# Syntactic Cut-Elimination and Backward Proof-Search for Tense Logic via Linear Nested Sequents

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# Motivation 1: Expressivity of linear nested sequents

Recent development: general methods for constructing analytic calculi for non-classical logics in various frameworks. E.g.:

- Modal logics
- Substructural logics
- Intermediate logics
- ...

using

- Sequents
- Nested sequents
- Labelled sequents
- Display calculi

By now these frameworks are (reasonably) well understood ...

$$\begin{array}{ccccc} & & \textit{low} \longleftarrow \text{expressivity} \longrightarrow \textit{high} & & \\ \text{Sequents} & \leq & \text{Nested sequents} & \leq & \left\{ \begin{array}{l} \text{Labelled sequents} \\ \text{Display calculi} \end{array} \right. \\ & & \textit{low} \longleftarrow \text{complexity} \longrightarrow \textit{high} & & \end{array}$$

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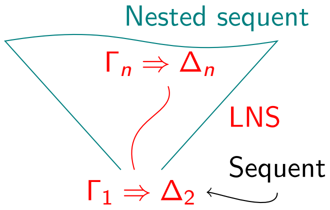
...But what about the stuff in between?

# Motivation 1: Expressivity of Linear nested sequents

Restricting nested sequents to a single branch yields linear nested sequents (LNS):

$$\Gamma_1 \Rightarrow \Delta_1 \nearrow \ldots \nearrow \Gamma_n \Rightarrow \Delta_n$$

(don't confuse with hypersequents!)




Linear nested sequents capture a number of logics, e.g.:

- normal modal logics $\mathsf{K} + \{\mathsf{D}, \mathsf{T}, 4\}$ [Masini:'92, L.:'15]
- non-normal modal logics [L., Pimentel:'19]
- linear temporal logics [Indrzejczak:'16; Baelde et al:'18]
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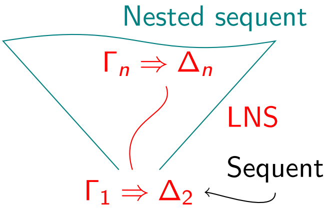


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But these logics either have a cut-free sequent system or have a "linear" semantics.

# Motivation 1: Expressivity of Linear nested sequents

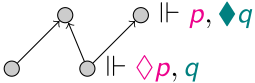
Question 1:

Are there "structurally interesting" examples of logics handled by linear nested sequents?

# Motivation 2: A minimal system for converse/symmetry

Modal tense logic Kt adds the converse modality $\blacksquare$ and its dual $\blacklozenge$ to normal modal logic K.

Symmetric modal logic KB collapses the modalities of Kt.




Modal tense logic and symmetric modal logic are captured (cut-free) in a number of frameworks, e.g.:

- nested sequents [Kashima:'94; Goré et al:'11; Brünnler:'09]
- display calculi [Wansing:'94]
- labelled sequents [Bonnette, Goré:'98]

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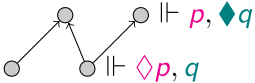


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- display calculi [Wansing:'94]
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But these frameworks use rather heavy machinery in the form of very expressive formalisms

# Motivation 2: A minimal system for converse/symmetry

Question 2:

What is the minimal structural extension of standard sequents suitable for handling converse/symmetry?

# Putting it together...

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Disclaimer: Here we mainly look at KB.

# Reminder: Modal logic KB

The formulae of modal logic are given by

$$\varphi ::= \mathsf{Var} \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi$$

The Hilbert-style presentation of normal modal logic KB is given by the axioms and rules for classical propositional logic and

$$\text{(k)}\ \ \Box(A \to B) \wedge \Box A \to \Box B \qquad \text{(b)}\ \ A \to \Box\neg\Box\neg A \qquad \frac{\vdash A}{\vdash \Box A}\ \text{nec}$$

Semantically, KB is given by the formulae valid in Kripke frames $\langle W, R, V\rangle$ with symmetric accessibility relation $R$.

Validity is defined via the standard clauses for truth at a world:

- $\langle W, R, V\rangle, w \Vdash \Box A$ iff $\forall v \in W\,(wRv \Rightarrow \langle W, R, V\rangle, v \Vdash A)$.
- local clauses for the propositional connectives

## Linear Nested Sequents for KB

A linear nested sequent (LNS) is a finite list of sequents, written

$$\Gamma_1 \Rightarrow \Delta_1 \nearrow \ldots \nearrow \Gamma_n \Rightarrow \Delta_n$$

and interpreted as $\bigwedge\Gamma_1 \to \bigvee\Delta_1 \vee \Box(\ldots\Box(\bigwedge\Gamma_n \to \bigvee\Delta_n)\ldots)$.

The system $\mathsf{LNS}^*_{\mathsf{KB}}$ is given by

$$\frac{}{\mathcal{G} \nearrow \Gamma, p \Rightarrow p, \Delta}\ (id) \qquad \frac{}{\mathcal{G} \nearrow \Gamma, \bot \Rightarrow \Delta}\ \bot_L$$

$$\frac{\mathcal{G} \nearrow \Gamma, A \Rightarrow \Delta, A \to B, B}{\mathcal{G} \nearrow \Gamma \Rightarrow \Delta, A \to B}\ \to_R$$

$$\frac{\mathcal{G} \nearrow \Gamma, A \to B, B \Rightarrow \Delta \quad \mathcal{G} \nearrow \Gamma, A \to B \Rightarrow \Delta, A}{\mathcal{G} \nearrow \Gamma, A \to B \Rightarrow \Delta}\ \to_L$$

$$\frac{\mathcal{G} \nearrow \Gamma \Rightarrow \Delta, \Box A \nearrow \epsilon \Rightarrow A}{\mathcal{G} \nearrow \Gamma \Rightarrow \Delta, \Box A}\ \Box_R \qquad \frac{\mathcal{G} \nearrow \Gamma, \Box A \Rightarrow \Delta \nearrow \Sigma, A \Rightarrow \Pi}{\mathcal{G} \nearrow \Gamma, \Box A \Rightarrow \Delta \nearrow \Sigma \Rightarrow \Pi}\ \Box^1_L$$

$$\frac{\mathcal{G}}{\mathcal{G} \nearrow \Gamma \Rightarrow \Delta}\ \mathsf{EW} \qquad \frac{\mathcal{G} \nearrow \Gamma, A \Rightarrow \Delta}{\mathcal{G} \nearrow \Gamma \Rightarrow \Delta \nearrow \Sigma, \Box A \Rightarrow \Pi}\ \Box^2_L$$

# Cut elimination: the trick

We want completeness via syntactic cut elimination

... but we cannot reduce the following cut:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta \nearrow \Xi \Rightarrow \Upsilon \nearrow \epsilon \Rightarrow A}{\Gamma \Rightarrow \Delta \nearrow \Xi \Rightarrow \Upsilon, \Box A}\ \Box_R \qquad \dfrac{\Sigma, A \Rightarrow \Theta}{\Sigma \Rightarrow \Pi \nearrow \Omega, \Box A \Rightarrow \Theta}\ \Box_L^2}{\Gamma, \Sigma \Rightarrow \Delta, \Pi \nearrow \Xi, \Omega \Rightarrow \Upsilon, \Theta}\ \text{cut}$$

Solution: Add a "superfluous premiss" to the $\Box$ right rule!

$$\dfrac{\mathcal{G} \nearrow \Gamma \Rightarrow \Delta, A \nearrow \Sigma \Rightarrow \Pi, \Box A \qquad \mathcal{G} \nearrow \Gamma \Rightarrow \Delta \nearrow \Sigma \Rightarrow \Pi, \Box A \nearrow \epsilon \Rightarrow A}{\mathcal{G} \nearrow \Gamma \Rightarrow \Delta \nearrow \Sigma \Rightarrow \Pi, \Box A}\ \Box_R^1$$

$$\dfrac{\Gamma \Rightarrow \Delta, \Box A \nearrow \epsilon \Rightarrow A}{\Gamma \Rightarrow \Delta, \Box A}\ \Box_R^2$$

Now cut elimination is "easy" and converting derivations to the original system is trivial.

# The (not so small) hiccup: Admissibility of Necessitation

Necessitation should be easy:
If $\vdash \epsilon \Rightarrow A$, then $\vdash \epsilon \Rightarrow \epsilon \nearrow \epsilon \Rightarrow A$, right?

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Unfortunately not...

$$\dfrac{\dfrac{\dfrac{\overline{p \Rightarrow p}\ (id)}{\epsilon \Rightarrow p \nearrow \Box p \Rightarrow \epsilon}\ \Box_L^2}{\epsilon \Rightarrow p \nearrow \epsilon \Rightarrow \neg\Box p}\ \neg_R}{\epsilon \Rightarrow p, \Box\neg\Box p}\ \Box_R^2$$

$$\dfrac{\dfrac{X}{\epsilon \Rightarrow \neg\Box p} \qquad \dfrac{\checkmark}{\epsilon \Rightarrow \epsilon \nearrow p, \Box\neg\Box p \nearrow \epsilon \Rightarrow \neg\Box p}}{\epsilon \Rightarrow \epsilon \nearrow \epsilon \Rightarrow p, \Box\neg\Box p}\ \Box_R^1$$

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Unfortunately not, but at least:
If $\vdash \epsilon \Rightarrow A$, then $\vdash \epsilon \Rightarrow \Box A \nearrow \epsilon \Rightarrow A$!

$$
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$$

$$
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$$

Main idea: "Reconstruct the old root of the LNS when needed"
(Warning: Proving termination is non-trivial.)

# An alternative: Completeness via countermodels

Consider root-first proof search, applying the rules in the order

- Termination rules: $(id)$ and $\bot_L$
- Propositional rules: $(\to_R)$ and $(\to_L)$ (with local loop check)
- Propagation rule: $\Box^1_L$ (with local loop check)
- Restart rule: $\Box^2_L$ (with local loop check)
- Box rules: $\Box^1_R$ and $\Box^2_R$ (backtrack over all the choices).

Note: This terminates.

Due to the restart rule, this revisits components multiple times.

Hence to construct a model from failed proof search we need to prune the search space. . .

# Pruning the search space

Main idea: Start with root-first proof search . . .

$$\dfrac{p, q \Rightarrow \Box\neg\Box p, \Box\neg\Box q \nearrow \Box q \Rightarrow \neg\Box q}{p, q \Rightarrow \Box\neg\Box p, \Box\neg\Box q \nearrow \epsilon \Rightarrow \neg\Box q}\ \neg_R$$

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$$p \Rightarrow \ldots \nearrow \epsilon \Rightarrow \neg\Box p \qquad \dfrac{p, q \Rightarrow \Box\neg\Box p, \Box\neg\Box q}{p \Rightarrow \Box\neg\Box p, \Box\neg\Box q \nearrow \epsilon \Rightarrow \neg\Box q}\ \Box_L^2, \neg_R$$

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Delete components introduced by the restart rule $\Box^2_L$ . . .

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# Pruning the search space

Main idea: Start with root-first proof search . . .
Delete components introduced by the restart rule $\Box^2_L$ . . .
Read off the model from topmost saturated components.

$$\frac{p, q \Rightarrow \Box\neg\Box p, \Box\neg\Box q \nearrow \Box q \Rightarrow \neg\Box q}{p, q \Rightarrow \Box\neg\Box p, \Box\neg\Box q \nearrow \epsilon \Rightarrow \neg\Box q}\ \neg R$$

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# Summary

We obtained

- Linear nested sequent calculi for Kt and KB
- syntactic cut elimination by modifying the box right rules
- countermodel construction from failed proof search
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Answer 1: Yes! Modal tense logic Kt and modal logic KB.

Question 2: What is the minimal structural extension of standard sequents suitable for handling converse/symmetry?
Answer 2: Not more than linear nested sequents!

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Answer 2: Not more than linear nested sequents!

Question 3: Ask away!

# Modal tense logic

Linear nested sequents for Kt also use the structural connective $\swarrow$ for the converse modality $\blacksquare$.

The formula interpretation $\iota$ is given by:

$$
\begin{array}{rcl}
\iota(\Gamma \Rightarrow \Delta) & := & \bigwedge\Gamma \to \bigvee\Delta \\
\iota(\Gamma \Rightarrow \Delta \nearrow \mathcal{G}) & := & \bigwedge\Gamma \to \bigvee\Delta \vee \Box\, \iota(\mathcal{G}) \\
\iota(\Gamma \Rightarrow \Delta \swarrow \mathcal{G}) & := & \bigwedge\Gamma \to \bigvee\Delta \vee \blacksquare\, \iota(\mathcal{G})
\end{array}
$$

# Modal tense logic

The system $\mathsf{LNS_{Kt}}$ contains the following modal rules:

$$\frac{\mathcal{G} \updownarrow \Gamma \Rightarrow \Delta, A \swarrow \Sigma \Rightarrow \Pi, \Box A \quad \mathcal{G} \updownarrow \Gamma \Rightarrow \Delta \swarrow \Sigma \Rightarrow \Pi, \Box A \nearrow \epsilon \Rightarrow A}{\mathcal{G} \updownarrow \Gamma \Rightarrow \Delta \swarrow \Sigma \Rightarrow \Pi, \Box A}\ \Box^1_R$$

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