,\wedge_{\mathbf{f}}}{[((B \wedge A) \vee (C \wedge A)) \mid B,C]}\,\vee_{\mathbf{f}}}{[((B \wedge A) \vee (C \wedge A)) \mid (B \vee C)]}\,\vee_{\mathbf{t}}}{[((B \wedge A) \vee (C \wedge A)) \mid ((B \vee C) \wedge A)]}\,\wedge_{\mathbf{t}}$$

Proposition 31 *The formulas* $(A \wedge (B \vee C))$ *and* $((A \wedge B) \vee (A \wedge C))$ *are equivalent.*

The problem is equivalent to proving the following sequents:

$$[(A \wedge (B \vee C)) \mid ((A \wedge B) \vee (A \wedge C))]$$
$$[((A \wedge B) \vee (A \wedge C)) \mid (A \wedge (B \vee C))]$$

Derivation of $[(A \wedge (B \vee C)) \mid ((A \wedge B) \vee (A \wedge C))]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } C}{[A,C \mid B,C]} \quad \genfrac{}{}{0pt}{}{\text{axiom for } A}{[A,C \mid A,B]}}{[A,C \mid B,(A \wedge C)]}\,\wedge_{\mathbf{t}} \quad \genfrac{}{}{0pt}{}{\text{axiom for } A}{[A,C \mid A,(A \wedge C)]}}{[A,C \mid (A \wedge B),(A \wedge C)]}\,\wedge_{\mathbf{t}}}{[A,C \mid ((A \wedge B) \vee (A \wedge C))]}\,\vee_{\mathbf{t}} \quad \dfrac{\dfrac{\genfrac{}{}{0pt}{}{\text{axiom for } B}{[A,B \mid B,(A \wedge C)]} \quad \genfrac{}{}{0pt}{}{\text{axiom for } A}{[A,B \mid A,(A \wedge C)]}}{[A,B \mid (A \wedge B),(A \wedge C)]}\,\wedge_{\mathbf{t}}}{[A,B \mid ((A \wedge B) \vee (A \wedge C))]}\,\vee_{\mathbf{t}}}{[A,(B \vee C) \mid ((A \wedge B) \vee (A \wedge C))]}\,\vee_{\mathbf{f}}}{[(A \wedge (B \vee C)) \mid ((A \wedge B) \vee (A \wedge C))]}\,\wedge_{\mathbf{f}}$$

Derivation of $[((A \wedge B) \vee (A \wedge C)) \mid (A \wedge (B \vee C))]$:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\begin{array}{c}\text{axiom for } C\\ [A, C \mid B, C]\end{array}}{[(A \wedge C) \mid B, C]}\,\wedge_{\mathbf{f}}
\quad
\dfrac{\begin{array}{c}\text{axiom for } B\\ [A, B \mid B, C]\end{array}}{[(A \wedge B) \mid B, C]}\,\wedge_{\mathbf{f}}
}{[((A \wedge B) \vee (A \wedge C)) \mid B, C]}\,\vee_{\mathbf{f}}
}{[((A \wedge B) \vee (A \wedge C)) \mid (B \vee C)]}\,\vee_{\mathbf{t}}
\quad
\dfrac{
\dfrac{\begin{array}{c}\text{axiom for } A\\ [A, C \mid A]\end{array}}{[(A \wedge C) \mid A]}\,\wedge_{\mathbf{f}}
\quad
\dfrac{\begin{array}{c}\text{axiom for } A\\ [A, B \mid A]\end{array}}{[(A \wedge B) \mid A]}\,\wedge_{\mathbf{f}}
}{[((A \wedge B) \vee (A \wedge C)) \mid A]}\,\vee_{\mathbf{f}}
}{[((A \wedge B) \vee (A \wedge C)) \mid (A \wedge (B \vee C))]}\,\wedge_{\mathbf{t}}
$$

Proposition 32 *The formulas* $((B \wedge C) \vee A)$ *and* $((B \vee A) \wedge (C \vee A))$ *are equivalent.*

The problem is equivalent to proving the following sequents:

$$[((B \wedge C) \vee A) \mid ((B \vee A) \wedge (C \vee A))]$$
$$[((B \vee A) \wedge (C \vee A)) \mid ((B \wedge C) \vee A)]$$

Derivation of $[((B \wedge C) \vee A) \mid ((B \vee A) \wedge (C \vee A))]$:

$$
\dfrac{
\dfrac{
\dfrac{
\begin{array}{c}\text{axiom for } A\\ [A \mid A, C]\end{array}
\quad
\dfrac{\begin{array}{c}\text{axiom for } C\\ [B, C \mid A, C]\end{array}}{[(B \wedge C) \mid A, C]}\,\wedge_{\mathbf{f}}
}{[((B \wedge C) \vee A) \mid A, C]}\,\vee_{\mathbf{f}}
}{[((B \wedge C) \vee A) \mid (C \vee A)]}\,\vee_{\mathbf{t}}
\quad
\dfrac{
\dfrac{
\begin{array}{c}\text{axiom for } A\\ [A \mid A, B]\end{array}
\quad
\dfrac{\begin{array}{c}\text{axiom for } B\\ [B, C \mid A, B]\end{array}}{[(B \wedge C) \mid A, B]}\,\wedge_{\mathbf{f}}
}{[((B \wedge C) \vee A) \mid A, B]}\,\vee_{\mathbf{f}}
}{[((B \wedge C) \vee A) \mid (B \vee A)]}\,\vee_{\mathbf{t}}
}{[((B \wedge C) \vee A) \mid ((B \vee A) \wedge (C \vee A))]}\,\wedge_{\mathbf{t}}
$$

Derivation of $[((B \vee A) \wedge (C \vee A)) \mid ((B \wedge C) \vee A)]$:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\begin{array}{c}\text{axiom for } A\\ [A \mid A, (B \wedge C)]\end{array}}{[A \mid ((B \wedge C) \vee A)]}\,\vee_{\mathbf{t}}
\quad
\dfrac{\begin{array}{c}\text{axiom for } A\\ [A, C \mid A, (B \wedge C)]\end{array}}{[A, C \mid ((B \wedge C) \vee A)]}\,\vee_{\mathbf{t}}
}{[A, (C \vee A) \mid ((B \wedge C) \vee A)]}\,\vee_{\mathbf{f}}
\quad
\dfrac{
\dfrac{\begin{array}{c}\text{axiom for } A\\ [A, B \mid A, (B \wedge C)]\end{array}}{[A, B \mid ((B \wedge C) \vee A)]}\,\vee_{\mathbf{t}}
\quad
\dfrac{
\dfrac{\begin{array}{c}\text{axiom for } C\\ [B, C \mid A, C]\end{array}\quad\begin{array}{c}\text{axiom for } B\\ [B, C \mid A, B]\end{array}}{[B, C \mid A, (B \wedge C)]}\,\wedge_{\mathbf{t}}
}{[B, C \mid ((B \wedge C) \vee A)]}\,\vee_{\mathbf{t}}
}{[B, (C \vee A) \mid ((B \wedge C) \vee A)]}\,\vee_{\mathbf{f}}
}{[(B \vee A), (C \vee A) \mid ((B \wedge C) \vee A)]}\,\vee_{\mathbf{f}}
}{[((B \vee A) \wedge (C \vee A)) \mid ((B \wedge C) \vee A)]}\,\wedge_{\mathbf{f}}
$$

Proposition 33 *The formulas* $(A \vee (B \wedge C))$ *and* $((A \vee B) \wedge (A \vee C))$ *are equivalent.*

The problem is equivalent to proving the following sequents:

$$[(A \vee (B \wedge C)) \mid ((A \vee B) \wedge (A \vee C))]$$
$$[((A \vee B) \wedge (A \vee C)) \mid (A \vee (B \wedge C))]$$

Derivation of $[(A \vee (B \wedge C)) \mid ((A \vee B) \wedge (A \vee C))]$:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\begin{array}{c}\text{axiom for } C\\ [B, C \mid A, C]\end{array}}{[(B \wedge C) \mid A, C]}\,\wedge_{\mathbf{f}}
\quad
\begin{array}{c}\text{axiom for } A\\ [A \mid A, C]\end{array}
}{[(A \vee (B \wedge C)) \mid A, C]}\,\vee_{\mathbf{f}}
}{[(A \vee (B \wedge C)) \mid (A \vee C)]}\,\vee_{\mathbf{t}}
\quad
\dfrac{
\dfrac{
\dfrac{\begin{array}{c}\text{axiom for } B\\ [B, C \mid A, B]\end{array}}{[(B \wedge C) \mid A, B]}\,\wedge_{\mathbf{f}}
\quad
\begin{array}{c}\text{axiom for } A\\ [A \mid A, B]\end{array}
}{[(A \vee (B \wedge C)) \mid A, B]}\,\vee_{\mathbf{f}}
}{[(A \vee (B \wedge C)) \mid (A \vee B)]}\,\vee_{\mathbf{t}}
}{[(A \vee (B \wedge C)) \mid ((A \vee B) \wedge (A \vee C))]}\,\wedge_{\mathbf{t}}
$$

Derivation of $[((A \vee B) \wedge (A \vee C)) \mid (A \vee (B \wedge C))]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{axiom for } C}{[B, C \mid A, C]} \quad \dfrac{\text{axiom for } B}{[B, C \mid A, B]}}{[B, C \mid A, (B \wedge C)]}\,\wedge_{\mathbf{t}}}{[B, C \mid (A \vee (B \wedge C))]}\,\vee_{\mathbf{t}} \quad \dfrac{\dfrac{\text{axiom for } A}{[A, B \mid A, (B \wedge C)]}}{[A, B \mid (A \vee (B \wedge C))]}\,\vee_{\mathbf{t}}}{[B, (A \vee C) \mid (A \vee (B \wedge C))]}\,\vee_{\mathbf{f}} \quad \dfrac{\dfrac{\dfrac{\text{axiom for } A}{[A, C \mid A, (B \wedge C)]}}{[A, C \mid (A \vee (B \wedge C))]}\,\vee_{\mathbf{t}} \quad \dfrac{\dfrac{\text{axiom for } A}{[A \mid A, (B \wedge C)]}}{[A \mid (A \vee (B \wedge C))]}\,\vee_{\mathbf{t}}}{[A, (A \vee C) \mid (A \vee (B \wedge C))]}\,\vee_{\mathbf{f}}}{\dfrac{[(A \vee B), (A \vee C) \mid (A \vee (B \wedge C))]}{[((A \vee B) \wedge (A \vee C)) \mid (A \vee (B \wedge C))]}\,\wedge_{\mathbf{f}}}\,\vee_{\mathbf{f}}$$

5 Interdefinability of connectives

Proposition 34 *The equality* $(A \rightarrow B) = (\neg A \vee B)$ *holds.*

The problem is equivalent to proving the following sequents:

$$[(A \rightarrow B) \mid (\neg A \vee B)]$$
$$[(\neg A \vee B) \mid (A \rightarrow B)]$$

Derivation of $[(A \rightarrow B) \mid (\neg A \vee B)]$:

$$\dfrac{\dfrac{\dfrac{\text{axiom for } B}{[B \mid B, \neg A]}}{[B \mid (\neg A \vee B)]}\,\vee_{\mathbf{t}} \quad \dfrac{\dfrac{\dfrac{\text{axiom for } A}{[A \mid A, B]}}{[\emptyset \mid A, B, \neg A]}\,\neg_{\mathbf{t}}}{[\emptyset \mid A, (\neg A \vee B)]}\,\vee_{\mathbf{t}}}{[(A \rightarrow B) \mid (\neg A \vee B)]}\,\rightarrow_{\mathbf{f}}$$

Derivation of $[(\neg A \vee B) \mid (A \rightarrow B)]$:

$$\dfrac{\dfrac{\dfrac{\text{axiom for } B}{[A, B \mid B]} \quad \dfrac{\dfrac{\text{axiom for } A}{[A \mid A, B]}}{[A, \neg A \mid B]}\,\neg_{\mathbf{f}}}{[A, (\neg A \vee B) \mid B]}\,\vee_{\mathbf{f}}}{[(\neg A \vee B) \mid (A \rightarrow B)]}\,\rightarrow_{\mathbf{t}}$$

Proposition 35 *The equality* $(A \rightarrow B) = \neg(A \wedge \neg B)$ *holds.*

The problem is equivalent to proving the following sequents:

$$[(A \rightarrow B) \mid \neg(A \wedge \neg B)]$$
$$[\neg(A \wedge \neg B) \mid (A \rightarrow B)]$$

Derivation of $[(A \rightarrow B) \mid \neg(A \wedge \neg B)]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{axiom for } B}{[A, B \mid B]} \quad \dfrac{\text{axiom for } A}{[A \mid A, B]}}{[A, (A \rightarrow B) \mid B]}\,\rightarrow_{\mathbf{f}}}{[A, \neg B, (A \rightarrow B) \mid \emptyset]}\,\neg_{\mathbf{f}}}{[(A \wedge \neg B), (A \rightarrow B) \mid \emptyset]}\,\wedge_{\mathbf{f}}}{[(A \rightarrow B) \mid \neg(A \wedge \neg B)]}\,\neg_{\mathbf{t}}$$

Derivation of $[\neg(A \wedge \neg B) \mid (A \to B)]$:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{axiom for } B\\ [A, B \mid B]\end{array}}{[B \mid (A \to B)]}\to_{\mathbf{t}}}{[\emptyset \mid \neg B, (A \to B)]}\neg_{\mathbf{t}} \quad \dfrac{\begin{array}{c}\text{axiom for } A\\ [A \mid A, B]\end{array}}{[\emptyset \mid A, (A \to B)]}\to_{\mathbf{t}}}{\dfrac{[\emptyset \mid (A \wedge \neg B), (A \to B)]}{[\neg(A \wedge \neg B) \mid (A \to B)]}\neg_{\mathbf{f}}}\wedge_{\mathbf{t}}$$

Proposition 36 *The equality* $(A \vee B) = ((A \to B) \to B)$ *holds.*

The problem is equivalent to proving the following sequents:

$$[(A \vee B) \mid ((A \to B) \to B)]$$
$$[((A \to B) \to B) \mid (A \vee B)]$$

Derivation of $[(A \vee B) \mid ((A \to B) \to B)]$:

$$\dfrac{\dfrac{\begin{array}{c}\text{axiom for } B\\ [B, (A \vee B) \mid B]\end{array} \quad \dfrac{\begin{array}{c}\text{axiom for } B\\ [B \mid A, B]\end{array} \quad \begin{array}{c}\text{axiom for } A\\ [A \mid A, B]\end{array}}{[(A \vee B) \mid A, B]}\vee_{\mathbf{f}}}{[(A \to B), (A \vee B) \mid B]}\to_{\mathbf{f}}}{[(A \vee B) \mid ((A \to B) \to B)]}\to_{\mathbf{t}}$$

Derivation of $[((A \to B) \to B) \mid (A \vee B)]$:

$$\dfrac{\dfrac{\begin{array}{c}\text{axiom for } B\\ [B \mid A, B]\end{array}}{[B \mid (A \vee B)]}\vee_{\mathbf{t}} \quad \dfrac{\dfrac{\begin{array}{c}\text{axiom for } A\\ [A \mid A, B]\end{array}}{[A \mid B, (A \vee B)]}\vee_{\mathbf{t}}}{[\emptyset \mid (A \to B), (A \vee B)]}\to_{\mathbf{t}}}{[((A \to B) \to B) \mid (A \vee B)]}\to_{\mathbf{f}}$$

Proposition 37 *The equality* $(A \vee B) = \neg(\neg A \wedge \neg B)$ *holds.*

The problem is equivalent to proving the following sequents:

$$[(A \vee B) \mid \neg(\neg A \wedge \neg B)]$$
$$[\neg(\neg A \wedge \neg B) \mid (A \vee B)]$$

Derivation of $[(A \vee B) \mid \neg(\neg A \wedge \neg B)]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{axiom for } B\\ [B \mid A, B]\end{array} \quad \begin{array}{c}\text{axiom for } A\\ [A \mid A, B]\end{array}}{[(A \vee B) \mid A, B]}\vee_{\mathbf{f}}}{[\neg B, (A \vee B) \mid A]}\neg_{\mathbf{f}}}{[\neg A, \neg B, (A \vee B) \mid \emptyset]}\neg_{\mathbf{f}}}{[(\neg A \wedge \neg B), (A \vee B) \mid \emptyset]}\wedge_{\mathbf{f}}}{[(A \vee B) \mid \neg(\neg A \wedge \neg B)]}\neg_{\mathbf{t}}$$

Derivation of $[\neg(\neg A \wedge \neg B) \mid (A \vee B)]$:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{axiom for } B\\ [B \mid A, B]\end{array}}{[B \mid (A \vee B)]}\vee_{\mathbf{t}}}{[\emptyset \mid \neg B, (A \vee B)]}\neg_{\mathbf{t}} \quad \dfrac{\dfrac{\begin{array}{c}\text{axiom for } A\\ [A \mid A, B]\end{array}}{[A \mid (A \vee B)]}\vee_{\mathbf{t}}}{[\emptyset \mid \neg A, (A \vee B)]}\neg_{\mathbf{t}}}{\dfrac{[\emptyset \mid (\neg A \wedge \neg B), (A \vee B)]}{[\neg(\neg A \wedge \neg B) \mid (A \vee B)]}\neg_{\mathbf{f}}}\wedge_{\mathbf{t}}$$

Proposition 38 *The equality* $(A \wedge B) = \neg(A \to \neg B)$ *holds.*

The problem is equivalent to proving the following sequents:

$$[(A \wedge B) \mid \neg(A \to \neg B)]$$
$$[\neg(A \to \neg B) \mid (A \wedge B)]$$

Derivation of $[(A \wedge B) \mid \neg(A \to \neg B)]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } B}{[A, B \mid B]}}{[A, B, \neg B \mid \emptyset]}\,\neg_{\mathbf{f}} \qquad \overset{\text{axiom for } A}{[A, B \mid A]}}{[A, B, (A \to \neg B) \mid \emptyset]}\,\to_{\mathbf{f}}}{[(A \wedge B), (A \to \neg B) \mid \emptyset]}\,\wedge_{\mathbf{f}}}{[(A \wedge B) \mid \neg(A \to \neg B)]}\,\neg_{\mathbf{t}}$$

Derivation of $[\neg(A \to \neg B) \mid (A \wedge B)]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } B}{[A, B \mid B]}}{[A \mid B, \neg B]}\,\neg_{\mathbf{t}}}{[\emptyset \mid B, (A \to \neg B)]}\,\to_{\mathbf{t}} \qquad \dfrac{\overset{\text{axiom for } A}{[A \mid A, \neg B]}}{[\emptyset \mid A, (A \to \neg B)]}\,\to_{\mathbf{t}}}{[\emptyset \mid (A \wedge B), (A \to \neg B)]}\,\wedge_{\mathbf{t}}}{[\neg(A \to \neg B) \mid (A \wedge B)]}\,\neg_{\mathbf{f}}$$

Proposition 39 *The equality* $(A \vee B) = \neg(\neg A \wedge \neg B)$ *holds.*

The problem is equivalent to proving the following sequents:

$$[(A \vee B) \mid \neg(\neg A \wedge \neg B)]$$
$$[\neg(\neg A \wedge \neg B) \mid (A \vee B)]$$

Derivation of $[(A \vee B) \mid \neg(\neg A \wedge \neg B)]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } B}{[B \mid A, B]} \qquad \overset{\text{axiom for } A}{[A \mid A, B]}}{[(A \vee B) \mid A, B]}\,\vee_{\mathbf{f}}}{[\neg B, (A \vee B) \mid A]}\,\neg_{\mathbf{f}}}{[\neg A, \neg B, (A \vee B) \mid \emptyset]}\,\neg_{\mathbf{f}}}{[(\neg A \wedge \neg B), (A \vee B) \mid \emptyset]}\,\wedge_{\mathbf{f}}}{[(A \vee B) \mid \neg(\neg A \wedge \neg B)]}\,\neg_{\mathbf{t}}$$

Derivation of $[\neg(\neg A \wedge \neg B) \mid (A \vee B)]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\text{axiom for } B}{[B \mid A, B]}}{[B \mid (A \vee B)]}\,\vee_{\mathbf{t}}}{[\emptyset \mid \neg B, (A \vee B)]}\,\neg_{\mathbf{t}} \qquad \dfrac{\dfrac{\overset{\text{axiom for } A}{[A \mid A, B]}}{[A \mid (A \vee B)]}\,\vee_{\mathbf{t}}}{[\emptyset \mid \neg A, (A \vee B)]}\,\neg_{\mathbf{t}}}{[\emptyset \mid (\neg A \wedge \neg B), (A \vee B)]}\,\wedge_{\mathbf{t}}}{[\neg(\neg A \wedge \neg B) \mid (A \vee B)]}\,\neg_{\mathbf{f}}$$

6 Metaconsequences

Proposition 40 *The following meta-consequence holds:*

$$P, Q \vdash R \quad / \quad P \vdash (Q \to R)$$

The problem is equivalent to proving the following sequents:

$$[P \mid Q, (Q \to R)]$$
$$[P, R \mid (Q \to R)]$$

Derivation of $[P \mid Q, (Q \to R)]$:

$$\dfrac{\dfrac{\text{axiom for } Q}{[P, Q \mid Q, R]}}{[P \mid Q, (Q \to R)]} \to_{\mathbf{t}}$$

Derivation of $[P, R \mid (Q \to R)]$:

$$\dfrac{\dfrac{\text{axiom for } R}{[P, Q, R \mid R]}}{[P, R \mid (Q \to R)]} \to_{\mathbf{t}}$$

Proposition 41 *The following meta-consequence holds:*

$$(P \wedge Q) \vdash R \quad / \quad P \vdash (Q \to R)$$

The problem is equivalent to proving the following sequents:

$$[P, R \mid (Q \to R)]$$
$$[P \mid (P \wedge Q), (Q \to R)]$$

Derivation of $[P, R \mid (Q \to R)]$:

$$\dfrac{\dfrac{\text{axiom for } R}{[P, Q, R \mid R]}}{[P, R \mid (Q \to R)]} \to_{\mathbf{t}}$$

Derivation of $[P \mid (P \wedge Q), (Q \to R)]$:

$$\dfrac{\dfrac{\dfrac{\text{axiom for } Q}{[P, Q \mid Q, R]}}{[P \mid Q, (Q \to R)]} \to_{\mathbf{t}} \qquad \dfrac{\text{axiom for } P}{[P \mid P, (Q \to R)]}}{[P \mid (P \wedge Q), (Q \to R)]} \wedge_{\mathbf{t}}$$

7 Program listing: `ex_classical.pl`

```
% Test file to check things in classical logic

% make sure MUltseq is loaded
:- ensure_loaded('../multseq/multseq').

% load sample properties
:- [properties].

% load the rules
:- load_logic('classical.msq').

% define standard Omap
:- setOmap([(neg)/(-),imp/(>),and/(/\),or/(\/),equiv/(=)]).
```

```
% check all properties and write report to out.tex

:- set_option(tex_output(verbose)).
:- set_option(tex_sequents(multidimensional)).
:- set_option(tex_rulenames(on)).

:- start_logging(ex_classical,'.tex').

:- print_tex(tex_title("Report␣on␣Classical␣Logic")).

:- print_tex(tex_logic).

:- print_tex(tex_paragraph(["We␣verify␣that␣all␣classical␣logic␣satisfies
␣␣␣␣␣␣some␣well-known␣properties␣(involving␣only␣$\\land$,␣$\\lor$,
␣␣␣␣␣␣$\\to$,␣$\\neg$,␣and␣$\\leftrightarrow$).␣We␣output␣proofs␣in
␣␣␣␣␣␣``multidimensional''␣format,␣which␣for␣classical␣logic␣means␣just
␣␣␣␣␣␣two␣sides␣to␣a␣sequent,
␣␣␣␣␣␣as␣usual."])).

:- print_tex(tex_section(["Bernays's␣axioms␣for␣classical␣logic"])).

:- (member(X,[bernays1,bernays2,bernays3,bernays4,bernays5,bernays6,bernays7,bernays8,
    bernays9,bernays10,bernays11,bernays12,bernays13,bernays14,bernays15,bernays16,bernays17
    ]), chkProp(X), fail; true).

:- print_tex(tex_section(["Classical␣tautologies␣not␣intuitionistically␣valid"])).

:- (member(X,[lem,weaklem,prelinearity,peirce]), chkProp(X), fail; true).

:- print_tex(tex_section(["Some␣popular␣consequences"])).

:- (member(X,[modusponens,modustollens,hyposyllogism,disjsyllogism,destrdilemma,
    constrdilemma,importation,exportation]), chkProp(X), fail; true).

:- print_tex(tex_section(["Some␣popular␣equivalences"])).

:- (member(X,[ldistrright,ldistrleft]), chkProp(X), fail; true).
% Here we switch and and or
:- (member(X,[ldistrright,ldistrleft]), chkProp([or/(/\),and/(\/)],X), fail; true).

:- print_tex(tex_section(["Interdefinability␣of␣connectives"])).

:- (member(X,[defimpor,defimpand,deforimp,deforand,defandimp,deforand]), chkProp(X), fail;
    true).

:- print_tex(tex_section(["Metaconsequences"])).

:- (member(X,[deductionthm,residuation]), chkProp(X), fail; true).

:- print_tex(tex_listing("ex_classical.pl")).

:- stop_logging.
```