Nested sequents: Intermediate logics and neighbourhoods

Björn Lellmann

Logic Seminar Wien
Oct 10, 2018

General methods in proof theory

Recent development: general methods for constructing analytic calculi for non-classical logics in various frameworks. E.g.:

- Modal logics
- Substructural logics
- Intermediate logics
- ...

using

- Sequents
- Hypersequents
- Labelled sequents
- Display calculi

By now these frameworks are (reasonably) well understood ...

$$\begin{array}{c} \textit{low} \longleftarrow \text{expressivity} \longrightarrow \textit{high} \\ \text{Sequents} \leq \text{Hypersequents} \leq \left\{ \begin{array}{l} \text{Labelled sequents} \\ \text{Display calculi} \end{array} \right. \\ \textit{low} \longleftarrow \text{complexity} \longrightarrow \textit{high} \end{array}$$

General methods in proof theory

Recent development: general methods for constructing analytic calculi for non-classical logics in various frameworks. E.g.:

- Modal logics
- Substructural logics
- Intermediate logics
- ...

using

- Sequents
- Hypersequents
- Labelled sequents
- Display calculi

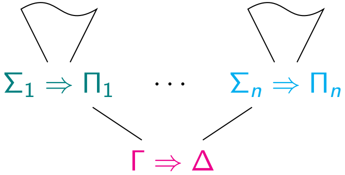
By now these frameworks are (reasonably) well understood ...

$$\begin{array}{c} \textit{low} \longleftarrow \text{expressivity} \longrightarrow \textit{high} \\ \text{Sequents} \leq \text{Hypersequents} \leq \,?\, \leq \left\{ \begin{array}{l} \text{Labelled sequents} \\ \text{Display calculi} \end{array} \right. \\ \textit{low} \longleftarrow \text{complexity} \longrightarrow \textit{high} \end{array}$$

...But what about the stuff in between?

Nested sequents

Nested sequents are trees of (multi-set based) sequents:



interpreted as $\bigwedge\Gamma \to \bigvee\Delta \vee (\bigwedge\Sigma_1 \to \bigvee\Pi_1^*) \vee \cdots \vee (\bigwedge\Sigma_n \to \bigvee\Pi_n^*)$

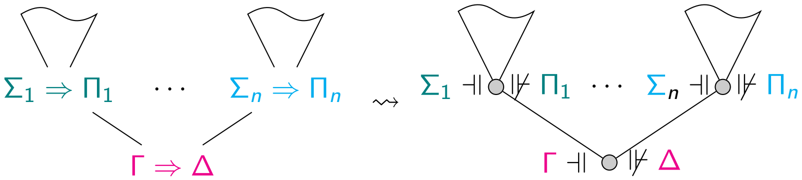
or $\bigwedge\Gamma \to \bigvee\Delta \vee \Box(\bigwedge\Sigma_1 \to \bigvee\Pi_1^*) \vee \cdots \vee \Box(\bigwedge\Sigma_n \to \bigvee\Pi_n^*)$.

A bit of history:

- Precursors: [Bull:'92], [Kashima:'94], [Masini:'92]
- Current form in modal logics: [Brünnler:'09], [Poggiolesi:'09]
- For intuitionistic modal logics: [Straßburger et al:'12 - now]
- Adapted to intuitionistic logic in [Fitting:'14]

Nested sequents

Nested sequents are trees of (multi-set based) sequents:



interpreted as $\bigwedge\Gamma \to \bigvee\Delta \vee (\bigwedge\Sigma_1 \to \bigvee\Pi_1^*) \vee \cdots \vee (\bigwedge\Sigma_n \to \bigvee\Pi_n^*)$

or $\bigwedge\Gamma \to \bigvee\Delta \vee \Box(\bigwedge\Sigma_1 \to \bigvee\Pi_1^*) \vee \cdots \vee \Box(\bigwedge\Sigma_n \to \bigvee\Pi_n^*)$.

Nested sequents give rise to models for intuitionistic and modal logic.

But what about intermediate and non-normal modal logics?

Part 1: Intermediate Logics

Reminder: Intermediate logics

The formulae of intermediate logics are given by

$$p \in \mathsf{Var} \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi$$

A frame $\mathcal{F} = (W, \preccurlyeq)$ has a reflexive transitive $\preccurlyeq \subseteq W \times W$.
Valuations σ satisfy:

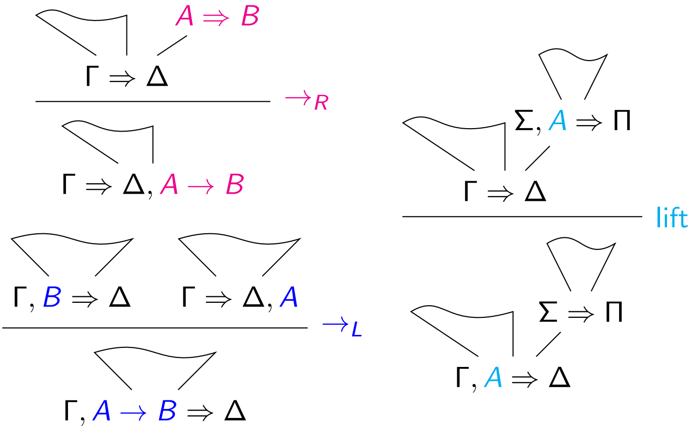
- monotonicity: $\mathcal{F}, \sigma, x \Vdash p$ and $x \preccurlyeq y$ then $\mathcal{F}, \sigma, y \Vdash p$
- local clauses for $\wedge, \vee, \bot$
- $\mathcal{F}, \sigma, x \Vdash A \to B$ iff
  $$\forall y \, (x \preccurlyeq y \Rightarrow (\mathcal{F}, \sigma, y \nVdash A \text{ or } \mathcal{F}, \sigma, y \Vdash B))$$

Intermediate logics are obtained by restricting the class of frames.

- Bd_k: depth at most k $(x_0 \preccurlyeq \ldots \preccurlyeq x_k \Rightarrow \bigvee_{i=1}^{k} x_{i-1} = x_i)$
- GD: linear frames $(x \preccurlyeq y \vee y \preccurlyeq x)$
- Jan: confluent frames $(\exists z \, (x \preccurlyeq z \wedge y \preccurlyeq z))$
- ...

Nested sequents for intuitionistic logic

Fitting's rules (applied anywhere inside the nested sequent):



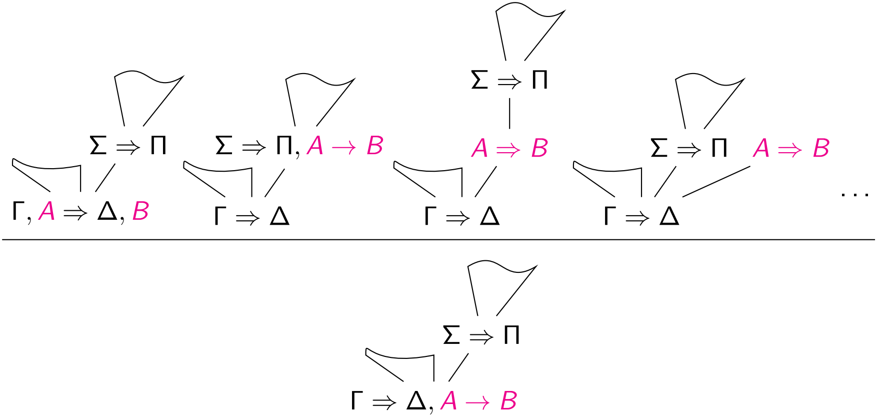
Together with local rules for $\wedge, \vee, \bot$, init, and contraction.

Problem: Rule $\rightarrow_R$ loses control over the structure of the models.

Our approach: Be more explicit

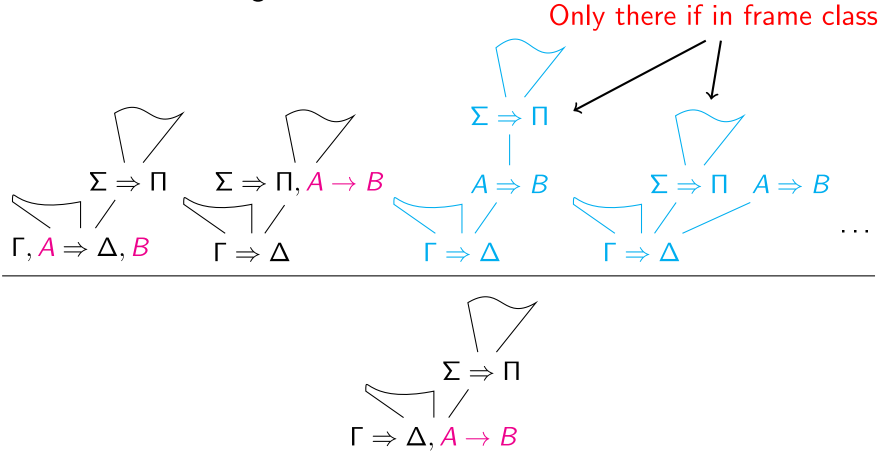
To regain control over the structure of the models we incorporate all different possibilities in the implication right rule

...



Our approach: Be more explicit

To regain control over the structure of the models we incorporate all different possibilities in the implication right rule and restrict according to the class of frames!



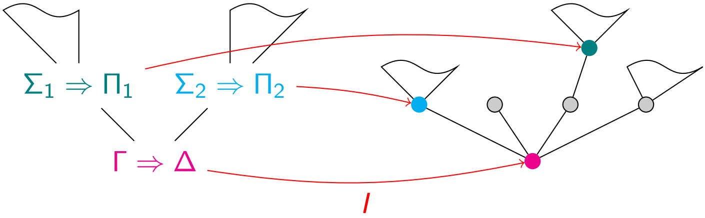
Injective nested sequents: Formally

A nested sequent is a finite directed labelled tree (T, ν), written

$$\Gamma \Rightarrow \Delta, [S_1], \ldots, [S_n]$$

with $n \geq 0$ and S_i nested sequents.
Its underlying tree is the reflexive transitive closure T^* of T.

An interpretation I of a nested sequent in a tree-like model is a tree-embedding of its underlying tree into the model:



In particular, an intepretation is injective!

The general rule scheme

The restriction of a set $\mathcal{S}$ of nested sequents to a frame-class F is

$$\mathcal{S} \restriction_{\mathsf{F}} = \{ (T, \nu) \in \mathcal{S} \mid T^* \in \mathsf{F} \}$$

For a suitable class F of frames, the $\to_R^{\mathsf{F}}$ rule is given by:

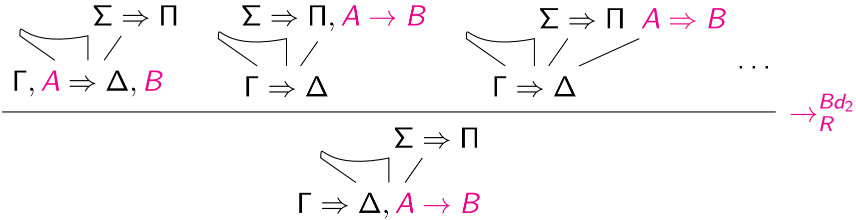
$$\dfrac{\left\{ \begin{array}{l} \nabla\{\Gamma, A \Rightarrow B, A \to B, \Delta, [\Sigma_1 \Rightarrow \Pi_1^*], \dots, [\Sigma_n \Rightarrow \Pi_n^*]\} \\ \nabla\{\Gamma \Rightarrow A \to B, \Delta, [\Sigma_1 \Rightarrow A \to B, \Pi_1^*], \dots, [\Sigma_n \Rightarrow \Pi_n^*]\} \\ \nabla\{\Gamma \Rightarrow A \to B, \Delta, [A \Rightarrow B, [\Sigma_1 \Rightarrow \Pi_1^*]], \dots, [\Sigma_n \Rightarrow \Pi_n^*]\} \\ \qquad \vdots \\ \nabla\{\Gamma \Rightarrow A \to B, \Delta, [\Sigma_1 \Rightarrow \Pi_1^*] \dots, [\Sigma_n \Rightarrow A \to B, \Pi_n^*]\} \\ \nabla\{\Gamma \Rightarrow A \to B, \Delta, [\Sigma_1 \Rightarrow \Pi_1^*] \dots, [A \Rightarrow B, [\Sigma \Rightarrow \Pi^*]]\} \\ \nabla\{\Gamma \Rightarrow A \to B, \Delta, [A \Rightarrow B], [\Sigma_1 \Rightarrow \Pi_1^*], \dots, [\Sigma_n \Rightarrow \Pi_n^*]\} \end{array} \right\} \restriction_{\mathsf{F}}}{\nabla\{\Gamma \Rightarrow A \to B, \Delta, [\Sigma_1 \Rightarrow \Pi_1^*], \dots, [\Sigma_n \Rightarrow \Pi_n^*]\}} \to_R^{\mathsf{F}}$$

Example: Bounded depth Bd_2

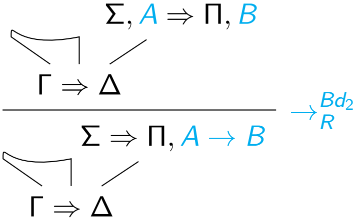
Reminder: Bd_2 frames have depth at most 2.

Thus the rules work only on nested sequents of depth ≤ 2.

The rule with principal formula in the root:



And the rule with principal formula in a leaf:



Example: Gödel-Dummett logic GD

Reminder: GD frames are linear: every node has ≤ 1 successor.

$$
\dfrac{
\begin{array}{c} \vdots \\ \Sigma \Rightarrow \Pi \\ | \\ \Gamma, A \Rightarrow \Delta, B \\ \vdots \end{array}
\quad
\begin{array}{c} \vdots \\ \Sigma \Rightarrow \Pi \\ | \\ A \Rightarrow B \\ | \\ \Gamma \Rightarrow \Delta \\ \vdots \end{array}
\quad
\begin{array}{c} \vdots \\ \Sigma \Rightarrow \Pi, A \to B \\ | \\ \Gamma \Rightarrow \Delta \\ \vdots \end{array}
}{
\begin{array}{c} \vdots \\ \Sigma \Rightarrow \Pi \\ | \\ \Gamma \Rightarrow \Delta, A \to B \\ \vdots \end{array}
}
\to_R^{GD}
\qquad
\dfrac{
\begin{array}{c} \Gamma, A \Rightarrow \Delta, B \\ \vdots \end{array}
\quad
\begin{array}{c} A \Rightarrow B \\ | \\ \Gamma \Rightarrow \Delta \\ \vdots \end{array}
}{
\begin{array}{c} \Gamma \Rightarrow \Delta, A \to B \\ \vdots \end{array}
}
\to_R^{GD}
$$

Example: Gödel-Dummett logic *GD*

Reminder: *GD* frames are linear: every node has ≤ 1 successor.

$$
\dfrac{
\begin{array}{c} \vdots \\ \Sigma \Rightarrow \Pi \\ | \\ A \Rightarrow B \\ | \\ \Gamma \Rightarrow \Delta \\ \vdots \end{array}
\qquad
\begin{array}{c} \vdots \\ \Sigma \Rightarrow \Pi, A \to B \\ | \\ \Gamma \Rightarrow \Delta \\ \vdots \end{array}
}{
\begin{array}{c} \vdots \\ \Sigma \Rightarrow \Pi \\ | \\ \Gamma \Rightarrow \Delta, A \to B \\ \vdots \end{array}
}\ \to_R^{GD'}
\qquad
\dfrac{
\begin{array}{c} A \Rightarrow B \\ | \\ \Gamma \Rightarrow \Delta \\ \vdots \end{array}
}{
\begin{array}{c} \Gamma \Rightarrow \Delta, A \to B \\ \vdots \end{array}
}\ \to_R^{GD'}
$$

Side note: In this case we can even omit more premisses.

Theorem (Kuznets-L.:AiML'18)

This linear nested sequent calculus for GD has syntactic cut elimination.

Completeness and countermodels

For a frame class F the calculus $\mathsf{G_F}$ has the rule $\to_R^{\mathsf{F}}$, Fitting's propositional and lift rules, and

$$\frac{\Gamma \Rightarrow B, \Delta^*, [\Sigma \Rightarrow B, \Pi^*]}{\Gamma \Rightarrow \Delta^*, [\Sigma \Rightarrow B, \Pi^*]}\ \mathsf{Lwr} \qquad \frac{\Gamma, A, A \Rightarrow B, A \to B, \Delta^*}{\Gamma, A \Rightarrow A \to B, \Delta^*}\ \to_R^*$$

Theorem (Kuznets-L.)

Let F *consist of tree-like frames which are linear or of depth at most 2. Then* $\mathsf{G_F}$ *is sound and complete for* F*, and failed proof-search yields a countermodel.*

Corollary

- *Proof search for* $\mathsf{GD}, \mathsf{Sm}, \mathsf{BD_2}, \mathsf{GSc}, \mathsf{CL}$ *is complexity-optimal.*
- GD *has the linear model property.*

Non-trivial application: Interpolation

A logic $\mathcal{L}$ has Craig interpolation if whenever $\mathcal{L} \vdash A(\vec{p}, \vec{q}) \to B(\vec{q}, \vec{r})$, then there is an interpolant $I(\vec{q})$ in the common language of A and B with

$$\mathcal{L} \vdash A(\vec{p}, \vec{q}) \to I(\vec{q}) \qquad \text{and} \qquad \mathcal{L} \vdash I(\vec{q}) \to B(\vec{q}, \vec{r})$$

It has Lyndon interpolation if the polarities of the $\vec{q}$ are the same in A, B, I.

Theorem (Maksimova:1977, nonconstructively)

There are exactly 7 intermediate logics with Craig interpolation.

Theorem (Kuznets-L.)

For $\mathrm{GD}, \mathrm{GSc}, \mathrm{Sm}, \mathrm{BD}_2$ *the injective nested sequent calculi yield constructive proofs of Craig interpolation. For* GD *we also obtain Lyndon interpolation.*

What do derivations look like?

. . . Let the implementation work that out!

Part 2: Non-normal modal logics

Monotone modal logic

The formulae of monotone modal logic M are given by

$$p \in \mathsf{Var} \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \langle\,]\varphi$$

A neighbourhood frame $\mathcal{F} = (W, \mathcal{N})$ has a neighbourhood function satisfying $\mathcal{N}(w) \subseteq \mathcal{P}(W)$ for every $w \in W$.
Valuations σ satisfy:

- local clauses for $\wedge, \vee, \to, \bot$.
- $\mathcal{F}, \sigma, w \Vdash \langle\,]A \quad \text{iff} \quad \exists \alpha \in \mathcal{N}(w)\, \forall v \in \alpha.\, \mathcal{F}, \sigma, v \Vdash A$

The axiomatisation of M is given by the rule

$$\frac{\vdash A \to B}{\vdash \langle\,]A \to \langle\,]B}$$

There is a linear nested sequent system for M [L-P'15]...
but it lacks a formula interpretation and countermodel generation

What's the problem with the formula interpretation?

Interpreting the nesting of nested sequents with τ and using Ackermann's Lemma we have the following equivalences:

$$\frac{\Gamma \Rightarrow \Delta, [\ \Rightarrow A] \quad \Gamma \Rightarrow \Delta, [\ \Rightarrow B]}{\Gamma \Rightarrow \Delta, [\ \Rightarrow A \wedge B]} \quad \leftrightsquigarrow \quad \overline{\tau(A) \wedge \tau(B) \Rightarrow \tau(A \wedge B)}$$

$$\overline{\Rightarrow [p \Rightarrow p]} \quad \leftrightsquigarrow \quad \overline{\Rightarrow \tau(p \to p)}$$

$$\frac{\Gamma \Rightarrow \Delta, [\ \Rightarrow A]}{\Gamma \Rightarrow \Delta, [\ \Rightarrow A \vee B]} \quad \leftrightsquigarrow \quad \overline{\tau(A) \Rightarrow \tau(A \vee B)}$$

Note that these are (equivalent to) the axioms of K. Hence:

"Deep" admissibility of the propositional rules implies normality of the interpretation of the nesting operator!

monotone modal logic

The formulae of monotone modal logic are given by

$$p \in \mathsf{Var} \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \langle\,]\varphi$$

A neighbourhood frame $\mathcal{F} = (W, \mathcal{N})$ has a neighbourhood function satisfying $\mathcal{N}(w) \subseteq \mathcal{P}(W)$ for every $w \in W$.
Valuations σ satisfy:

- local clauses for $\wedge, \vee, \to, \bot$.
- $\mathcal{F}, \sigma, w \Vdash \langle\,]A$ iff $\exists \alpha \in \mathcal{N}(w)\, \forall v \in \alpha.\, \mathcal{F}, \sigma, v \Vdash A$

Bimodal monotone modal logic

The formulae of bimodal monotone modal logic aka. Brown's Ability Logic are given by

$$p \in \mathsf{Var} \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \langle]\varphi \mid []\varphi$$

A neighbourhood frame $\mathcal{F} = (W, \mathcal{N})$ has a neighbourhood function satisfying $\mathcal{N}(w) \subseteq \mathcal{P}(W)$ for every $w \in W$.
Valuations σ satisfy:

- local clauses for $\wedge, \vee, \to, \bot$.
- $\mathcal{F}, \sigma, w \Vdash \langle]A$ iff $\exists \alpha \in \mathcal{N}(w)\, \forall v \in \alpha.\ \mathcal{F}, \sigma, v \Vdash A$
- $\mathcal{F}, \sigma, w \Vdash []A$ iff $\forall \alpha \in \mathcal{N}(w)\, \forall v \in \alpha.\ \mathcal{F}, \sigma, v \Vdash A$

Brown's ability interpretation [Brown:'88]:

$\langle]A$: "The agent can reliably bring about A"
$[]A$: "The agent will bring about A"

Bimodal nested sequents

A bimodal nested sequent is a structure

$$\Gamma \Rightarrow \Delta, [S_1], \ldots, [S_n], \langle \Sigma_1 \Rightarrow \Pi_1 \rangle, \ldots, \langle \Sigma_m \Rightarrow \Pi_m \rangle$$

with $n, m \geq 0$ where the S_i are bimodal nested sequents.
Its formula interpretation ι is

$$\bigwedge \Gamma \rightarrow \bigvee \Delta \;\vee\; \bigvee_{i=1}^{n} [\,]\iota(S_i) \;\vee\; \bigvee_{j=1}^{m} \langle\,](\bigwedge \Sigma_j \rightarrow \bigvee \Pi_j)$$

The calculus for bimodal M

The calculus contains the (classical) propositional rules plus:

$$\frac{\Gamma \Rightarrow \Delta, [\Rightarrow A]}{\Gamma \Rightarrow \Delta, [\,]A}\ [\,]_R \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]}{\Gamma, [\,]A \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}\ [\,]_L$$

$$\frac{\Gamma \Rightarrow \Delta, \langle \Rightarrow A\rangle}{\Gamma \Rightarrow \Delta, \langle\,]A}\ \langle\,]_R \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]}{\Gamma, \langle\,]A \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi\rangle}\ \langle\,]_L$$

$$\frac{\Gamma \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}{\Gamma \Rightarrow \Delta, [\,]A, \langle \Sigma \Rightarrow \Pi\rangle}\ W$$

Rules are applied anywhere except inside $\langle . \rangle$.

Theorem

The rules are sound wrt. the formula interpretation.

The calculus for bimodal M

The calculus contains the (classical) propositional rules plus:

$$\frac{\Gamma \Rightarrow \Delta, [\ \Rightarrow A]}{\Gamma \Rightarrow \Delta, [\,]A}\ [\,]_R \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]}{\Gamma, [\,]A \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}\ [\,]_L$$

$$\frac{\Gamma \Rightarrow \Delta, \langle\ \Rightarrow A\rangle}{\Gamma \Rightarrow \Delta, \langle\,]A}\ \langle\,]_R \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]}{\Gamma, \langle\,]A \Rightarrow \Delta, \langle\Sigma \Rightarrow \Pi\rangle}\ \langle\,]_L$$

$$\frac{\Gamma \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}{\Gamma \Rightarrow \Delta, [\,]A, \langle\Sigma \Rightarrow \Pi\rangle}\ W$$

Rules are applied anywhere except inside $\langle . \rangle$.

Bonus: Restricting the language specifies the calculus to the standard (linear) nested sequent calculus for modal logic K

The calculus for bimodal M

The calculus contains the (classical) propositional rules plus:

$$\frac{\Gamma \Rightarrow \Delta, [\ \Rightarrow A]}{\Gamma \Rightarrow \Delta, [\,]A}\ [\,]_R \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]}{\Gamma, [\,]A \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}\ [\,]_L$$

$$\frac{\Gamma \Rightarrow \Delta, \langle\ \Rightarrow A\rangle}{\Gamma \Rightarrow \Delta, \langle\,]A}\ \langle\,]_R \qquad \frac{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]}{\Gamma, \langle\,]A \Rightarrow \Delta, \langle\Sigma \Rightarrow \Pi\rangle}\ \langle\,]_L$$

$$\frac{\Gamma \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]}{\Gamma \Rightarrow \Delta, [\,]A, \langle\Sigma \Rightarrow \Pi\rangle}\ W$$

Rules are applied anywhere except inside $\langle . \rangle$.

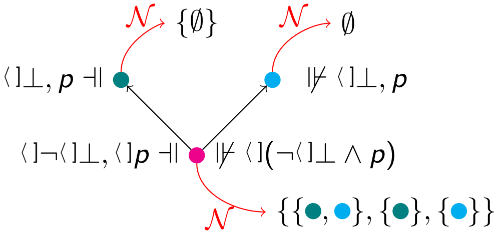
Bonus: Restricting the language specifies the calculus to the standard (linear) nested sequent calculus for modal logic K or the (linear) nested sequent calculus for monomodal M

What about countermodels?

Using an annotated version of the calculus, underivable sequents give rise to countermodels: E.g.

$$\langle\,]\neg\langle\,]\bot, \langle\,]p \Rightarrow \langle\,](\neg\langle\,]\bot \wedge \langle\,]p),\ [\,\langle\,]\bot, p \Rightarrow\,],\ [\,\Rightarrow \langle\,]\bot, p\,]$$

yields



Theorem

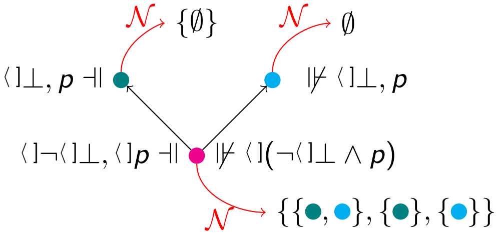
The calculus for bimodal M *is cut-free complete and failed proof search yields a countermodel.*

What about countermodels?

Using an annotated version of the calculus, underivable sequents give rise to countermodels: E.g.

$$\langle\,]\neg\langle\,]\bot, \langle\,]p \Rightarrow \langle\,](\neg\langle\,]\bot \wedge \langle\,]p),\ [\langle\,]\bot, p \Rightarrow\,],\ [\,\Rightarrow \langle\,]\bot, p\,]$$

yields



Corollary (Bonus)

The calculi for K *and monomodal* M *are cut-free complete and failed proof search yields a countermodel.*

What do derivations look like?

. . . Let the implementation work that out!

Suming up

Injective nested sequents for intermediate logics yield:

- uniform calculi for a number of logics based on semantics;
- optimal decision procedures with countermodel construction;
- constructive interpolation proofs.

Bimodal nested sequents for monotone modal logic yield:

- an internal calculus;
- support for countermodel construction;
- the basis for a general treatment of non-normal modal logics

Thank You!