Graphical Construction of Cut-free Sequent Systems

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#### Brief Recap: Modal Logics

We are interested in modal logics in the broad sense. Ingredients:

Formulae, e.g.

$$A_1,\ldots,A_n \ni \mathcal{F} ::= \begin{array}{c} p \mid \neg A_1 \mid A_1 \land A_2 \mid A_1 \lor A_2 \mid A_1 \to A_2 \\ \mid \Box A_1 \mid \Diamond A_1 \mid A_1 \preccurlyeq A_2 \mid \heartsuit(A_1,\ldots,A_n) \mid \ldots \end{array}$$

- Semantics: e.g. Kripke or sphere or coalgebraic semantics (we don't care about that now)
- A sound and complete proof system to produce the theorems of the logic (we do care about that now)

#### Question

Which formulae are theorems of a given modal logic? How do we find out?

The Proof System: Sequents and Sequent Rules

We use sequents  $\Gamma \vdash \Delta$  with  $\Gamma, \Delta$  multisets of formulae. (Read as the formula  $\bigwedge \Gamma \rightarrow \bigvee \Delta$ .)

Our sequent systems have axioms  $\overline{\Gamma, A \vdash A, \Delta}$  and the rules G3*cp* of classical propositional logic:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \land B \vdash \Delta} \land_{L}, \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \land B} \land_{R}, \frac{\Gamma \vdash \Delta, A}{\Gamma, \neg A \vdash \Delta} \neg_{L}$$
$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \lor B \vdash \Delta} \lor_{L}, \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \lor B} \land_{L}, \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \neg_{R}$$
$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \operatorname{con}_{L}, \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A} \operatorname{con}_{R}, \frac{\Gamma \vdash \Delta, A \land A, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi} \operatorname{cut}$$

#### Shallow Rules

In addition to the rules of propositional logic our systems have shallow rules, i.e. rules of the form

$$\frac{\Gamma, \Gamma_1 \vdash \Delta, \Delta_1 \quad \dots \quad \Gamma, \Gamma_\ell \vdash \Delta, \Delta_\ell \quad \Sigma_1 \vdash \Pi_1 \quad \dots \quad \Sigma_k \vdash \Pi_k}{\Gamma, \heartsuit_1(\vec{p}), \dots, \heartsuit_n(\vec{p}) \vdash \Delta, \heartsuit_1'(\vec{p}), \dots, \heartsuit_m'(\vec{p})}$$

with contextual premisses  $\Gamma_i \vdash \Delta_i$  and non-contextual premisses  $\Sigma_j \vdash \Pi_j$  consisting of variables in  $\vec{p}$ .

Examples

$$\frac{p_1, \dots, p_n \vdash q}{\Gamma, \Box p_1, \dots, \Box p_n \vdash \Box q, \Delta} \ \mathsf{K}_n \ , \quad \frac{\Gamma, p \vdash \Delta}{\Gamma, \Box p \vdash \Delta} \ \mathsf{T}$$

#### Theorem

Every modal logic given by a finite set of axioms without nested modalities in a Hilbert system has a corresponding (sound and complete) sequent system of shallow rules.

#### The Cut Rule And Cut Elimination

Unfortunately, the systems rely on the cut rule for completeness:

$$\frac{\Gamma\vdash\Delta,A\quad A,\Sigma\vdash\Pi}{\Gamma,\Sigma\vdash\Delta,\Pi} \text{ cut }$$

We would like to get rid of the cut rule since derivations with cuts are not analytic and we have to "invent" the cut formula in backwards proof search.

Standard procedure for cut elimination: permute the cuts up until they fall off the leaves. E.g.:



But: We need enough rules to be able to do so!

## Saturated Rule Sets and Generic Cut Elimination

#### A set ${\mathcal R}$ of shallow rules is

- cut closed if cuts between principal formulae of rules in R can be replaced by cuts on their premisses and an application of a rule in R
- contraction closed if contractions of principal formulae of rules in R can be replaced by contractions of their premisses and an application of a rule in R
- saturated if it is cut and contraction closed.

#### Theorem

If  $\mathcal{R}$  is saturated, then we have cut elimination for  $G\mathcal{R}$ .

#### Theorem

If  ${\mathcal R}$  is saturated, then derivability in  ${\sf G}{\mathcal R}$  is decidable in polynomial space.

Problem: How to construct saturated rule sets? Idea: Saturate the rule set under cuts between rules: Slogan: "Cut the conclusion, cut the premisses". E.g. for *K*:

$$\begin{array}{c} A_1, A_2 \vdash B \\ \hline \Box A_1, \Box A_2 \vdash \Box B \end{array}$$

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$$\begin{array}{c} A_1, A_2 \vdash B \\ \hline \Box A_1, \Box A_2 \vdash \Box B \end{array} \quad \begin{array}{c} B, A_3 \vdash C \\ \hline \Box B, \Box A_3 \vdash \Box C \end{array}$$

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 $\begin{array}{c|c} A_1, A_2, A_3 \vdash \mathcal{C} \\ \hline A_1, A_2 \vdash \mathcal{B} & B, A_3 \vdash \mathcal{C} \\ \hline \Box A_1, \Box A_2 \vdash \Box \mathcal{B} & \Box \mathcal{B}, \Box A_3 \vdash \Box \mathcal{C} \\ \hline \Box A_1, \Box A_2, \Box A_3 \vdash \Box \mathcal{C} \end{array}$ 

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(And similarly for contraction.)

Continuing like this we see the rule pattern  $\frac{A_1, \ldots, A_n \vdash B}{\Box A_1, \ldots, \Box A_n \vdash \Box B}$ .

Easy enough for K, but cutting rules and spotting patterns quickly becomes complicated for other rules.

Can we represent sequents and sequent rules more intuitively?

▶ Represent sequents by doodles: Take ⊢ as an arrow with multiple heads and tails:

$$A, B, C \vdash D, D$$

$$\begin{array}{ccc} A \vdash B \\ \overline{\Gamma, \Box A \vdash \Box B, \Delta} & K_1 \\ \downarrow \Box & B \\ \downarrow \Gamma, \Delta \\ \downarrow \Gamma, \Delta \end{array}$$
Thus cut 
$$\begin{array}{ccc} \Gamma \vdash \Delta, A & A, \Sigma \vdash \Pi \\ \overline{\Gamma, \Sigma \vdash \Delta, \Pi} & \text{becomes} \end{array} \begin{array}{c} & \Gamma, \Delta \\ A \\ \downarrow \\ \Sigma, \Pi \end{array}$$

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$$\begin{array}{ccc} A \vdash B \\ \overline{\Gamma, \Box A \vdash \Box B, \Delta} & K_1 & \begin{array}{c} \Box & A \\ \overline{\Gamma, \Box A \vdash \Box B, \Delta} & K_1 \\ \overline{\Gamma, \Delta} & \begin{array}{c} \Box & B \\ \overline{\Gamma, \Delta} & \end{array} \end{array}$$
Thus cut  $\begin{array}{c} \Gamma \vdash \Delta, A & A, \Sigma \vdash \Pi \\ \overline{\Gamma, \Sigma \vdash \Delta, \Pi} & \text{becomes} \end{array} \begin{bmatrix} \Gamma, \Delta \\ A & \\ \overline{\Sigma, \Pi} & \end{array}$ 
Slogan: "Connect heads and tails and yank the wire!"

Consider conditional logic V with the entrenchment connective  $\preccurlyeq$  (read  $A \preccurlyeq B$  as: "A is at least as possible as B").

Translating the axioms into rules and saturating under cuts and contractions of rules yields a saturated rule set. A typical step:



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Some doodles are still a bit hard to read:

#### So we make them 3D:





#### Theorem.

The sequent system  $G\mathcal{R}_V$  for the conditional logic V with  $R_{n,m}$  shown on the right for  $n \ge 1$ ,  $m \ge 0$  has cut elimination and allows backwards proof search in polynomial space.



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## Summing Up

If you want to construct a cut-free sequent system for a non-iterative modal logic

- translate the axioms into shallow rules
- represent the rules as rule doodles
- start doodling and spot the pattern
- and get PSPACE for free!

# Thanks!