A Dichotomy Theorem for the Classes $\mathrm{W[P]}(\mathcal{C})$

Björn Lellmann

Department of Computing – Imperial College London

ICCSW 2011

Motivation

Problem

Many NP*-hard problems are solved efficiently in practice.*

Ways of dealing with this:

- Average case complexity
- *Parameterised complexity*.

Parameterised Problems

Fix a finite alphabet Σ.

Definition

A parameterised problem is a subset of $\Sigma^* \times \mathbb{N}$.

Example

MC(LTL)	
Instance:	A finite Kripke structure $\mathcal{K}$, a state v of $\mathcal{K}$, an LTL-formula φ.
Problem:	Decide whether $\mathcal{K}, v \models \varphi$.

The problem MC(LTL) is decidable in time $2^{\mathcal{O}(|\varphi|)} \cdot \|\mathcal{K}\|$.

Parameterised Problems

Fix a finite alphabet Σ.

Definition

A parameterised problem is a subset of $\Sigma^* \times \mathbb{N}$.

Example

p-MC(LTL)	
Instance:	A finite Kripke structure $\mathcal{K}$, a state v of $\mathcal{K}$, an LTL-formula φ.
Parameter:	Length of φ.
Problem:	Decide whether $\mathcal{K}, v \models \varphi$.

The problem p-MC(LTL) is decidable in time $2^{\mathcal{O}(|\varphi|)} \cdot \|\mathcal{K}\|$.

A Parameterised Analogue of P

Notation

- n for length of input
- k for parameter

Definition

A parameterised problem is in the class FPT, if it is solvable by a deterministic Turing machine in time $p(n) \cdot f(k)$ for some polynomial p and computable f.

Example

The problem p-MC(LTL) is in FPT.

A Parameterised Analogue of NP

Definition

A parameterised problem is in the class $\mathrm{W[P]}$ if it is solvable by a nondeterministic Turing machine in time $p(n) \cdot f(k)$ with only $\log n \cdot h(k)$ nondeterministic bits for some polynomial p and computable f and h.

Example

The problem

p-WSAT(CIRC)	
Instance:	A boolean circuit and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Decide whether the circuit is satisfiable by a valuation with weight k.

is in $\mathrm{W[P]}$.

Parameterised Reductions And Completeness

Definition

Let P and Q be parameterised problems. A mapping $R : \Sigma^* \times \mathbb{N} \rightarrow \Sigma^* \times \mathbb{N}$ is an *fpt*-reduction from P to Q iff

- For all inputs (x, k): $((x, k) \in P \Leftrightarrow R(x, k) \in Q)$
- R is computable in time $p(n) \cdot f(k)$ for a polynomial p and a computable f
- There is a computable g such that if $R(x, k) = (x', k')$, then $k' \leq g(k)$.

Definition

A parameterised problem Q is $\mathrm{W[P]}$-complete if it is in $\mathrm{W[P]}$ and every problem in $\mathrm{W[P]}$ is *fpt*-reducible to it.

Characterising the Intractable Problems with Circuits

Fact

The problem

p-$\mathrm{WS}\textsc{at}$*(CIRC)*	
Instance:	*A boolean circuit and* $k \in \mathbb{N}$.
Parameter:	k.
Problem:	*Decide whether the circuit is satisfiable by a valuation with weight* k.

is $\mathrm{W[P]}$*-complete.*

Question

What happens, if the circuits use other than the boolean connectives?

Generalizing the Boolean Connectives

Definition

A connective is a ptime-computable function $C : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$.

Example

Visualisation of the *majority* and *parity* connectives:

$$\begin{array}{cccccccc}
\#\bot & \uparrow & & & & & & \\
 & 0 & & & & & & \\
 & 0 & 0 & & & & & \\
 & 0 & 0 & 0 & & & & \\
 & 0 & 0 & 0 & 1 & & & \\
 & 0 & 0 & 1 & 1 & 1 & & \\
 & 0 & 1 & 1 & 1 & 1 & 1 & \rightarrow \\
 & & & & & & & \#\top
\end{array}$$

majority

$$\begin{array}{cccccccc}
\#\bot & \uparrow & & & & & & \\
 & 0 & & & & & & \\
 & 0 & 1 & & & & & \\
 & 0 & 1 & 0 & & & & \\
 & 0 & 1 & 0 & 1 & & & \\
 & 0 & 1 & 0 & 1 & 0 & & \\
 & 0 & 1 & 0 & 1 & 0 & 1 & \rightarrow \\
 & & & & & & & \#\top
\end{array}$$

parity

The Classes $\mathrm{W[P](C)}$

Let $\mathcal{C}$ be a class of connectives.

p-WSAT(CIRC)($\mathcal{C}$)	
Instance:	A circuit, whose gates are labelled with connectives from $\mathcal{C}$, and $k \in \mathbb{N}$.
Parameter:	k.
Problem:	Decide whether the circuit is satisfiable by a valuation with weight k.

Definition

A parameterised problem is in the class $\mathrm{W[P]}(\mathcal{C})$ if it is *fpt*-reducible to the problem p-WSAT(CIRC)($\mathcal{C}$).

A Dichotomy Theorem

Theorem

Let $\mathcal{C}$ be a class of connectives. Then

$$\mathrm{W[P]}(\mathcal{C}) = \mathrm{W[P]} \textit{ or } \mathrm{W[P]}(\mathcal{C}) = \mathrm{FPT}.$$

A Dichotomy Theorem

Theorem

Let $\mathcal{C}$ be a class of connectives. Then

$$\mathrm{W[P]}(\mathcal{C}) = \mathrm{W[P]} \text{ or } \mathrm{W[P]}(\mathcal{C}) = \mathrm{FPT}.$$

Idea of Proof

Fact

Let $\mathcal{C}$ be a class of connectives. Then

$$\mathrm{FPT} \subseteq \mathrm{W[P]}(\mathcal{C}) \subseteq \mathrm{W[P]}.$$

A Dichotomy Theorem

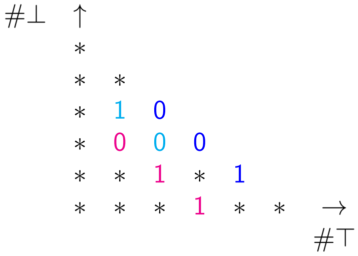
Theorem

Let $\mathcal{C}$ be a class of connectives. Then

$$\mathrm{W[P]}(\mathcal{C}) = \mathrm{W[P]} \ \textit{or} \ \mathrm{W[P]}(\mathcal{C}) = \mathrm{FPT}.$$

Idea of Proof

If $\mathcal{C}$ can simulate at least two of the boolean connectives $\wedge$, $\vee$, $\neg$, then $\mathrm{W[P]}(\mathcal{C}) = \mathrm{W[P]}$, otherwise $\mathrm{W[P]}(\mathcal{C}) = \mathrm{FPT}$.



Summary

- Problems intractable in the classical sense may be tractable in the parameterised sense (in FPT)
- The class $\mathrm{W[P]}$ of problems intractable in the parameterised sense is characterised in terms of boolean circuits
- Generalising the boolean connectives yields the classes $\mathrm{W[P]}(\mathcal{C})$
- The classes $\mathrm{W[P]}(\mathcal{C})$ equal $\mathrm{W[P]}$ or collapse to FPT.

Thank you for your attention!