Combining Monotone and Normal Modal Logic in Nested Sequents – with Countermodels

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Modal logics: A success story

Fact

Many problems in Computer Science are modelled in Modal Logic.

Examples

- Epistemic logics: $\mathcal{K}(A)$ ... "the agent knows A is the case"
- Deontic logics: $\mathcal{O}(A)$ ... "A ought to be the case"
- ...

In particular, modal logics often have nice reasoning systems a.k.a. calculi with strong connections to

- Syntax: useful for proving theorems
- Semantics: useful for finding countermodels.

Modal logics: A success story (normally?)

… But not all applications might satisfy normality:

Epistemic logics: $\mathcal{K}(A)$ … "the agent knows that A is the case"

- $\mathcal{K}(\top)$ … "the agent knows all tautologies"

Deontic logics: $\mathcal{O}(A)$ … "A ought to be the case"

- $\mathcal{O}(\text{go}) \wedge \mathcal{O}(\neg\text{go}) \rightarrow \mathcal{O}(\text{go} \wedge \neg\text{go})$ … "in presence of conflicting obligations, $\bot$ ought to be the case"

So…

Can we find good calculi for non-normal modal logics?

Monotone modal logic

The formulae of monotone modal logic M are given by

$$p \in \mathrm{Var} \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \langle]\varphi$$

A neighbourhood frame $\mathcal{F} = (W, \mathcal{N})$ has a neighbourhood function satisfying $\mathcal{N}(w) \subseteq \mathcal{P}(W)$ for every $w \in W$.
Valuations $[\![.]\!]$ satisfy:

- local clauses for $\wedge, \vee, \rightarrow, \bot$.
- $\mathcal{F}, [\![.]\!], w \Vdash \langle]A \quad \text{iff} \quad \exists \alpha \in \mathcal{N}(w)\, \forall v \in \alpha.\, \mathcal{F}, [\![.]\!], v \Vdash A$

The axiomatisation of M is given by propositional logic and the rule

$$\frac{\vdash A \rightarrow B}{\vdash \langle]A \rightarrow \langle]B}$$

Reasoning in monotone modal logic

There are some calculi for M:

Syntactical calculi:

- Sequent calculi [Lavendhomme, Lucas:2000, Indrzejczak:2005]
- (Labelled Tableaux [Indrzejczak:2007])

Pro: Good for reasoning, formula interpretation

Con: Bad for countermodels

Semantical calculi:

- Labelled sequent calculi [Negri:2017, Dalmonte et al:2018]

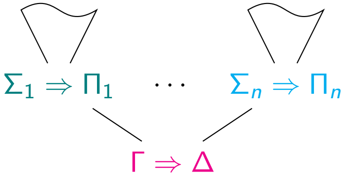
Pro: Good for countermodels

Con: Bad for reasoning, no formula interpretation

Can we get the best of both worlds?

Nested sequents to the rescue!

Nested sequents are trees of (multi-set based) sequents:



interpreted in normal modal logics as

$$\bigwedge\Gamma \to \bigvee\Delta \vee \Box(\bigwedge\Sigma_1 \to \bigvee\Pi_1^*) \vee \cdots \vee \Box(\bigwedge\Sigma_n \to \bigvee\Pi_n^*).$$

A bit of history:

- Precursors: [Bull:'92], [Kashima:'94], [Masini:'92]
- Current form in modal logics: [Brünnler:'09], [Poggiolesi:'09]
- For intuitionistic modal logics: [Straßburger et al:'12 - now]
- Adapted to intuitionistic logic in [Fitting:'14]

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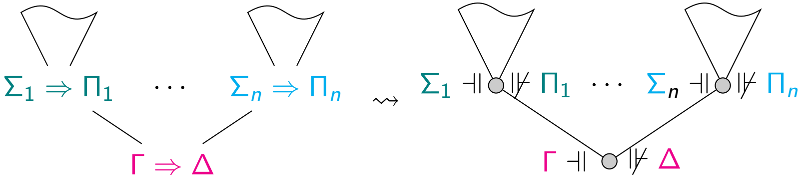

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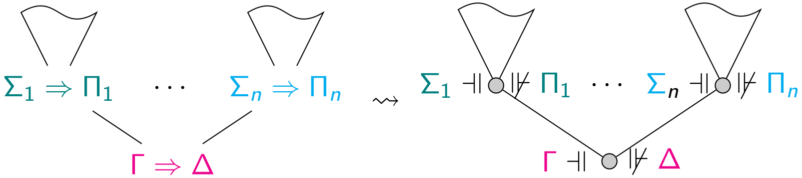

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Nested sequents give rise to models for normal modal logic.

But:

- "deep applicability" of the rules implies normality of the formula interpretation.
- How to construct countermodels for non-normal logics?

monotone modal logic

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Bimodal monotone modal logic

The formulae of bimodal monotone modal logic aka. Brown's Ability Logic are given by

$$p \in \mathsf{Var} \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \langle]\varphi \mid []\varphi$$

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- $\mathcal{F}, [\![.]\!], w \Vdash \langle]A$ iff $\exists \alpha \in \mathcal{N}(w)\, \forall v \in \alpha.\, \mathcal{F}, [\![.]\!], v \Vdash A$
- $\mathcal{F}, [\![.]\!], w \Vdash []A$ iff $\forall \alpha \in \mathcal{N}(w)\, \forall v \in \alpha.\, \mathcal{F}, [\![.]\!], v \Vdash A$

Brown's ability interpretation [Brown:'88]:

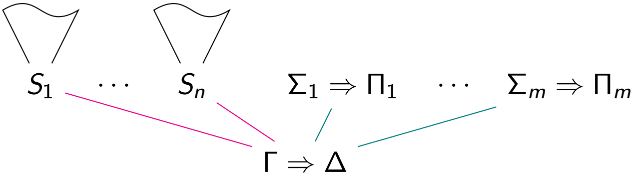
$\langle]A$: "The agent can reliably bring about A"
$[]A$: "The agent will bring about A"

Bimodal nested sequents

A bimodal nested sequent is a structure

$$\Gamma \Rightarrow \Delta, [S_1], \ldots, [S_n], \langle \Sigma_1 \Rightarrow \Pi_1 \rangle, \ldots, \langle \Sigma_m \Rightarrow \Pi_m \rangle$$

with $n, m \geq 0$ where the S_i are bimodal nested sequents.
As a tree:



Its formula interpretation ι is

$$\bigwedge\Gamma \to \bigvee\Delta \;\lor\; \bigvee_{i=1}^{n} [\,]\iota(S_i) \;\lor\; \bigvee_{j=1}^{m} \langle\,](\bigwedge\Sigma_j \to \bigvee\Pi_j)$$

The calculus for bimodal M

The calculus contains the (classical) propositional rules plus:

$$\frac{\Gamma \Rightarrow \Delta, [\Rightarrow A]}{\Gamma \Rightarrow \Delta, [\,]A}\ [\,]_R \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]}{\Gamma, [\,]A \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}\ [\,]_L$$

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$$\frac{\Gamma \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}{\Gamma \Rightarrow \Delta, [\,]A, \langle\Sigma \Rightarrow \Pi\rangle}\ \mathsf{I}$$

Rules are applied anywhere except inside $\langle . \rangle$.

Theorem

The rules are sound wrt. the formula interpretation and (a variant of) the calculus has cut elimination.

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Rules are applied anywhere except inside $\langle . \rangle$.

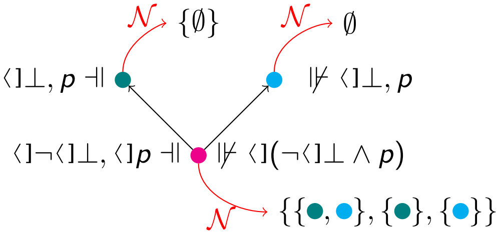
Bonus: Restricting the language specifies the calculus to the standard (linear) nested sequent calculus for modal logic K or the (linear) nested sequent calculus for monomodal M

What about countermodels?

Using an annotated version of the calculus, underivable sequents give rise to countermodels: E.g.

$$\langle]\neg\langle]\bot, \langle]p \Rightarrow \langle](\neg\langle]\bot \wedge \langle]p),\ [\,\langle]\bot, p \Rightarrow\,],\ [\,\Rightarrow \langle]\bot, p\,]$$

yields



Theorem

The calculus for bimodal M *is cut-free complete and failed proof search yields a countermodel.*

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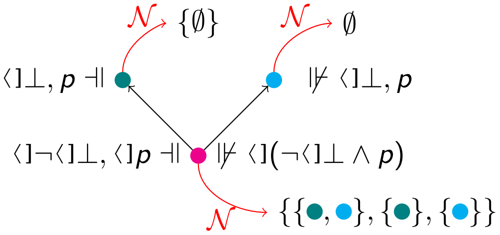

Corollary (Bonus)

The calculi for K *and monomodal* M *are cut-free complete and failed proof search yields a countermodel.*

What do derivations look like?

…Let the implementation work that out!
(`http://subsell.logic.at/bprover/nnProver/`)

Input sequent: $\Rightarrow (([\forall\forall](a \to b) \wedge \langle\exists\forall\rangle a) \to \langle\exists\forall\rangle b)$
Derivation found!

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\dfrac{}{[\forall\forall](a \to b), \langle\exists\forall\rangle a \Rightarrow \langle a \Rightarrow b\rangle^{\mathsf{f}}, \langle \Rightarrow b\rangle^{\mathsf{u}}, \langle\exists\forall\rangle b\,[b, a \overset{a}{\Rightarrow} b]}\ \mathsf{init}
\quad
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}{[\forall\forall](a \to b), \langle\exists\forall\rangle a \Rightarrow \langle a \Rightarrow b\rangle^{\mathsf{f}}, \langle \Rightarrow b\rangle^{\mathsf{u}}, \langle\exists\forall\rangle b}\ \mathsf{jump}
}{[\forall\forall](a \to b), \langle\exists\forall\rangle a \Rightarrow \langle \Rightarrow b\rangle^{\mathsf{u}}, \langle\exists\forall\rangle b}\ \langle\exists\forall\rangle_L
}{[\forall\forall](a \to b), \langle\exists\forall\rangle a \Rightarrow \langle\exists\forall\rangle b}\ \langle\exists\forall\rangle_R
}{([\forall\forall](a \to b) \wedge \langle\exists\forall\rangle a) \Rightarrow \langle\exists\forall\rangle b}\ \wedge_L
}{\Rightarrow (([\forall\forall](a \to b) \wedge \langle\exists\forall\rangle a) \to \langle\exists\forall\rangle b)}\ \to_R
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Input sequent:

$$\Rightarrow (((\langle\exists\forall\rangle(((\langle\exists\forall\rangle c \wedge d) \rightarrow ([\forall\forall]a \vee (\langle\exists\forall\rangle b \vee f))) \wedge (\langle\exists\forall\rangle(\langle\exists\forall\rangle m \rightarrow (k \vee (\langle\exists\forall\rangle l \vee \langle\exists\forall\rangle n))) \wedge [\forall\forall](((\langle\exists\forall\rangle x \wedge [\forall\forall]y) \rightarrow \langle\exists\forall\rangle(\langle\exists\forall\rangle z \wedge \langle\exists\forall\rangle w)))) \rightarrow ([\forall\forall](g \vee [\forall\forall]h) \vee (\langle\exists\forall\rangle\langle\exists\forall\rangle\langle\exists\forall\rangle\langle\exists\forall\rangle i \vee \langle\exists\forall\rangle([\forall\forall]o \rightarrow \langle\exists\forall\rangle p))))$$

Countermodel found!

Suming up

Bimodal nested sequents for monotone modal logic yield:

- an internal calculus;
- syntactic cut elimination;
- support for countermodel construction;
- the basis for a general treatment of non-normal modal logics (see paper for first steps...);
- an implementation including countermodel generation