Interpolation for Intermediate Logics via Hyper- and Linear Nested Sequents

Roman Kuznets and Björn Lellmann

TU Wien

AiML 2018, Bern
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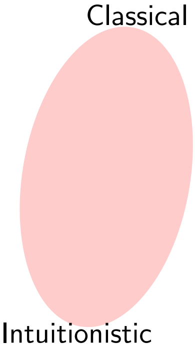

(mainly Gödel logic)

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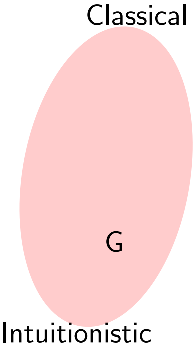

Reminder: Interpolation for Intermediate Logics

Definition: A logic $\mathcal{L}$ has Craig interpolation if whenever $\mathcal{L} \vdash A(\vec{p}, \vec{q}) \to B(\vec{q}, \vec{r})$, then there is an interpolant $I(\vec{q})$ in the common language of A and B with

$$\mathcal{L} \vdash A(\vec{p}, \vec{q}) \to I(\vec{q}) \qquad \text{and} \qquad \mathcal{L} \vdash I(\vec{q}) \to B(\vec{q}, \vec{r})$$

Theorem (Maksimova:1977)

There are exactly 7 intermediate logics with Craig interpolation.

But: Maksimova's proof is non-constructive.

Question: Can we extend proof-theoretic methods for constructing interpolants to these logics, in particular Gödel Logic G?

Reminder: Interpolation for Intermediate Logics

Definition: A logic $\mathcal{L}$ has Lyndon interpolation if whenever $\mathcal{L} \vdash A(\vec{p}, \vec{q}) \to B(\vec{q}, \vec{r})$, then there is an interpolant $I(\vec{q})$ in the common language of A and B with

$$\mathcal{L} \vdash A(\vec{p}, \vec{q}) \to I(\vec{q}) \qquad \text{and} \qquad \mathcal{L} \vdash I(\vec{q}) \to B(\vec{q}, \vec{r})$$

such that the polarities of the $\vec{q}$ are the same in A, B, I.

Theorem (Maksimova:2014)

The logics $\mathsf{Int}, \mathsf{KC}, \mathsf{LP}_2, \mathsf{LS}, \mathsf{CI}$ *have Lyndon-interpolation.*

Question: What about the other two, in particular Gödel logic G?

Gödel logic

The formulae of intermediate logics are given by

$$\Phi ::= \mathsf{Var} \mid \bot \mid \top \mid \Phi \vee \Phi \mid \Phi \wedge \Phi \mid \Phi \rightarrow \Phi$$

Negation is defined as $\neg A \equiv A \rightarrow \bot$.

Frames are tuples $(W, \leq)$ where $\leq \subseteq W \times W$ is reflexive, transitive and antisymmetric. A model $\mathcal{M}$ extends a frame by a valuation $V : W \rightarrow 2^{\mathsf{Var}}$ with $w \leq v \Rightarrow V(w) \subseteq V(v)$.

Truth of a formula is written $\mathcal{M}, w \Vdash A$ and defined by

$$\mathcal{M}, w \Vdash A \rightarrow B \quad \text{iff} \quad \forall v \geq w \, (\mathcal{M}, v \nVdash A \text{ or } \mathcal{M}, v \Vdash B)$$

Gödel logic G is the set of formulae valid in all linear frames, i.e., frames with: $\forall v, w \, (v \leq w \text{ or } w \leq v)$

What about Hypersequents?

A hypersequent is a finite multiset $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ of sequents with formula interpretation

$$(\bigwedge \Gamma_1 \to \bigvee \Delta_1) \vee \ldots \vee (\bigwedge \Gamma_n \to \bigvee \Delta_n) \, .$$

The hypersequent calculus for G has the communication rule:

$$\frac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta \qquad \mathcal{G} \mid \Sigma_1, \Sigma_2 \Rightarrow \Pi}{\mathcal{G} \mid \Gamma_1, \Sigma_1 \Rightarrow \Delta \mid \Gamma_2, \Sigma_2 \Rightarrow \Pi} \text{ com}$$

Intuition: "Pick the larger world" (bottom-up)

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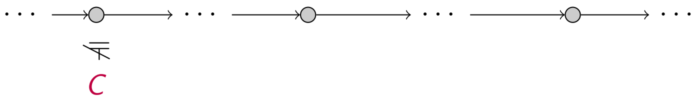

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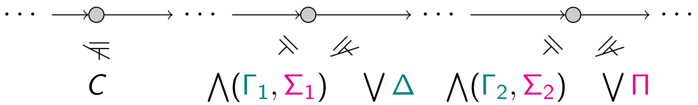

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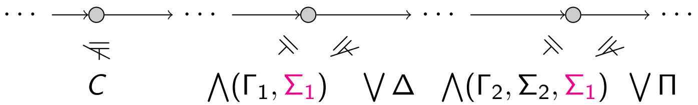

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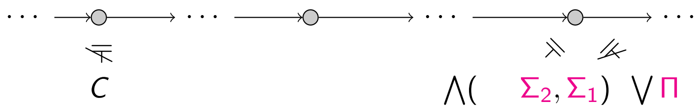

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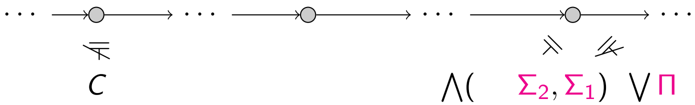

Problem for interpolation: this forgets the structure of the worlds

Can we keep the structure of linear models?

Linear nested sequents for G

A Linear nested sequent is a finite list of sequents $\Gamma_1 \Rightarrow \Delta_1 /\!\!/ \Gamma_2 \Rightarrow \Delta_2 /\!\!/ \ldots /\!\!/ \Gamma_n \Rightarrow \Delta_n$ with formula interpretation

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The calculus $\mathsf{LNS_G}$ contains the implication-right rule

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A \Rightarrow B /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H} \quad \mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi, A \to B /\!\!/ \mathcal{H}}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, A \to B /\!\!/ \Sigma \Rightarrow \Pi /\!\!/ \mathcal{H}} \to_R^2$$

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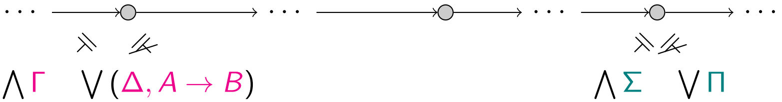

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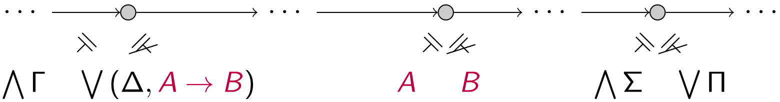

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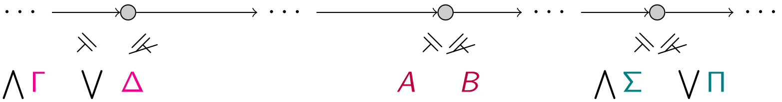

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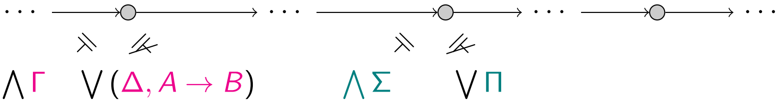

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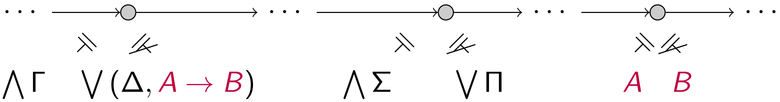

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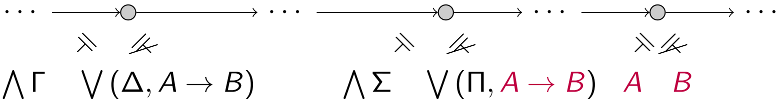

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$\bigwedge \Gamma \quad \bigvee (\Delta, A \to B) \qquad \bigwedge \Sigma \quad \bigvee (\Pi, A \to B)$

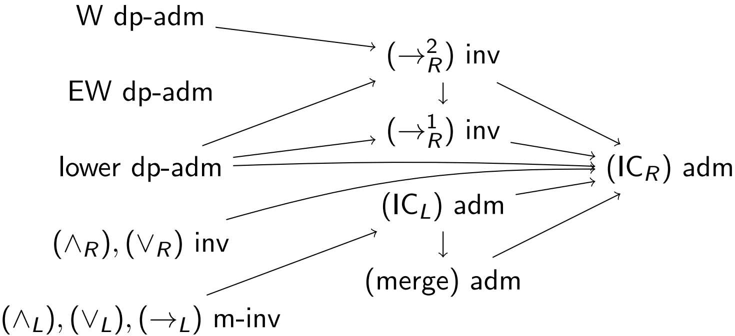
. . . hence the structure of the interpolants can be preserved!

Soundness and cut-free completeness

Theorem (KL:2018)

$\mathsf{LNS_G}$ *is sound and complete for* G.

Proof idea: Soundness as usual.
Completeness via a series of admissibility and invertibility lemmas followed by cut elimination:



But what about interpolation?

Interpolation proof-theoretically

Sequent case

- Find interpolants for each sequent in a sequent derivation top-to-bottom
- Interpolants are formulas
- Interpolation statement can be represented as two sequents

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Generalized sequents,

e.g., linear nested sequents, hypersequents, etc.

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- Interpolation statement has to be more complex than generalized sequents

Semantics respecting linear nested structure

$$\mathcal{M}, w_1, \ldots, w_n \vDash \Gamma_1 \Rightarrow \Delta_1 \mathbin{/\!/} \ldots \mathbin{/\!/} \Gamma_n \Rightarrow \Delta_n$$

$$\Longleftrightarrow$$

$$(\exists A_i \in \Gamma_i) \quad w_i \nVdash A_i \qquad \text{or} \qquad (\exists B_i \in \Delta_i) \quad w_i \Vdash B_i$$

where $w_1 \leq \cdots \leq w_n$ are worlds from linear $\mathcal{M}$

Semantics respecting linear nested structure

$$\mathcal{M}, w_1, \ldots, w_n \models \Gamma_1 \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n \Rightarrow \Delta_n$$

$$\Longleftrightarrow$$

$$(\exists A_i \in \Gamma_i) \quad w_i \not\Vdash A_i \qquad \text{or} \qquad (\exists B_i \in \Delta_i) \quad w_i \Vdash B_i$$

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Theorem (Completeness, KL:2018)

$$\mathsf{LNS_G} \vdash \Gamma_1 \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n \Rightarrow \Delta_n$$

iff

$$(\forall \textit{linear } \mathcal{M})(\forall w_1 \leq \cdots \leq w_n \in \mathcal{M})$$
$$\mathcal{M}, w_1, \ldots, w_n \models \Gamma_1 \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n \Rightarrow \Delta_n$$

Interpolants of formulas evaluated at different worlds

Already used for classical hypersequents, nested sequents, etc.

For $w_1 \leq \cdots \leq w_n \in \mathcal{M}$ for a linear model $\mathcal{M}$

$$w_1, \ldots, w_n \vDash A^{(i)} \quad \text{iff} \quad w_i \Vdash A$$

$$w_1, \ldots, w_n \vDash \mho_1 \owedge \mho_2 \quad \text{iff} \quad w_1, \ldots, w_n \vDash \mho_1 \text{ and } w_1, \ldots, w_n \vDash \mho_2$$

$$w_1, \ldots, w_n \vDash \mho_1 \ovee \mho_2 \quad \text{iff} \quad w_1, \ldots, w_n \vDash \mho_1 \text{ or } w_1, \ldots, w_n \vDash \mho_2$$

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$$w_1, \ldots, w_n \vDash \mho_1 \otimes \mho_2 \quad \text{iff} \quad w_1, \ldots, w_n \vDash \mho_1 \text{ and } w_1, \ldots, w_n \vDash \mho_2$$

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New for intuitionistic case

$$w_1, \ldots, w_n \vDash \overline{A}^{(i)} \quad \text{iff} \quad w_i \nVdash A$$

(Classically, $\overline{A}^{(i)}$ is expressible as $(\neg A)^{(i)}$.)

Componentwise interpolation statement

Definition

A multiformula $\mho$ componentwise interpolates (CW-interpolates) a split linear nested sequent

$$\Gamma_1; \Pi_1 \Rightarrow \Delta_1; \Sigma_1 \mathbin{/\!/} \dots \mathbin{/\!/} \Gamma_n; \Pi_n \Rightarrow \Delta_n; \Sigma_n$$

iff

- $\mho$ only uses $A^{(k)}$ and $\overline{A}^{(k)}$ with $k \leq n$;
- $\mho$ only uses positive (negative) propositional atoms that occur positively (negatively) in both

$$\Gamma_1 \Rightarrow \Delta_1 \mathbin{/\!/} \dots \mathbin{/\!/} \Gamma_n \Rightarrow \Delta_n \quad \text{and} \quad \Pi_1 \Rightarrow \Sigma_1 \mathbin{/\!/} \dots \mathbin{/\!/} \Pi_n \Rightarrow \Sigma_n;$$

- for any $w_1 \leq \cdots \leq w_n$ in a linear frame

$$w_1, \dots, w_n \not\models \mho \quad \Longrightarrow \quad w_1, \dots, w_n \models \Gamma_1 \Rightarrow \Delta_1 \mathbin{/\!/} \dots \mathbin{/\!/} \Gamma_n \Rightarrow \Delta_n;$$
$$w_1, \dots, w_n \models \mho \quad \Longrightarrow \quad w_1, \dots, w_n \models \Pi_1 \Rightarrow \Sigma_1 \mathbin{/\!/} \dots \mathbin{/\!/} \Pi_n \Rightarrow \Sigma_n.$$

Componentwise interpolation example

$$p; \Rightarrow /\!\!/ \Rightarrow; p /\!\!/ ; q \Rightarrow; q \;\leftarrow\; p^{(1)}$$

$$p; \Rightarrow /\!\!/ \Rightarrow; p /\!\!/ ; q \Rightarrow; q \;\leftarrow\; p^{(2)}$$

$$p; \Rightarrow /\!\!/ \Rightarrow; p /\!\!/ ; q \Rightarrow; q \not\leftarrow p^{(3)}$$

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But...

Useful answer

$$\bigcircledwedge_{i=1}^{m} \left(\overline{C_i}^{(1)} \circledvee D_i{}^{(1)}\right) \text{ is a CW-interpolant of } A; \Rightarrow; B$$

implies

$$\bigwedge_{i=1}^{m} (C_i \to D_i) \text{ is a Lyndon interpolant of } A \to B.$$

Applications of the method

To linear nested sequents

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Applying the method to well-known hypersequents for Gödel logic does not succeed:

$$\frac{p;\ \Rightarrow p; \qquad ; q \Rightarrow ; q}{; q \Rightarrow p; \mid p; \ \Rightarrow ; q}$$

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Applying the method to well-known hypersequents for Gödel logic does not succeed:

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Thus, CW interpolation is strictly stronger than Lyndon one.

Thank you

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Simple interpolation transformation

$$\frac{\widetilde{\mathcal{G}} \,/\!/\, \widetilde{\Gamma} \Rightarrow \Delta ; \Pi /\!/ ; A \Rightarrow ; B \;\leftarrow\; \bigotimes_{j=1}^{m} \left(\overline{C_j}^{(n)} \oslash D_j^{(n)} \oslash \bigovee_{l=1}^{n-1} \left(\overline{E_{jl}}^{(l)} \oslash F_{jl}^{(l)} \right) \right)}{\widetilde{\mathcal{G}} \,/\!/\, \widetilde{\Gamma} \Rightarrow \Delta ; \Pi , A \to B \;\leftarrow\; \bigotimes_{j=1}^{m} \left((C_j \to D_j)^{(n-1)} \oslash \bigovee_{l=1}^{n-1} \left(\overline{E_{jl}}^{(l)} \oslash F_{jl}^{(l)} \right) \right)}$$

Complex interpolation transformation

$$\dfrac{\widetilde{\mathcal{G}} \,/\!/\, \widetilde{\Gamma} \Rightarrow \Delta; \Theta \,/\!/\, \widetilde{\Sigma} \Rightarrow \Pi; \Lambda, A \to B \,/\!/\, \widetilde{\mathcal{H}} \leftarrow \mho \qquad \widetilde{\mathcal{G}} \,/\!/\, \widetilde{\Gamma} \Rightarrow \Delta; \Theta /\!/; A \Rightarrow; B \,/\!/\, \widetilde{\Sigma} \Rightarrow \Pi; \Lambda \,/\!/\, \widetilde{\mathcal{H}} \leftarrow \bigwedge_{j=1}^{m} \left(\overline{C_j}^{(n)} \vee D_j^{(n)} \vee \bigvee_{l \neq n} (\overline{E_{jl}}^{(l)} \vee F_{jl}^{(l)}) \right)}{\widetilde{\mathcal{G}} \,/\!/\, \widetilde{\Gamma} \Rightarrow \Delta; \Theta, A \to B \,/\!/\, \widetilde{\Sigma} \Rightarrow \Pi; \Lambda \,/\!/\, \widetilde{\mathcal{H}} \longleftarrow \mho \wedge \bigwedge_{j=1}^{m} \left(\bigvee_{l=1}^{n-1} (\overline{E_{jl}}^{(l)} \vee F_{jl}^{(l)}) \vee (C_j \to D_j)^{(n-1)} \vee \overline{C_j}^{(n)} \vee \bigvee_{l=n}^{n+k} (\overline{E_{j,l+1}}^{(l)} \vee F_{j,l+1}^{(l)}) \right)}$$

The LNS system

The system LNS$_G$ is based on [Fitting:2014, Indrzejczak:2016].

Some further rules:

$$\frac{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/ A \Rightarrow B}{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta, A \to B} \to_R^1$$

$$\frac{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/ A \Rightarrow B /\!/ \Sigma \Rightarrow \Pi /\!/ \mathcal{H} \quad \mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/ \Sigma \Rightarrow \Pi, A \to B /\!/ \mathcal{H}}{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta, A \to B /\!/ \Sigma \Rightarrow \Pi /\!/ \mathcal{H}} \to_R^2$$

$$\frac{\mathcal{G} /\!/ \Gamma, B \Rightarrow \Delta /\!/ \mathcal{H} \qquad \mathcal{G} /\!/ \Gamma, A \to B \Rightarrow \Delta, A /\!/ \mathcal{H}}{\mathcal{G} /\!/ \Gamma, A \to B \Rightarrow \Delta /\!/ \mathcal{H}} \to_L$$

$$\frac{\mathcal{G} /\!/ \Gamma, A \Rightarrow \Delta /\!/ \Sigma, A \Rightarrow \Pi /\!/ \mathcal{H}}{\mathcal{G} /\!/ \Gamma, A \Rightarrow \Delta /\!/ \Sigma \Rightarrow \Pi /\!/ \mathcal{H}} \text{ Lift}$$

All other rules are local.