Constructing Cut-free Sequent Systems with Context Restrictions for Modal Logics Based on Classical or Intuitionistic Logic

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11 January 2013
ICLA, Chennai

Motivation

Fact:

There are a many modal logics: K, KT, K4, S4, CK, CS4 . . . .

For deciding these logics we often use backwards proof search or the subformula property in "good" sequent systems.

But coming up with such a "good" sequent system is not easy!

Question:

Is there a generic method of constructing "good" sequent systems?

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But coming up with such a "good" sequent system is not easy!

Question:

Is there a generic method of constructing "good" sequent systems?

- What is a sequent system for a modal logic?
- What is a good sequent system?
- How to generically construct good sequent systems?

What is a sequent system for a modal logic?

Basics

Here we consider intuitionistic propositional modal logics. (But everything works in the classical case as well!)

Formulae are defined as usual:

$$A_1, \dots, A_n \ni \mathcal{F} ::= \begin{array}{l} p \mid \bot \mid A_1 \wedge A_2 \mid A_1 \vee A_2 \mid A_1 \to A_2 \\ \mid \Box A_1 \mid \Diamond A_1 \mid \heartsuit(A_1, \dots, A_n) \mid \dots \end{array}$$

We use asymmetric sequents $\Gamma \vdash \delta$ with Γ a multiset of formulae and δ a formula. Intended interpretation: $\bigwedge \Gamma \to \delta$.

Our sequent systems have axioms $\overline{\Gamma, A \vdash A}$, the structural rules

$$\frac{\Gamma \vdash \delta}{A, \Gamma \vdash \delta}\ \mathsf{W}, \qquad \frac{\Gamma, A, A \vdash \delta}{\Gamma, A \vdash \delta}\ \mathsf{Con}_L, \qquad \frac{\Gamma \vdash A \quad A, \Sigma \vdash \pi}{\Gamma, \Sigma \vdash \pi}\ \mathsf{Cut},$$

the propositional rules, the congruence rules for all operators $\heartsuit$:

$$\frac{A_1 \vdash B_1 \quad B_1 \vdash A_1 \quad \dots \quad A_n \vdash B_n \quad B_n \vdash A_n}{\Gamma, \heartsuit(A_1, \dots, A_n) \vdash \heartsuit(B_1, \dots, B_n)}$$

and additional rules of a specific format.

Rules with Context Restrictions

A context restriction is a tuple $\langle F_\ell; F_r \rangle$ of sets of formulae. It restricts a sequent $\Gamma \vdash \delta$ by allowing only substitution instances of formulae from F_ℓ (resp. F_r) in Γ (resp. δ).

A rule with context restrictions is of the form

$$\frac{(\Gamma_1 \vdash \delta_1; \mathcal{C}_1) \quad \dots \quad (\Gamma_n \vdash \delta_n; \mathcal{C}_n)}{\Sigma \vdash \pi}$$

with principal formulae $\Sigma \vdash \pi \in \mathit{Seq}(\mathit{Mod}(\mathit{Var}))$ and premisses $\Gamma_i \vdash \delta_i \in \mathit{Seq}(\mathit{Var})$ with associated context restrictions $\mathcal{C}_i$.

In an application of such a rule a premiss with associated restriction $\mathcal{C}_i$ carries over only the context restricted according to $\mathcal{C}_i$ from the conclusion.

Examples of Rules with Context Restrictions

Our rule format captures many standard rules for modal logics, e.g.:

$$\frac{A, B \vdash C}{\Gamma, \Box A, \Box B \vdash \Box C}\ K_2 \qquad \frac{(A, B \vdash C\ ; \langle \emptyset; \emptyset \rangle)}{\Box A, \Box B \vdash \Box C}$$

$$\frac{\Gamma, A \vdash \delta}{\Gamma, \Box A \vdash \delta}\ \mathsf{T}_\Box \qquad \frac{(A \vdash\ ; \langle \{p\}; \{p\} \rangle)}{\Box A \vdash}$$

$$\frac{\Box\Gamma \vdash A}{\Sigma, \Box\Gamma \vdash \Box A}\ 4_\Box \qquad \frac{(\vdash A\ ; \langle \{\Box p\}; \emptyset \rangle)}{\vdash \Box A}$$

We often use the more suggestive notation on the left.

What is a good sequent system for a modal logic?

Cut Elimination

The structural rules Con_L and Cut are bad for backwards proof search, since they give rise to infinite search trees. Also, Cut sabotages the subformula property.

$$\frac{\Gamma, A, A \vdash \delta}{\Gamma, A \vdash \delta}\ \mathrm{Con}_L, \qquad \frac{\Gamma \vdash A \quad A, \Sigma \vdash \pi}{\Gamma, \Sigma \vdash \pi}\ \mathrm{Cut}$$

Thus in a good sequent system these rules should be admissible: the system should derive the same sequents if we drop these rules.

Idea:

Extract general conditions on the rule sets from the standard proofs which guarantee admissibility of Cut and Con_L.

Cut (and Contraction) as Operations on Rules

Cuts between rules:

Slogan: Cut the conclusion, cut the premisses, be liberal!

Example:

$$\frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A} \qquad \frac{A, B \vdash C}{\Box A, \Box B \vdash \Box C}$$

Contractions of rules:

Slogan: Contract the conclusion, contract the premisses!
Example:

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The General Conditions: Saturation

A set $\mathcal{R}$ of rules is

- principal-cut closed if cuts between rules from $\mathcal{R}$ are $\mathcal{R}$ConW-derivable;
- context-cut closed if whenever context restrictions of R and Q overlap on A (i.e. if $A \in F_r^R \cap F_\ell^Q$) , then the principal formulae and all restrictions of one rule satisfy all restrictions of the other rule overlapping on A;
- mixed-cut closed if whenever a principal formula A of R satisfies a context restriction of Q then all restrictions and principal formulae of R satisfy this restriction;
- contraction closed if contractions of rules from $\mathcal{R}$ are in $\mathcal{R}$; [1]
- saturated if it is all of the above.

[1] Compare the closure condition in [Negri, von Plato 2001]

Generic Cut Elimination

Theorem (Generic Cut Elimination)

In saturated rule sets the cut rule can be eliminated.

Copying all formulae obeying the associated context restriction into the premisses, yields admissibility of Contraction. For tractable rule sets (codes of applicable rules / their premisses can be computed in pspace from the conclusion / the rule code) we also have

Theorem (Complexity)

For saturated and tractable sets of rules with restrictions the derivability problem is in EXPTIME.

How to generically construct good sequent systems?

Constructing Cut-free Calculi

Lemma (Cuts preserve soundness)

If G2ip $\in \mathcal{R}$*, then cuts between rules in* $\mathcal{R}$ *are* $\mathcal{R}$ConCut*-derivable.*

This suggests the following heuristic:

1. Saturate the rules under cuts and contractions (guarantees principal-cut closure and contraction closure)
2. check context- and mixed-cut closure and tractability

The heuristic has been applied e.g. in the construction of cut-free systems for several conditional logics including $\mathbb{V}_{\preccurlyeq}$ and $\mathbb{VA}_{\preccurlyeq}$.

Question: How do we get the rules to start with?

Similar approach as for cut elimination: find criteria guaranteeing translatability of axioms from Hilbert-systems.

Translating Axioms: Nesting Depth 1

Consider $\mathsf{CS4}_\Box = \mathsf{CK}_\Box + (\Box p \to p) + (\Box p \to \Box\Box p)$.
(With standard rules for $\mathsf{CK}_\Box$.)

First take axiom $\Box p \to p$.

We take the axiom ...

$$\Box p \to p$$

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We take the axiom, turn it into a zero-premiss rule . . .

$$\Box p \to p \quad \rightsquigarrow \quad \frac{}{\vdash \Box p \to p}$$

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resolve propositional logic and turn top-level propositional variables into contextual premisses

$$\Box p \to p \quad \rightsquigarrow \quad \frac{}{\vdash \Box p \to p} \quad \rightsquigarrow \quad \frac{}{\Box p \vdash p} \quad \rightsquigarrow \quad \frac{\Gamma, p \vdash \delta}{\Gamma, \Box p \vdash \delta}\ \mathsf{T}_\Box$$

introducing the restriction $\langle \{p\}, \{p\} \rangle$ in the last step.

This strategy yields criteria for when a non-nested axiom translates into a shallow rule (set) over intuitionistic logic. (See paper.)

Translating Axioms: Higher Nesting Depth

Now consider axiom $\Box p \to \Box\Box p$.

Key observation: $\Box p$ occurs on top level and under exactly one $\Box$.

Again first resolve propositional logic.

$$\frac{}{\Box p \vdash \Box\Box p}$$

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Now consider axiom $\Box p \to \Box\Box p$.

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Again first resolve propositional logic. Take the occurrence of $\Box p$ under $\Box$, substitute a fresh variable q for this

$$\frac{}{\Box p \vdash \Box\Box p} \quad \rightsquigarrow \quad \frac{\Box p \vdash q \quad q \vdash \Box p}{\Box p \vdash \Box q}$$

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Now computing multiple cuts with instances of the K-rule gives:

$$\frac{\Box p_1 \vdash q_1}{\Box p_1 \vdash \Box q_1}, \quad \frac{q_1, \dots, q_n \vdash r}{\Box q_1, \dots, \Box q_n \vdash \Box r} \quad \rightsquigarrow \quad \frac{\Box p_1, q_2, \dots, q_n \vdash r}{\Box p_1, \Box q_2, \dots, \Box q_n \vdash \Box r}$$

$$\rightsquigarrow \quad \dots \quad \rightsquigarrow \frac{\Box p_1, \dots, \Box p_n \vdash r}{\Box p_1, \dots, \Box p_n \vdash \Box r} \quad \rightsquigarrow \quad \frac{\Box\Gamma \vdash r}{\Box\Gamma \vdash \Box r}\ 4_\Box$$

This gives a purely syntactic construction of the rules for CS4.
(And again a strategy for translation.)

Summary

- A rule format capturing most standard systems
- General (sufficient) conditions for Cut Elimination
- General criteria / tools for translating axioms into rules
- All results for both classical and intuitionistic frameworks

Thank you!