Combining monotone and normal modal logic in nested sequents – with countermodels*

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Abstract. We introduce nested sequent calculi for bimodal monotone modal logic, aka. Brown's ability logic, a natural combination of non-normal monotone modal logic M and normal modal logic K. The calculus generalises in a natural way previously existing calculi for both mentioned logics, has syntactical cut elimination, and can be used to construct countermodels in the neighbourhood semantics. We then consider some extensions of interest for deontic logic. An implementation is also available.

Keywords: Modal Logic · Non-normal Modal Logic · Ability Logic · Nested Sequents · Countermodel Generation

1 Introduction

The *nested sequent framework* has been very successfully used to provide analytic calculi for a large number of logics. In the context of normal modal logics, it enabled modular calculi for all logics in the modal cube [3,18], for tense logics [10], and for intuitionistic and constructive modal logics [15]. One of the main advantages of this framework is that while it is a purely syntactic extension of the sequent framework with a structural connective for the modal box, the tree structure of nested sequents is also closely related to the semantics of modal logics, in particular to the underlying tree structure of Kripke models. Due to this aspect, nested sequent calculi often lend themselves to direct methods of countermodel construction: Usually, if proof search fails, it returns a saturated unprovable nested sequent from which the countermodel can be read off directly. However the full power and flexibility of this framework so far has not yet been harnessed in the context of *non-normal* modal logics. An initial attempt at doing so indeed yielded modular calculi for a number of non-normal modal logics in the framework of *linear* nested sequents [13,14]. Unfortunately, these calculi neither facilitated countermodel construction, nor was it possible to provide a formula interpretation of the linear nested sequents in the language of the logic.

Here we propose an approach to rectify this situation by considering *bimodal* versions of the non-normal modal logics. Such logics seem to have been considered first in [1] in the form of *ability logics*. In this framework, the neighbourhood semantics of monotone modal logic is interpreted by the "can" of ability. Intuitively,

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the neighbourhood function maps a world to a set of neighbourhoods, which correspond to actions available to an agent. If there is an action available such that a certain proposition is true after every possible execution of this action, i.e., true in every world in the corresponding neighbourhood, then the agent can reliably bring about this proposition. This interpretation then gives rise to a second modality interpreting the "will" of ability: If a proposition is true after every available action, i.e., true in every world of every neighbourhood of a particular world, then the agent will unavoidably bring about this proposition. Crucially and very conveniently, this second modality turns out to be normal, which lets us exploit the standard connection between nesting in nested sequents and the successor relation in Kripke models. Moreover, this induced second modality does not depend on the original ability interpretation of non-normal monotone modal logic, and hence its usefulness extends far beyond that particular context.

Using this approach reformulated in terms of one of the most fundamental non-normal modal logics, *monotone modal logic* M [4,7,17], we here present a nested sequent calculus for its bimodal version biM, which combines M with normal modal logic K. Notably, the nested sequents have a formula interpretation in the bimodal language, and the calculus facilitates the construction of countermodels from failed proof search in a slightly modified version. Since biM is a reformulation of Brown's ability logic, this immediately yields a nested sequent calculus for the latter. An additional benefit is that the calculus conservatively extends both the standard nested sequent calculus for K from [3,18] and the nested sequent calculus for M from [13,14]. A prototype implementation of proof search and countermodel construction using the calculus is available under `http://subsell.logic.at/bprover/nnProver/`.

In terms of related work, while the presented calculus is mainly intended as a foundation for nested sequents for monotone modal logics in general, it also seems to be the first sequent-style calculus for biM resp. Brown's ability logic. There are of course a number of calculi for the monomodal logics M and K. The standard sequent calculus for M was introduced in [11], where it was also used to generate countermodels. However, due to the fact that the sequent structure is too poor to capture the information necessary to construct neighbourhood functions, the countermodel generation is rather more involved than in the nested sequent framework. Based on *op.cit.*, sequent calculi for various extensions of M were given in [8] and later converted to the prefixed tableaux framework in [9]. The latter is interesting in that successor labels in these calculi correspond to the K-modality. However, the investigated logics are still only the purely monomodal non-normal fragments. Finally, calculi for non-normal logics including M in the framework of *labelled sequents* have been introduced recently in [16,6]. They are modular, facilitate syntactic cut elimination and can be used for countermodel construction, but due to the inherent semantical character of labelled sequents and the restriction to the monomodal language they lack a formula interpretation.

The article is structured as follows. In Sec. 2 we recall bimodal monotone modal logic and introduce the base calculus. In Sec. 3 we show syntactical cut elimination in a slight variant of the calculus, and Sec. 4 contains the countermodel

$(\mathsf{C}\langle\rangle)$ $\langle\rangle(A \vee B) \to (\langle\rangle A \vee \langle\rangle B)$	$(\mathsf{RM}\langle\rangle)$ $\vdash A \to B / \vdash \langle\rangle A \to \langle\rangle B$
(V) $\langle](A \vee B) \to (\langle\rangle A \vee \langle]B)$	$(\mathsf{RM}\langle])$ $\vdash A \to B / \vdash \langle]A \to \langle]B$
(W) $\langle\rangle A \to ([]B \to \langle]B)$	$(\mathsf{RN}[])$ $\vdash A / \vdash []A$

Fig. 1. The modal axioms and rules for biM from [1]. In addition, the full axiomatisation contains the standard axioms and rules of classical propositional logic.

$(\mathsf{C}[])$ $[]A \wedge []B \to [](A \wedge B)$	$(\mathsf{RM}[])$ $\vdash A \to B / \vdash []A \to []B$
(V') $\langle](A \to B) \wedge []A \to \langle]B$	$(\mathsf{RM}\langle])$ $\vdash A \to B / \vdash \langle]A \to \langle]B$
(W') $[]B \to \langle]B \vee []C$	$(\mathsf{RN}[])$ $\vdash A / \vdash []A$

Fig. 2. The reformulation of the modal axioms and rules for biM.

construction from failed proof search. Some extensions are considered in Sec. 5, followed by a short description of the implementation in Sec. 6 and the conclusion.

2 The basic system

The set $\mathcal{F}$ of *formulae* of bimodal monotone modal logic is given by the following grammar, built over a set $\mathcal{V}$ of propositional variables:

$$\mathcal{F} ::= \bot \mid \mathcal{V} \mid \mathcal{F} \to \mathcal{F} \mid \langle]\mathcal{F} \mid []\mathcal{F}$$

The remaining propositional connectives are defined as usual. The semantics are given in terms of *neighbourhood semantics*, following [1,2,4,17]:

Definition 1. *A* neighbourhood model *is a tuple* $\mathfrak{M} = (W, \mathcal{N}, [\![.]\!])$ *consisting of a universe* W*, a* neighbourhood function $\mathcal{N} : W \to 2^{2^W}$*, and a* valuation $[\![.]\!] : \mathcal{V} \to 2^W$*. The* truth set of a formula A *in a model, written as* $[\![A]\!]$*, extends* $[\![.]\!]$ *by the propositional clauses* $[\![\bot]\!] = \emptyset$ *and* $[\![A \to B]\!] = [\![A]\!]^c \cup [\![B]\!]$ *together with*

- $[\![\langle]A]\!] = \{w \in W \mid$ *exists* $\alpha \in \mathcal{N}(w)$ *s.t. for all* $v \in \alpha : v \in [\![A]\!]\}$
- $[\![[]A]\!] = \{w \in W \mid$ *for all* $\alpha \in \mathcal{N}(w)$ *and for all* $v \in \alpha : v \in [\![A]\!]\}$

We write $\mathfrak{M}, w \Vdash A$ *for* $w \in [\![A]\!]$ *and call* A valid*, if* $[\![A]\!] = W$ *for every model.*

The dual connectives are defined via $[\rangle A \equiv \neg\langle]\neg A$ and $\langle\rangle A \equiv \neg[]\neg A$. The original axiomatisation for *bimodal monotone logic* biM from [1] (called $\mathcal{V}$ there) is given in Fig. 1, its reformulation using only $\langle]$ and $[]$ in Fig. 2. Note that for a model $\mathfrak{M} = (W, \mathcal{N}, [\![.]\!])$ and world $w \in W$ we have that $\mathfrak{M}, w \Vdash []A$ if and only if for all $v \in \bigcup \mathcal{N}(w)$ we have $\mathfrak{M}, v \Vdash A$. Hence $[]$ is a normal K-type modality. The fact that the modality $\langle]$ is a monotone modality follows immediately from the semantics, since if for $\alpha \in \mathcal{N}(w)$ we have $\alpha \subseteq [\![A]\!]$ and $[\![A]\!] \subseteq [\![B]\!]$, i.e., $A \to B$ is valid, then we also have $\alpha \subseteq [\![B]\!]$. Thus if $A \to B$ is valid, then so is $\langle]A \to \langle]B$. This can also be read off the axiomatisation in Fig. 2, since $(\mathsf{C}[])$, $(\mathsf{RM}[])$, $(\mathsf{RN}[])$ is an axiomatisation of K, and $(\mathsf{RM}\langle])$ gives monotonicity of $\langle]$.

$$\frac{}{\mathcal{S}\{\Gamma, p \Rightarrow p, \Delta\}}\ \mathsf{init} \qquad \frac{}{\mathcal{S}\{\Gamma, \bot \Rightarrow \Delta\}}\ \bot_L$$

$$\frac{\mathcal{S}\{\Gamma, A \Rightarrow \Delta, B\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, A \to B\}}\ \to_R \qquad \frac{\mathcal{S}\{\Gamma, B \Rightarrow \Delta\} \quad \mathcal{S}\{\Gamma \Rightarrow \Delta, A\}}{\mathcal{S}\{\Gamma, A \to B \Rightarrow \Delta\}}\ \to_L$$

$$\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\ \Rightarrow A]\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, []A\}}\ []_R \qquad \frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]\}}{\mathcal{S}\{\Gamma, []A \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]\}}\ []_L$$

$$\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle\ \Rightarrow A\rangle\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle]A\}}\ \langle]_R \qquad \frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]\}}{\mathcal{S}\{\Gamma, \langle]A \Rightarrow \Delta, \langle\Sigma \Rightarrow \Pi\rangle\}}\ \langle]_L$$

$$\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, []A, \langle\Sigma \Rightarrow \Pi\rangle\}}\ \mathsf{I}$$

$$\frac{\mathcal{S}\{\Gamma, A, A \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma, A \Rightarrow \Delta\}}\ \mathsf{ICL} \qquad \frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, A, A\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, A\}}\ \mathsf{ICR} \qquad \frac{\mathcal{S}\{\Gamma \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma, \Sigma \Rightarrow \Delta, \Pi\}}\ \mathsf{W}$$

Fig. 3. The nested sequent rules of the calculus $\mathcal{N}_{\mathsf{M}}$ for the bimodal system biM.

To obtain a calculus for bimodal monotone logic, we extend the ordinary sequent structure by the two structural connectives $\langle . \rangle$ and $[.]$ in the succedent, corresponding to the logical connectives $\langle]$ and $[]$, respectively:

Definition 2. *A* nested sequent *is an expression*

$$\Gamma \Rightarrow \Delta, \langle \Sigma_1 \Rightarrow \Pi_1 \rangle, \dots, \langle \Sigma_n \Rightarrow \Pi_n \rangle, [\mathcal{S}_1], \dots, [\mathcal{S}_m]$$

where $\Gamma, \Delta, \Sigma_i, \Pi_i$ are multisets of formulae, and the $\mathcal{S}_j$ are nested sequents. The formula interpretation *of a nested sequent $\mathcal{S}$ is written $\iota(\mathcal{S})$ and given by*

$$\iota(\mathcal{S}) = \bigwedge \Gamma \to \bigvee \Delta \vee \bigvee\nolimits_{i=1}^{n} \langle](\bigwedge \Sigma_i \to \bigvee \Pi_i) \vee \bigvee\nolimits_{j=1}^{m} []\,\iota(\mathcal{S}_j)\ .$$

Intuitively, a nested sequent is a tree, where each node is labelled with an expression $\Gamma \Rightarrow \Delta, \langle \Sigma_1 \Rightarrow \Pi_1 \rangle, \dots, \langle \Sigma_n \Rightarrow \Pi_n \rangle$ and is called a *component* of the nested sequent, and the successor relation corresponds to the nesting operator $[.]$. To shorten presentation we slightly abuse notation and sometimes take the succedent of a sequent to contain nested sequents as well, i.e., we might write $\Gamma \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]$ instead of $\Gamma \Rightarrow \Delta', \langle \Omega \Rightarrow \Theta \rangle, [\Sigma \Rightarrow \Pi], [\Xi \Rightarrow \Upsilon]$. The rules of the nested sequent calculus intuitively then can be applied at any node of the nested sequent. Syntactically, this uses the notion of a *context* as follows.

Definition 3 (Nested sequent context). *A* nested sequent context *is a nested sequent with a* hole $\{.\}$*, defined by* $\mathcal{S}\{.\} ::= \{.\} \mid \Gamma \Rightarrow \Delta, [\mathcal{S}\{.\}]$.

Note that $\langle . \rangle$ never contains $\{.\}$. This ensures non-normality of its interpretation $\langle]$ by preventing application of the propositional rules inside $\langle . \rangle$.

Definition 4. *The rules of the nested sequent calculus $\mathcal{N}_{\mathsf{M}}$ are given in Fig. 3. A* derivation *in $\mathcal{N}_{\mathsf{M}}$ is a finite tree where each node is labelled with a nested sequent, and the label of each node results from the labels of its successors by an application of a rule from $\mathcal{N}_{\mathsf{M}}$. The* depth *of a derivation is the maximal number of nodes in its branches minus one, and the* conclusion *is the label of its root.*

Note that the fragment of $\mathcal{N}_{\mathsf{M}}$ without the rules $\langle]_R, \langle]_L, \mathsf{I}$ is the standard two-sided nested sequent calculus for K from [3,18]. Similarly, the fragment of $\mathcal{N}_{\mathsf{M}}$ without the rules $[]_L, []_R, \mathsf{I}$ is the full nested version of the linear nested sequent calculus for M from [13,14]. Hence, since the semantics are easily transferred, completeness of the full calculus implies completeness of the fragments for K and M respectively. The novel rule I corresponds to axiom (W') and is the necessary link between those two systems. The first step is to show soundness of the system.

Theorem 5. *The calculus $\mathcal{N}_{\mathsf{M}}$ is* sound *for* biM *wrt. the formula interpretation ι, i.e., if a nested sequent $\mathcal{S}$ is derivable in $\mathcal{N}_{\mathsf{M}}$, then $\iota(\mathcal{S})$ is valid in* biM.

Proof. This follows as usual by an induction on the depth of the derivation of $\mathcal{S}$ from the fact that all rules preserve soundness wrt. the formula interpretation ι. For all rules apart from $\langle]_L$ and I this is standard or trivial.

For $\langle]_L$, assume that $\mathcal{S}\{.\} = \Gamma_1 \Rightarrow \Delta_1, [\dots[\Gamma_n \Rightarrow \Delta_n, [\{.\}]]\dots]$, and that $\iota(\mathcal{S}\{\Gamma, \langle]A \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi\rangle\})$ is falsified in $\mathfrak{M} = (W, \mathcal{N}, [\![.]\!])$ at w. If the contradiction comes from the context, i.e., $\iota(\mathcal{S}\{\ \Rightarrow\ \})$ is falsified at w, then also the interpretation of the premiss is falsified at w. Otherwise we have sequences $x_1, \dots, x_{n+1}$ of worlds and $\alpha_1, \dots, \alpha_{n+1}$ of neighbourhoods with

- $w = x_1$
- $x_{i+1} \in \alpha_{i+1} \in \mathcal{N}(x_i)$ for $i = 1, \dots, n$
- $\mathfrak{M}, x_i \Vdash \bigwedge \Gamma_i \wedge \neg \bigvee \iota(\Delta_i)$ for $1 \leq i \leq n$
- $\mathfrak{M}, x_{n+1} \Vdash \bigwedge \Gamma \wedge \langle]A \wedge \neg\iota(\Delta) \wedge [\rangle(\bigwedge \Sigma \wedge \neg \bigvee \Pi)$

where $\iota(\Delta)$ is the natural interpretation of Δ, potentially including further nesting operators. From the last item we obtain a neighbourhood $\alpha \in \mathcal{N}(x_{n+1})$ with $\alpha \subseteq [\![A]\!]$. Due to the fact that $\mathfrak{M}, x_{n+1} \Vdash [\rangle(\bigwedge \Sigma \wedge \neg \bigvee \Pi)$ we then obtain a world $y \in \alpha$ such that $\mathfrak{M}, y \Vdash \bigwedge \Sigma \wedge A \wedge \neg \bigvee \Pi$. Hence we have $\mathfrak{M}, x_{n+1} \Vdash \langle\rangle(\bigwedge \Sigma \wedge A \wedge \neg \bigvee \Pi)$ and the formula interpretation $\iota(\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle \Sigma, A \Rightarrow \Pi\rangle\})$ of the premiss of $\langle]_L$ is also falsified in $\mathfrak{M}, w$.

For the rule I, suppose the formula interpretation $\iota(\mathcal{S}\{\Gamma \Rightarrow \Delta, []A, \langle \Sigma \Rightarrow \Pi\rangle\})$ of the conclusion is falsified in $\mathfrak{M} = (W, \mathcal{N}, [\![.]\!])$ at w. Then as above $\iota(\mathcal{S}\{\ \Rightarrow\ \})$ is falsified at w or there is a world $v \in W$ such that $\mathfrak{M}, v$ falsifies $\bigwedge \Gamma \to \bigvee \iota(\Delta) \vee []A \vee \langle](\bigwedge \Sigma \to \bigvee \Pi)$. Thus, in particular, we have $\mathfrak{M}, v \Vdash \langle\rangle\neg A$ and $\mathfrak{M}, v \Vdash [\rangle(\bigwedge \Sigma \wedge \neg \bigvee \Pi)$. Since $\mathfrak{M}, v \Vdash \langle\rangle\neg A$, there is an $\alpha \in \mathcal{N}(v)$ with $\alpha \neq \emptyset$. For this α then there exists a $x \in \alpha$ with $\mathfrak{M}, x \Vdash \bigwedge \Sigma \wedge \neg \bigvee \Pi$. Thus, in particular we have $\mathfrak{M}, v \Vdash \langle\rangle(\bigwedge \Sigma \wedge \neg \bigvee \Pi)$, and hence $\mathfrak{M}, v \not\Vdash [](\bigwedge \Sigma \to \bigvee \Pi) = \iota([\Sigma \Rightarrow \Pi])$. Hence the formula interpretation of the premiss is also falsified in $\mathfrak{M}, w$. □

We can prove completeness of the calculus in a number of different ways. Perhaps the easiest way is via a detour through the corresponding sequent calculi.

Theorem 6. *The calculus $\mathcal{N}_{\mathsf{M}}$ is complete for* biM*, i.e., if a formula A is valid, then the nested sequent $\Rightarrow A$ is derivable in $\mathcal{N}_{\mathsf{M}}$.*

Proof (Sketch). First, observe that in the ordinary sequent system $\mathsf{G}_{\mathsf{biM}}$ given by the standard propositional rules of $\mathsf{G3c}$ of [19] together with the three rules

$$\frac{\Gamma \Rightarrow B}{\Sigma, []\Gamma \Rightarrow []B, \Pi} \qquad \frac{\Gamma \Rightarrow B}{\Sigma, []\Gamma \Rightarrow []A, \langle]B, \Pi} \qquad \frac{\Gamma, A \Rightarrow B}{\Sigma, []\Gamma, \langle]A \Rightarrow \langle]B, \Pi}$$

and the cut rule all axioms and rules of biM are derivable. Hence $\mathsf{G}_{\mathsf{biM}}$ is complete in presence of cut. It also has cut elimination, as can be seen by straightforward adaption of the standard proof [19], or by checking that it satisfies the criteria for cut elimination from [12]. Completeness of $\mathcal{N}_{\mathsf{M}}$ then follows by simulating derivations in the sequent system in a leaf node of the nested sequents as in [13,14]. In particular, the second and third modal rules above are simulated as follows, abbreviating multiple rule applications by a double line:

$$\dfrac{\dfrac{\dfrac{\Sigma \Rightarrow \Pi, [\Gamma \Rightarrow B]}{\Sigma, [\,]\Gamma \Rightarrow \Pi, [\ \Rightarrow B]}\ [\,]_L}{\Sigma, [\,]\Gamma \Rightarrow [\,]A, \Pi, \langle\ \Rightarrow B\rangle}\ \mathsf{I}}{\Sigma, [\,]\Gamma \Rightarrow [\,]A, \langle\,]B, \Pi}\ \langle\,]_R \qquad \dfrac{\dfrac{\dfrac{\Sigma \Rightarrow \Pi, [\Gamma, A \Rightarrow B]}{\Sigma, [\,]\Gamma \Rightarrow \Pi, [A \Rightarrow B]}\ [\,]_L}{\Sigma, [\,]\Gamma, \langle\,]A \Rightarrow \Pi, \langle\ \Rightarrow B\rangle}\ \langle\,]_L}{\Sigma, [\,]\Gamma, \langle\,]A \Rightarrow \langle\,]B, \Pi}\ \langle\,]_R$$

The simulation of the remaining modal rule is similar but easier. □

Note that analogously to the results in [13,14] the proof of the previous theorem further shows that completeness of the calculus is preserved if we restrict the nested sequents to be *linear*, i.e., to consist only of a single branch, and stipulate that all rules are *end-active*, i.e., only work in the last component:

Corollary 7. *The end-active linear version of* $\mathcal{N}_{\mathsf{M}}$ *is complete for* biM. □

Like the ordinary sequent calculus constructed in the proof, the end-active linear version of $\mathcal{N}_{\mathsf{M}}$ could be used to obtain an optimal PSPACE-complexity result. However, since this already follows using standard techniques and backwards proof search in the ordinary sequent system, we omit the proof.

Theorem 8. *The problem of deciding whether a formula is a theorem of* biM *is* PSPACE*-complete.* □

While the end-active linear version of $\mathcal{N}_{\mathsf{M}}$ is more suitable for space-efficient proof search, it is not ideal for constructing countermodels to underivable sequents. Hence in the following we consider the full nested version.

3 Cut elimination

An alternative completeness proof is given by showing cut elimination. For this we move to the cumulative or *kleene'd* variant of the calculus, where all principal formulae and structures are copied into the premiss(es). The resulting *kleene'd calculus* $\mathcal{N}_{\mathsf{M}}^k$ is given in Fig. 4. Note that it contains the structural version I^s of the interaction rule I. To show equivalence to the base calculus, we show admissibility of the internal and external structural rules, including $\mathsf{ICL}, \mathsf{ICR}, \mathsf{W}$. The proof for the internal rules is by standard induction on the depth of the derivation:

Lemma 9. *The following rules are admissible in* $\mathcal{N}_{\mathsf{M}}^k$*:*

$$\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma, \Sigma \Rightarrow \Delta, \Pi\}} \qquad \frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi\rangle\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle \Sigma, \Omega \Rightarrow \Pi, \Theta\rangle\}} \qquad \frac{\mathcal{S}\{\Gamma, A, A \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma, A \Rightarrow \Delta\}} \qquad \frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, A, A\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, A\}}$$

$$\frac{}{\mathcal{S}\{\Gamma, p \Rightarrow p, \Delta\}}\ \mathsf{init}^i \qquad \frac{}{\mathcal{S}\{\Gamma, \bot \Rightarrow \Delta\}}\ \bot_L^i$$

$$\frac{\mathcal{S}\{\Gamma, A \Rightarrow \Delta, A \to B, B\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, A \to B\}}\ \to_R^i \qquad \frac{\mathcal{S}\{\Gamma, A \to B, B \Rightarrow \Delta\} \quad \mathcal{S}\{\Gamma, A \to B \Rightarrow \Delta, A\}}{\mathcal{S}\{\Gamma, A \to B \Rightarrow \Delta\}}\ \to_L^i$$

$$\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\,]A, [\Rightarrow A]\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\,]A\}}\ [\,]_R^i \qquad \frac{\mathcal{S}\{\Gamma, [\,]A \Rightarrow \Delta, [\Sigma, A \Rightarrow \Pi]\}}{\mathcal{S}\{\Gamma, [\,]A \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]\}}\ [\,]_L^i$$

$$\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle\,]A, \langle \Rightarrow A\rangle\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle\,]A\}}\ \langle\,]_R^i \qquad \frac{\mathcal{S}\{\Gamma, \langle\,]A \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi\rangle, [\Sigma, A \Rightarrow \Pi]\}}{\mathcal{S}\{\Gamma, \langle\,]A \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi\rangle\}}\ \langle\,]_L^i$$

$$\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\,]A, \langle \Sigma \Rightarrow \Pi\rangle, [\Sigma \Rightarrow \Pi]\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\,]A, \langle \Sigma \Rightarrow \Pi\rangle\}}\ \mathsf{I}^i \qquad \frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi\rangle, [\Sigma \Rightarrow \Pi], [\Omega \Rightarrow \Theta]\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi\rangle, [\Omega \Rightarrow \Theta]\}}\ \mathsf{I}^s$$

Fig. 4. The kleene'd version $\mathcal{N}_{\mathsf{M}}^k$ of the calculus

Lemma 10. *The* merge rules *are admissible in* $\mathcal{N}_{\mathsf{M}}^k$*:*

$$\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\Sigma \Rightarrow \Pi], [\Omega \Rightarrow \Theta]\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\Sigma, \Omega \Rightarrow \Pi, \Theta]\}}\ \mathsf{mrg}_{[]} \qquad \frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi\rangle, \langle \Omega \Rightarrow \Theta\rangle\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle \Sigma, \Omega \Rightarrow \Pi, \Theta\rangle\}}\ \mathsf{mrg}_{\langle\rangle}$$

Proof. By induction on the depth of the derivation. The only non-standard cases are for $\mathsf{mrg}_{\langle\rangle}$ with last applied rule $\langle\,]_L^i$ or I^i. Here we apply the induction hypothesis, followed by admissibility of weakening and the original rule. □

Lemma 11. *The calculi* $\mathcal{N}_{\mathsf{M}} + \mathsf{mrg}_{\langle\rangle}$ *and* $\mathcal{N}_{\mathsf{M}}^k$ *are equivalent, i.e., a sequent* $\Rightarrow A$ *is derivable in* $\mathcal{N}_{\mathsf{M}}$ *plus* $\mathsf{mrg}_{\langle\rangle}$ *iff it is derivable in* $\mathcal{N}_{\mathsf{M}}^k$*.*

Proof. For one direction, using admissibility of weakening and contraction it is straightforward to transform a derivation in $\mathcal{N}_{\mathsf{M}}$ into a derivation in $\mathcal{N}_{\mathsf{M}}^k$.

For the other direction we transform derivations in $\mathcal{N}_{\mathsf{M}}^k$ into derivations in $\mathcal{N}_{\mathsf{M}}$ using contraction and merge, where $\mathsf{mrg}_{[]}$ is shown admissible in $\mathcal{N}_{\mathsf{M}}$ by induction on the depth of the derivation. The only tricky part is the transformation for the rule I^s. For this we use the fact that we can permute applications of I^s below applications of the other rules of $\mathcal{N}_{\mathsf{M}}^k$ (the proof is rather straightforward by going through all the cases), until a formula of the shape $[\,]A$ appears in the succedent. At this point we transform the application of I^s into an application of I creating the same formula $[\,]A$, followed by an application of contraction. □

Note that this lemma shows equivalence only with $\mathcal{N}_{\mathsf{M}}$ extended with the merge rule $\mathsf{mrg}_{\langle\rangle}$. While it would be possible to either make this rule part of $\mathcal{N}_{\mathsf{M}}$ from the outset, or to modify $\mathcal{N}_{\mathsf{M}}$ so that it becomes admissible, the advantage of the current formulation is the direct link to the end-active linear version (Cor. 7). To state the cut rule, we use the following notion adapted from [18]:

Definition 12. *For two nested sequents with holes*

$$\mathcal{S}\{\ \} = \Gamma_1 \Rightarrow \Delta_1, [\ldots[\Gamma_n \Rightarrow \Delta_n, [\{\ \}]]\ldots]$$
$$\mathcal{S}'\{\ \} = \Sigma_1 \Rightarrow \Pi_1, [\ldots[\Sigma_n \Rightarrow \Pi_n, [\{\ \}]]\ldots]$$

the merge *is the nested sequent with hole*

$$(\mathcal{S} \oplus \mathcal{S}')\{\ \} := \Gamma_1, \Sigma_1 \Rightarrow \Delta_1, \Pi_1, [\dots[\Gamma_n, \Sigma_n \Rightarrow \Delta_n, \Pi_n, [\{\ \}]] \dots]$$

obtained by "zipping" together the two nested sequents along the path from the root to the hole. Note that the hole is at the same depth in both nested sequents.

Using this notion, the *cut rule* then is the following rule:

$$\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, A\} \quad \mathcal{S}'\{A, \Sigma \Rightarrow \Pi\}}{(\mathcal{S} \oplus \mathcal{S}')\{\Gamma, \Sigma \Rightarrow \Delta, \Pi\}}\ \mathsf{cut}_1$$

In order to reduce cuts on $\langle]$-formulae, we also eliminate the *auxiliary cut rule*:

$$\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle \Omega \Rightarrow \Theta, A\rangle\} \quad \mathcal{S}'\{\Sigma \Rightarrow \Pi, [A, \Xi \Rightarrow \Upsilon]\}}{(\mathcal{S} \oplus \mathcal{S}')\{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \langle \Omega, \Xi \Rightarrow \Theta, \Upsilon\rangle\}}\ \mathsf{cut}_2$$

Soundness of these rules can be shown directly, but also follows from the fact that they are admissible in the cut-free calculus. Note that we only permit cut on components at the same depth of the nested sequents. While often this necessitates the addition or admissibility of certain structural rules [3,18], here the situation is simpler due to the fact that the axiomatisation of biM does not involve axioms of mixed modal rank such as $\mathsf{4}, \mathsf{5}$ or T.

Theorem 13. *The cut rules* cut_1 *and* cut_2 *are admissible in the calculus* $\mathcal{N}^k_{\mathsf{M}}$*.*

Proof. The proof is for both statements simultaneously by induction on the tuples (c, d) in lexicographic ordering, where c is the *complexity* of the cut formula, i.e., its length, and d is the *depth* of the cut, i.e., the sum of the depth of the derivations of the premisses of the cut. Call these tuples the *measure* of the corresponding application of cut. The proof of the statement for cut_1 with measure (c, d) uses the statements for cut_1 with measure (c, k) with $k < d$ and for cut_1 and cut_2 with measure $(c - 1, n)$ for arbitrary n. The proof of the statement for cut_2 with measure (c, d) uses the statements for cut_2 with measure (c, k) with $k < d$ and for cut_1 with measure (c, n) with arbitrary n.

The general strategy is to permute applications of cut up into the left premiss until the cut formula is principal, then up into the right premiss until it is principal here as well and can be reduced. We apply *cross-cuts* to eliminate the cut formula from the context. The cases for the zero-premiss rules are standard.

Cut formula contextual on the left. For cut_1, the cut is permuted as usual into the premiss(es) of the last applied rule in the derivation of the left premiss of the cut and eliminated by induction hypothesis on the depth.

For cut_2, we consider the case where the nesting $\langle . \rangle$ containing the cut formula in the left premiss of the application of cut_2 is active in the last rule of that derivation. If the last rule is $\langle]^i_L$, we have:

$$\frac{\dfrac{\mathcal{S}\{\Gamma, \langle]B \Rightarrow \Delta, \langle \Omega \Rightarrow \Theta, A\rangle, [\Omega, B \Rightarrow \Theta, A]\}}{\mathcal{S}\{\Gamma, \langle]B \Rightarrow \Delta, \langle \Omega \Rightarrow \Theta, A\rangle\}}\ \langle]^i_L \quad \mathcal{S}'\{\Sigma \Rightarrow \Pi, [A, \Xi \Rightarrow \Upsilon]\}}{(\mathcal{S} \oplus \mathcal{S}')\{\Gamma, \langle]B, \Sigma \Rightarrow \Delta, \Pi, \langle \Omega, \Xi \Rightarrow \Theta, \Upsilon\rangle\}}\ \mathsf{cut}_2$$

We first apply cut_2 with lower depth to the premiss of $\langle]^i_L$ and $\mathcal{S}'\{\Sigma \Rightarrow \Pi, [A, \Xi \Rightarrow \Upsilon]\}$ to obtain $(\mathcal{S} \oplus \mathcal{S}')\{\Gamma, \langle]B, \Sigma \Rightarrow \Delta, \Pi, \langle \Omega, \Xi \Rightarrow \Theta, \Upsilon \rangle, [\Omega, B \Rightarrow \Theta, A]\}$. Now an application of cut_1 with possibly higher depth but the same complexity yields

$$((\mathcal{S} \oplus \mathcal{S}') \oplus \mathcal{S}')\{\Gamma, \langle]B, \Sigma^2 \Rightarrow \Delta, \Pi^2, \langle \Omega, \Xi \Rightarrow \Theta, \Upsilon \rangle, [\Omega, B, \Xi \Rightarrow \Theta, \Upsilon]\} \ .$$

An application of $\langle]^i_L$ followed by admissibility of contraction and merge then gives the result. The cases for the rules I^i and I^s are analogous. The cases where the nesting is not active are even simpler.

Cut formula principal on the left and contextual on the right. Since the cut formula is principal on the left and no rule has a principal formula inside the nesting $\langle . \rangle$, we are dealing with the case of cut_1 only. Thus as usual we permute the cut into the premisses of the last rule of the derivation of the right premiss of the cut and eliminate it using the induction hypothesis on the depth.

Principal-principal: The cases where the cut formula is propositional are as usual. In case the cut formula is $\langle]A$, we have:

$$\dfrac{\dfrac{\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle]A, \langle \Rightarrow A \rangle\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle]A\}}\,\langle]^i_R \qquad \dfrac{\mathcal{S}'\{\langle]A, \Sigma \Rightarrow \Pi, \langle \Omega \Rightarrow \Theta \rangle, [A, \Omega \Rightarrow \Theta]\}}{\mathcal{S}'\{\langle]A, \Sigma \Rightarrow \Pi, \langle \Omega \Rightarrow \Theta \rangle\}}\,\langle]^i_L}{(\mathcal{S} \oplus \mathcal{S}')\{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \langle \Omega \Rightarrow \Theta \rangle\}}\,\mathsf{cut}_1$$

First we apply cross-cuts (i.e., the induction hypothesis on the lower depth) to the premiss of $\langle]^i_R$ and the conclusion of $\langle]^i_L$ and vice-versa to obtain derivations of the two nested sequents $(\mathcal{S} \oplus \mathcal{S}')\{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \langle \Rightarrow A \rangle, \langle \Omega \Rightarrow \Theta \rangle\}$ and $(\mathcal{S} \oplus \mathcal{S}')\{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \langle \Omega \Rightarrow \Theta \rangle, [A, \Omega \Rightarrow \Theta]\}$. Then we apply the induction hypothesis on the smaller complexity for cut_2 to these two to obtain

$$(\mathcal{S} \oplus \mathcal{S}') \oplus (\mathcal{S} \oplus \mathcal{S}')\{\Gamma^2, \Sigma^2 \Rightarrow \Delta^2, \Pi^2, \langle \Omega \Rightarrow \Theta \rangle, \langle \Omega \Rightarrow \Theta \rangle, \langle \Omega \Rightarrow \Theta \rangle\}$$

Now admissibility of $\mathsf{mrg}_{[]}$, $\mathsf{mrg}_{\langle\rangle}$ and contraction yields the result.

Suppose that the cut formula is $[\,]A$ with last applied rules $[\,]^i_R$ and $[\,]^i_L$:

$$\dfrac{\dfrac{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\,]A, [\Rightarrow A]\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\,]A\}}\,[\,]^i_R \qquad \dfrac{\mathcal{S}'\{[\,]A, \Sigma \Rightarrow \Pi, [A, \Omega \Rightarrow \Theta]\}}{\mathcal{S}'\{[\,]A, \Sigma \Rightarrow \Pi, [\Omega \Rightarrow \Theta]\}}\,[\,]^i_L}{(\mathcal{S} \oplus \mathcal{S}')\{\Gamma, \Sigma \Rightarrow \Delta, \Pi, [\Omega \Rightarrow \Theta]\}}\,\mathsf{cut}_1$$

Again, applying cross-cuts gives $(\mathcal{S} \oplus \mathcal{S}')\{\Gamma, \Sigma \Rightarrow \Delta, \Pi, [A, \Omega \Rightarrow \Theta]\}$ and $(\mathcal{S} \oplus \mathcal{S}')\{\Gamma, \Sigma \Rightarrow \Delta, \Pi, [\Rightarrow A], [\Omega \Rightarrow \Theta]\}$. Now an application of cut_1 with smaller complexity gives $(\mathcal{S} \oplus \mathcal{S}') \oplus (\mathcal{S} \oplus \mathcal{S}')\{\Gamma^2, \Sigma^2 \Rightarrow \Delta^2, \Pi^2, [\Omega \Rightarrow \Theta], [\Omega \Rightarrow \Theta]\}$ and using admissibility of merge and contraction we are done.

If the cut formula is $[\,]A$ with last applied rules I^i and $[\,]_L$ we have

$$\dfrac{\dfrac{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\,]A, \langle \Omega \Rightarrow \Theta \rangle, [\Omega \Rightarrow \Theta]\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\,]A, \langle \Omega \Rightarrow \Theta \rangle\}}\,\mathsf{I}^i \qquad \dfrac{\mathcal{S}'\{[\,]A, \Sigma \Rightarrow \Pi, [A, \Xi \Rightarrow \Upsilon]\}}{\mathcal{S}'\{[\,]A, \Sigma \Rightarrow \Pi, [\Xi \Rightarrow \Upsilon]\}}\,[\,]^i_L}{(\mathcal{S} \oplus \mathcal{S}')\{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \langle \Omega \Rightarrow \Theta \rangle, [\Xi \Rightarrow \Upsilon]\}}\,\mathsf{cut}_1$$

This is converted into

$$\dfrac{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\,]A, \langle \Omega \Rightarrow \Theta \rangle, [\Omega \Rightarrow \Theta]\} \quad \dfrac{\mathcal{S}'\{[\,]A, \Sigma \Rightarrow \Pi, [A, \Xi \Rightarrow \Upsilon]\}}{\mathcal{S}'\{[\,]A, \Sigma \Rightarrow \Pi, [\Xi \Rightarrow \Upsilon]\}}\ [\,]^i_L}{\dfrac{(\mathcal{S} \oplus \mathcal{S}')\{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \langle \Omega \Rightarrow \Theta \rangle, [\Omega \Rightarrow \Theta], [\Xi \Rightarrow \Upsilon]\}}{(\mathcal{S} \oplus \mathcal{S}')\{\Gamma, \Sigma \Rightarrow \Delta, \Pi, \langle \Omega \Rightarrow \Theta \rangle, [\Xi \Rightarrow \Upsilon]\}}\ \mathsf{I}^s}\ \mathsf{cut}_1$$

and we are done using the induction hypothesis on the depth. □

4 Completeness via countermodel generation

From a semantical point it is more informative to show completeness by constructing countermodels from a failed proof search. For this we slightly modify the system $\mathcal{N}^k_{\mathsf{M}}$ in two ways. First, to make the construction of a successor world more explicit, we split the nesting operator $\langle . \rangle$ into an *unfinished* version $\langle . \rangle^{\mathsf{u}}$ and a *finished* version $\langle . \rangle^{\mathsf{f}}$, adding an explicit jump rule which constructs a $[.]$-successor out of a finished $\langle . \rangle^{\mathsf{f}}$-successor as in [13,14]. To facilitate the construction of the neighbourhoods, we further add *annotations* to the components:

Definition 14. *An* annotated nested sequent *is an expression*

$$\Gamma \overset{\Xi}{\Rightarrow} \Delta, \langle \Sigma_1 \Rightarrow \Pi_1 \rangle^{\mathsf{u}}, \dots, \langle \Sigma_n \Rightarrow \Pi_n \rangle^{\mathsf{u}},$$
$$\langle \Omega_1 \Rightarrow \Theta_1 \rangle^{\mathsf{f}}, \dots, \langle \Omega_m \Rightarrow \Theta_m \rangle^{\mathsf{f}}, [\mathcal{S}_1], \dots, [\mathcal{S}_k]$$

where the annotation Ξ *is a multiset of formulae, and the $\mathcal{S}_i$ are annotated nested sequents. For a component v we write $\mathsf{an}(v)$ for the annotation of this component.*

Again, we can identify an annotated nested sequent with a labelled tree, and we call each node labelled with an expression $\Gamma \overset{\Xi}{\Rightarrow} \Delta, \langle \Sigma_1 \Rightarrow \Pi_1 \rangle^{\mathsf{u}}, \dots, \langle \Sigma_n \Rightarrow \Pi_n \rangle^{\mathsf{u}}$, $\langle \Omega_1 \Rightarrow \Theta_1 \rangle^{\mathsf{f}}, \dots, \langle \Omega_m \Rightarrow \Theta_m \rangle^{\mathsf{f}}$ a *component* of the annotated nested sequent. The main intuition for the annotations is that they store information on how a component of a nested sequent was created during backwards proof search. This information will the be used in the countermodel construction to collect all successors of a component with the same annotation into one neighbourhood of the component. Finally, we drop the structural version of the interaction rule. The resulting system $\mathcal{N}^a_{\mathsf{M}}$ is given in Fig. 5. Note that the annotations only store information on how a component of a nested sequent in the proof search was created, but do not influence proof search per se. Building on this, the proof of the following Lemma shows that, modulo the structural rules, derivations in the annotated and plain systems are easily converted into each other.

Lemma 15. *The systems $\mathcal{N}_{\mathsf{M}}$ and $\mathcal{N}^a_{\mathsf{M}}$ are equivalent, i.e.: A nested sequent $\Rightarrow A$ is derivable in $\mathcal{N}_{\mathsf{M}}$ if and only if $\overset{\emptyset}{\Rightarrow} A$ is derivable in $\mathcal{N}^a_{\mathsf{M}}$.*

$$\frac{}{\mathcal{S}\{\Gamma, p \stackrel{\Upsilon}{\Rightarrow} p, \Delta\}}\ \mathsf{init}^a \qquad \frac{}{\mathcal{S}\{\Gamma, \bot \stackrel{\Upsilon}{\Rightarrow} \Delta\}}\ \bot_L^a$$

$$\frac{\mathcal{S}\{\Gamma, A \stackrel{\Upsilon}{\Rightarrow} B, A \to B, \Delta\}}{\mathcal{S}\{\Gamma \stackrel{\Upsilon}{\Rightarrow} A \to B, \Delta\}}\ \to_R^a \qquad \frac{\mathcal{S}\{\Gamma, A \to B, B \stackrel{\Upsilon}{\Rightarrow} \Delta\} \quad \mathcal{S}\{\Gamma, A \to B \stackrel{\Upsilon}{\Rightarrow} A, \Delta\}}{\mathcal{S}\{\Gamma, A \to B \stackrel{\Upsilon}{\Rightarrow} \Delta\}}\ \to_L^a$$

$$\frac{\mathcal{S}\{\Gamma \stackrel{\Upsilon}{\Rightarrow} \Delta, [\,]A, [\ \stackrel{\emptyset}{\Rightarrow} A]\}}{\mathcal{S}\{\Gamma \stackrel{\Upsilon}{\Rightarrow} \Delta, [\,]A\}}\ [\,]_R^a \qquad \frac{\mathcal{S}\{\Gamma, [\,]A \stackrel{\Upsilon}{\Rightarrow} \Delta, [\Sigma, A \stackrel{\Xi}{\Rightarrow} \Pi]\}}{\mathcal{S}\{\Gamma, [\,]A \stackrel{\Upsilon}{\Rightarrow} \Delta, [\Sigma \stackrel{\Xi}{\Rightarrow} \Pi]\}}\ [\,]_L^a$$

$$\frac{\mathcal{S}\{\Gamma \stackrel{\Upsilon}{\Rightarrow} \Delta, \langle\,]A, \langle\ \Rightarrow A\rangle^{\mathsf{u}}\}}{\mathcal{S}\{\Gamma \stackrel{\Upsilon}{\Rightarrow} \Delta, \langle\,]A\}}\ \langle\,]_R^a \qquad \frac{\mathcal{S}\{\Gamma, \langle\,]A \stackrel{\Upsilon}{\Rightarrow} \Delta, \langle \Sigma \Rightarrow \Pi\rangle^{\mathsf{u}}, \langle \Sigma, A \Rightarrow \Pi\rangle^{\mathsf{f}}\}}{\mathcal{S}\{\Gamma, \langle\,]A \stackrel{\Upsilon}{\Rightarrow} \Delta, \langle \Sigma \Rightarrow \Pi\rangle^{\mathsf{u}}\}}\ \langle\,]_L^a$$

$$\frac{\mathcal{S}\{\Gamma \stackrel{\Upsilon}{\Rightarrow} \Delta, [\,]A, \langle \Sigma \Rightarrow \Pi\rangle^{\mathsf{u}}, \langle \Sigma \Rightarrow \Pi\rangle^{\mathsf{f}}\}}{\mathcal{S}\{\Gamma \stackrel{\Upsilon}{\Rightarrow} \Delta, [\,]A, \langle \Sigma \Rightarrow \Pi\rangle^{\mathsf{u}}\}}\ \mathsf{I}^a \qquad \frac{\mathcal{S}\{\Gamma \stackrel{\Upsilon}{\Rightarrow} \Delta, \langle \Sigma \Rightarrow \Pi\rangle^{\mathsf{f}}, [\Sigma \stackrel{\Sigma}{\Rightarrow} \Pi]\}}{\mathcal{S}\{\Gamma \stackrel{\Upsilon}{\Rightarrow} \Delta, \langle \Sigma \Rightarrow \Pi\rangle^{\mathsf{f}}\}}\ \mathsf{jump}^a$$

Fig. 5. The invertible annotated variant $\mathcal{N}_{\mathsf{M}}^a$ of the system

Proof. To convert derivations in $\mathcal{N}_{\mathsf{M}}$ into derivations in $\mathcal{N}_{\mathsf{M}}^a$, we first convert them into derivations in $\mathcal{N}_{\mathsf{M}}^k$ using Lem. 11, noting that the result does not contain the rule I^s. Hence we can convert the resulting derivation into a derivation in $\mathcal{N}_{\mathsf{M}}^a$ bottom-up, starting from the conclusion, replacing all the rules with their respective counterparts. The rules $\langle\,]_L^i$ and I^i are replaced with their annotated versions followed by jump^a. In the other direction, derivations in $\mathcal{N}_{\mathsf{M}}^a$ are converted into derivations in $\mathcal{N}_{\mathsf{M}}$ by deleting all the annotations and applications of jump^a, replacing the rules $\langle\,]_L^a$ and I^a by $\langle\,]_L$ and I respectively, and using contraction to remove additional copies of the principal formulae. □

As usual, we then construct a model from an annotated nested sequent which is saturated under the application of all the rules, defined as follows.

Definition 16. *An annotated nested sequent $\mathcal{S}$ is* saturated *if for each of its components $\Gamma \stackrel{\Xi}{\Rightarrow} \Delta$ the following hold:*

1. $\Gamma \cap \Delta \neq \emptyset$
2. $\bot \notin \Gamma$
3. $A \to B \in \Gamma$ *implies* $B \in \Gamma$ *or* $A \in \Delta$
4. $A \to B \in \Delta$ *implies* $A \in \Gamma$ *and* $B \in \Delta$
5. $\langle\,]A \in \Delta$ *implies* $\langle\ \Rightarrow A\rangle^{\mathsf{u}} \in \Delta$
6. $\langle\,]A \in \Gamma$ *and* $\langle \Sigma \Rightarrow \Pi\rangle^{\mathsf{u}} \in \Delta$ *implies* $\langle \Sigma, A \Rightarrow \Pi\rangle^{\mathsf{f}} \in \Delta$
7. $\langle \Sigma \Rightarrow \Pi\rangle^{\mathsf{f}} \in \Delta$ *implies there are* Ω, Θ *such that* $[\Sigma, \Omega \stackrel{\Sigma}{\Rightarrow} \Pi, \Theta] \in \Delta$
8. $[\,]A \in \Delta$ *implies there are* Σ, Ω, Θ *with* $[\Omega \stackrel{\Sigma}{\Rightarrow} A, \Theta] \in \Delta$
9. $[\,]A \in \Gamma$ *and* $[\Omega \stackrel{\Sigma}{\Rightarrow} \Theta] \in \Delta$ *implies* $A \in \Omega$.
10. $[\,]A \in \Delta$ *and* $\langle \Sigma \Rightarrow \Pi\rangle^{\mathsf{u}} \in \Delta$ *implies* $\langle \Sigma, \Omega \Rightarrow \Pi, \Theta\rangle^{\mathsf{f}} \in \Delta$ *for some* Ω, Θ.

The difficulty in building a model from a saturated nested sequent then lies in constructing the set of neighbourhoods for each world. We do this by collecting successor worlds into sets according to their annotations. A bit of care needs to be taken, depending on whether there is a formula of the form $\langle] A$ in the succedent of the component or not. Formally:

Definition 17. *Let $\mathcal{S}$ be a saturated nested sequent. The* model generated by $\mathcal{S}$ *is the model $\mathfrak{M}^{\mathcal{S}} = (W, \mathcal{N}, [\![.]\!])$ with*

- *W the set of components (nodes) of $\mathcal{S}$*
- *if $w \in W$, then $w \in [\![p]\!]$ iff w is a component $\Gamma \overset{\Sigma}{\Rightarrow} \Delta$ with $p \in \Gamma$*
- *$\mathcal{N}(w)$ is defined as follows. Let $\mathcal{C}_w$ be the set of immediate successors of w, and let $\ell[\mathcal{C}_w]$ be the set of annotations of nodes in $\mathcal{C}_w$. Then let*

$$\mathcal{L}_w := \{\, \{v \in \mathcal{C}_w \mid \mathsf{an}(v) = \Sigma\} \mid \Sigma \in \ell[\mathcal{C}_w]\}$$

Now, $\mathcal{N}(w)$ is defined as $(\mathcal{L}_w \cup \{\mathcal{C}_w\}) \smallsetminus \{\emptyset\}$ if there is a formula $\langle] A \in \Delta$, and $\mathcal{L}_w \cup \{\mathcal{C}_w\} \cup \{\emptyset\}$ otherwise.

Thus, disregarding the empty set, the set of neighbourhoods of a node in a nested sequent includes the set of all its children (to make the construction work for the normal modality $[]$), as well as every set of children with the same annotation. Whether it contains the empty set or not depends on whether there is a formula $\langle] A$ in its succedent. By construction we have:

Lemma 18 (Model Lemma). *If $\mathcal{S}$ is saturated, then the model generated by $\mathcal{S}$ is a neighbourhood model.* □

Non-derivable nested sequents then yield a saturated nested sequent via a standard proof search procedure, given as follows.

Definition 19. *The* proof search procedure *in $\mathcal{N}^a_{\mathsf{M}}$ is defined by application of the rules of $\mathcal{N}^a_{\mathsf{M}}$ in an arbitrary but fixed order, unless the conclusion of a potential rule application already satisfies the saturation condition corresponding to this rule. An annotated nested sequent is* minimal *if it can be obtained from an annotated nested sequent $\Gamma \overset{\emptyset}{\Rightarrow} \Delta$ by the proof search procedure.*

Lemma 20. *The proof search procedure terminates and either yields a derivation or a saturated annotated nested sequent.*

Proof. Every backwards application of a rule adds a formula or a sequent inside a nesting operator. Since the maximal modal nesting depth of formulae decreases in every nesting operator, and since by the saturation conditions no formula or sequent is created twice in the same component, the procedure terminates. □

The final ingredient for showing that the model generated from a saturated nested sequent obtained from proof search really is a model then is the following.

Lemma 21. *Let $\mathcal{S}$ be a minimal annotated nested sequent and $\Gamma \overset{\Xi}{\Rightarrow} \Delta$ be a component of $\mathcal{S}$. Then $\Xi \subseteq \Gamma$.*

Proof. Since in a minimal annotated nested sequent new components are only constructed via the jump^a rule which has identical label and sequent in the premiss, or via the $[\,]_R$ rule, which creates the empty label. □

Lemma 22 (Truth Lemma). *If $\mathcal{S}$ is saturated and minimal and w is a component of $\mathcal{S}$ containing $\Gamma \stackrel{\Xi}{\Rightarrow} \Delta$, then for every formula A:*

1. *$A \in \Gamma$ implies $\mathfrak{M}^{\mathcal{S}}, w \Vdash A$*
2. *$A \in \Delta$ implies $\mathfrak{M}^{\mathcal{S}}, w \nVdash A$.*

Proof. By induction on the complexity of A for both statements simultaneously.

If A is atomic, an implication or $\bot$, then the statement follows as usual.

Suppose that $A = \langle\,]B$ and $A \in \Gamma$. We need to show that $\mathfrak{M}^{\mathcal{S}}, w \Vdash \langle\,]B$, i.e., that there is an $\alpha \in \mathcal{N}(w)$ with $\alpha \subseteq [\![B]\!]$.

Case 1: There is no formula of the shape $\langle\,]C$ in Δ. Then by definition we have $\emptyset \in \mathcal{N}(w)$. But $\emptyset \subseteq [\![B]\!]$, and hence $\mathfrak{M}^{\mathcal{S}}, w \Vdash \langle\,]B$.

Case 2: There is a formula of the shape $\langle\,]C$ in Δ. Then

$$\mathcal{N}(w) = \{\{v \in \mathcal{C}_w \mid \mathsf{an}(v) = \Sigma\} \mid \Sigma \in \ell[\mathcal{C}_w]\} \cup \{\mathcal{C}_w\} \cup \{\emptyset\}$$

Since $\langle\,]C \in \Delta$, by saturation we have $\langle \Rightarrow C\rangle^{\mathsf{u}} \in \Delta$ for that same C. Then again by saturation and the fact that $\langle\,]B \in \Gamma$ we have $\langle B \Rightarrow C\rangle^{\mathsf{f}} \in \Delta$, and hence also there are Ω, Θ with $[B, \Omega \stackrel{B}{\Rightarrow} \Theta] \in \Delta$. Thus the set $\alpha := \{v \in \mathcal{C}_w \mid \mathsf{an}(v) = B\}$ is nonempty. By Lem. 21 we have for every component $[\Gamma' \stackrel{B}{\Rightarrow} \Delta']$ from α that $B \in \Gamma'$. Hence by induction hypothesis $\alpha \subseteq [\![B]\!]$, and thus $\mathfrak{M}^{\mathcal{S}}, w \Vdash \langle\,]B$.

Suppose that $A = \langle\,]B$ and $A \in \Delta$. Then by definition of $\mathcal{N}$ we have that $\emptyset \notin \mathcal{N}(w)$. We need to show that $\mathfrak{M}^{\mathcal{S}} \nVdash \langle\,]B$.

Case 1: w has no children. Then $\mathcal{N}(w) = \emptyset$ and hence $\mathfrak{M}^{\mathcal{S}}, w \nVdash \langle\,]B$.

Case 2: w has a child. Then

$$\mathcal{N}(w) = (\{\{v \in \mathcal{C}_w \mid \mathsf{an}(v) = \Sigma\} \mid \Sigma \in \ell[\mathcal{C}_w]\} \cup \{\mathcal{C}_w\}) \setminus \{\emptyset\}$$

is non-empty. Let $\alpha \in \mathcal{N}(w)$. Then there is an annotation $\Sigma \in \ell[\mathcal{C}_w]$ with $\alpha = \{v \in \mathcal{C}_w \mid \mathsf{an}(v) = \Sigma\}$, or $\alpha = \mathcal{C}_w$. We need to show that there is a $v \in \alpha$ with $\mathfrak{M}^{\mathcal{S}}, v \Vdash \neg B$, i.e., that $\alpha \nsubseteq [\![B]\!]$. We show this for $\alpha = \{v \in \mathcal{C}_w \mid \mathsf{an}(v) = \Sigma\}$. The statement for the second case then follows from the fact that every such set is a subset of $\mathcal{C}_w$, and that for every $v \in \mathcal{C}_w$ we have $v \in \{x \in \mathcal{C}_w \mid \mathsf{an}(x) = \mathsf{an}(v)\}$. So suppose $\alpha = \{v \in \mathcal{C}_w \mid \mathsf{an}(v) = \Sigma\}$. The only ways a successor can be created is by the rules $[\,]^a_R$ or jump^a. If $\Sigma = \emptyset$, then there must be a formula $[\,]D \in \Delta$, since either the rule $[\,]^a_R$ or the rule l^a must have been applied. But then by saturation and the fact that both $[\,]D$ and $\langle\,]B$ are in Δ, we have that $[\stackrel{\emptyset}{\Rightarrow} B] \in \Delta$ as well. By induction hypothesis, at this world B is false, and hence we have $\alpha \nsubseteq [\![B]\!]$. If in contrast $\Sigma \neq \emptyset$, then the component must have been created by jump^a, and hence there must be a $\langle\Sigma \Rightarrow \Pi\rangle^{\mathsf{f}} \in \Delta$. Moreover, there must be a formula C with $\Sigma = C, \Sigma'$ such that $\langle\,]C \in \Gamma$ and $\langle\Sigma' \Rightarrow \Pi\rangle^{\mathsf{u}} \in \Delta$. Note that due to the shape of the rules we have $\Sigma' = \emptyset$. Then, since $\langle\,]B \in \Delta$, by saturation we also have $\langle \Rightarrow B\rangle^{\mathsf{u}} \in \Delta$, and together with the previous also $\langle \Sigma \Rightarrow B\rangle^{\mathsf{u}} \in \Delta$. Then by saturation we also have $[\Sigma, \Omega \stackrel{\Sigma}{\Rightarrow} B, \Theta] \in \Delta$ for some Ω, Θ. By the induction hypothesis, the formula B is false at this world, and since the annotation is Σ, we have $\alpha \not\subseteq [\![B]\!]$. So in any case $\mathfrak{M}^{\mathcal{S}}, w \not\Vdash \langle]B$.

Suppose that $A = [\,]B$ and $A \in \Gamma$. We need to show that $\mathfrak{M}^{\mathcal{S}}, w \Vdash [\,]B$. If $\mathcal{N}(w) = \emptyset$ this is trivial. Assume that $\mathcal{N}(w) \neq \emptyset$ and take $\alpha \in \mathcal{N}(w)$. If $\alpha = \emptyset$, again the statement is trivial, so assume $\alpha \neq \emptyset$. By definition of $\mathcal{N}(w)$ this means that $\alpha \subseteq \mathcal{C}_w$. The children of w are exactly the nested sequents $[\Omega \stackrel{\Sigma}{\Rightarrow} \Theta] \in \Delta$, and for these by saturation and $[\,]B \in \Gamma$ we have $B \in \Omega$. Thus by induction hypothesis we have $\mathcal{C}_w \subseteq [\![B]\!]$, and hence also $\alpha \subseteq [\![B]\!]$. Thus $\mathfrak{M}^{\mathcal{S}}, w \Vdash [\,]B$.

Finally, suppose that $A = [\,]B$ and $A \in \Delta$. We need to show that $\mathfrak{M}^{\mathcal{S}}, w \not\Vdash [\,]B$. By saturation and $[\,]B \in \Delta$ we have that $[\Omega \stackrel{\Sigma}{\Rightarrow} B, \Theta] \in \Delta$ for some Σ, Ω, Θ. By induction hypothesis at this world B is false, and since it is a member of $\mathcal{C}_w$ and $\mathcal{C}_w \in \mathcal{N}(w)$ we have that $\mathfrak{M}^{\mathcal{S}}, w \not\Vdash [\,]B$. □

Putting everything together we thus obtain:

Theorem 23. *Proof search on input* $\stackrel{\emptyset}{\Rightarrow} A$ *yields either a derivation or a saturated minimal nested sequent* $\mathcal{S}$ *with root* w *such that* $\mathfrak{M}^{\mathcal{S}}, w \not\Vdash A$. □

5 Extensions

A number of possible axiomatic extensions of biM have been considered in [1]. Here we highlight some of these, shown in Fig. 6 together with the corresponding semantic condition and the ordinary sequent rules beyond those of $\mathsf{G_{biM}}$ obtained by converting the axioms into rules and closing the rule set under cuts as in [12]. Note that in the bimodal system the condition that $\emptyset \notin \mathcal{N}(w)$ is expressed by the two different axioms $\mathsf{n}_{\langle]}$ and $\mathsf{d}_{\langle]}$. These extensions are particularly interesting from the point of view of *deontic logic*, since they capture different readings of the "ought implies can" principle, where $\langle]A$ is read as "one ought to bring about A" and $[\,]A$ as "necessarily A". Note that the presence of two modalities permits a more fine-grained analysis of this principle than is possible in monomodal logics. The plain and annotated nested sequent rules are shown in Fig. 7, the kleene'd versions are as expected, copying the nesting of rule $N_{\langle]}$ into the premiss. The corresponding nested sequent calculi are given by $\mathcal{N}_{\mathsf{M}} + P_{\langle]}$ for both the axioms $\mathsf{n}_{\langle]}$ and $\mathsf{d}_{\langle]}$, by $\mathcal{N}_{\mathsf{M}} + N_{\langle]}$ for the axiom $\mathsf{d}_{[\rangle}$, and by $\mathcal{N}_{\mathsf{M}} + N_{\langle]} + D_{[\,]}$ for the axiom $\mathsf{d}_{[\,]}$. Note that we use structural versions of the rules instead of additional logical rules to enable smoother cut elimination proofs.

Lemma 24. *The plain rules are sound for the logics with the corresponding frame conditions under the interpretation* ι.

Proof. For $P_{\langle]}$: Suppose that the interpretation of the conclusion is falsified in $\mathfrak{M}, w$, not due to the context. Then as in Thm. 5 there is a world $v \in W$ such that $\mathfrak{M}, v \not\Vdash \iota(\Gamma \Rightarrow \Delta)$. Since $\emptyset \notin \mathcal{N}(v)$ by assumption, we have $\mathfrak{M}, v \Vdash \neg\langle]\bot$. But

$\mathsf{n}_{\langle]}$: $\neg\langle]\bot$	$\emptyset \notin \mathcal{N}(w)$	$\dfrac{\Gamma, C \Rightarrow}{[]\Gamma, \langle]C \Rightarrow}$	
$\mathsf{d}_{\langle]}$: $\neg(\langle]A \wedge []\neg A)$	$\emptyset \notin \mathcal{N}(w)$	$\dfrac{\Gamma, C \Rightarrow}{[]\Gamma, \langle]C \Rightarrow}$	
$\mathsf{d}_{[\rangle}$: $[]A \to \langle]A$	$\mathcal{N}(w) \neq \emptyset$	$\dfrac{\Gamma \Rightarrow B}{[]\Gamma \Rightarrow \langle]B}$	
$\mathsf{d}_{[]}$: $\neg([]A \wedge []\neg A)$	$\exists \alpha \in \mathcal{N}(w).\, \alpha \neq \emptyset$	$\dfrac{\Gamma \Rightarrow}{[]\Gamma \Rightarrow}$ +	$\dfrac{\Gamma, C \Rightarrow}{[]\Gamma, \langle]C \Rightarrow}$

Fig. 6. Axiomatic extensions of the bimodal system from [1] with corresponding semantic conditions and direct translation into sequent rule.

$$\dfrac{\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle \Rightarrow \rangle\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta\}}\ P_{\langle]} \qquad \dfrac{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\Sigma \Rightarrow \Pi]\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi \rangle\}}\ N_{\langle]} \qquad \dfrac{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\ \Rightarrow]\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta\}}\ D_{[]}$$

$$\dfrac{\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle \Rightarrow \rangle^{\mathsf{u}}\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta\}}\ P^{a}_{\langle]} \quad \dfrac{\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi \rangle^{\mathsf{u}}, \langle \Sigma \Rightarrow \Pi \rangle^{\mathsf{f}}\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi \rangle^{\mathsf{u}}\}}\ N^{a}_{\langle]} \quad \dfrac{\mathcal{S}\{\Gamma \Rightarrow \Delta, [\overset{\emptyset}{\Rightarrow}]\}}{\mathcal{S}\{\Gamma \Rightarrow \Delta\}}\ D^{a}_{[]}$$

Fig. 7. The nested rules for the extensions in their plain and annotated versions.

since $\langle]\bot$ is equivalent to $\iota(\langle \Rightarrow \rangle)$, we have that $\mathfrak{M}, v$ falsifies $\iota(\Gamma \Rightarrow \Delta, \langle \Rightarrow \rangle)$. Hence $\mathfrak{M}, w$ falsifies the formula interpretation of the premiss.

For $N_{\langle]}$: Suppose that the interpretation of the conclusion is falsified in $\mathfrak{M}, w$, again not due to the context. Then there is a world v falsifying $\iota(\Gamma \Rightarrow \Delta, \langle \Sigma \Rightarrow \Pi \rangle)$. Hence in particular we have $\mathfrak{M}, v \not\Vdash \langle](\bigwedge \Sigma \to \bigvee \Pi)$. This together with the assumption that $\mathcal{N}(v) \neq \emptyset$ yields that there is $\alpha \in \mathcal{N}(v)$ and a world $x \in \alpha$ with $\mathfrak{M}, x \Vdash \bigwedge \Sigma \wedge \neg \bigvee \Pi$. Hence $\mathfrak{M}, v \not\Vdash [](\bigwedge \Sigma \to \bigvee \Pi)$, and so the interpretation of the premiss is falsified in $\mathfrak{M}, w$ as well.

The proof for $D_{[]}$ is as for nested sequents for modal logic KD. □

All the methods used to show completeness for the base calculus $\mathcal{N}_{\mathsf{M}}$ and its variants can be adapted to show completeness for the calculi for the extensions as well. First, it is straightforward to simulate the sequent rules of Fig. 6 in the plain versions of the corresponding nested calculi as in the proof of Thm. 6, giving:

Theorem 25. *The plain nested systems are complete.* □

Similarly, the cut elimination proof of Thm. 13 extends readily to the kleene'd versions of the calculi. The only non-trivial case is where the cut formula is contextual on the left in the conclusion of the rule $N_{\langle]}$. This is treated as the case for $\langle]^{i}_{L}$, giving:

Theorem 26. *The rules* $\mathsf{cut}_1, \mathsf{cut}_2$ *are admissible in the kleene'd systems.* □

Perhaps the most interesting extension is that for countermodel generation. For this we need to extend the saturation conditions of Def. 16 with the following, depending on whether the corresponding rule is in the system:

($P_{\langle]}$) There is a $\langle \Sigma \Rightarrow \Pi \rangle^{\mathsf{u}} \in \Delta$

($N_{\langle]}$) $\langle \Sigma \Rightarrow \Pi \rangle^{\mathsf{u}} \in \Delta$ implies there are Ω, Θ with $\langle \Sigma, \Omega \Rightarrow \Pi, \Theta \rangle^{\mathsf{f}} \in \Delta$

($D_{[\,]}$) $\Gamma \cup \Delta \neq \emptyset$ implies there are Σ, Ω, Θ with $[\Omega \overset{\Sigma}{\Rightarrow} \Pi] \in \Delta$.

Note that the condition ($D_{[\,]}$) incorporates a loop check, preventing an infinite sequence of new components. Because of this, for the system with the rules $D_{[\,]}$ and $N_{\langle]}$ we need to slightly adapt the definition of the neighbourhood function in the generated model (Def. 17), so that $\mathcal{N}(w)$ is defined as $(\mathcal{L}_w \cup \{\mathcal{C}_w\}) \setminus \{\emptyset\}$ if there is a formula $\langle]A \in \Delta$, otherwise as $\mathcal{L}_w \cup \{\mathcal{C}_w\} \cup \{\emptyset\}$ if $\Gamma \cup \Delta = \emptyset$ and $\{\{w\}\}$ if $\Gamma \cup \Delta = \emptyset$. This ensures that when a component has no successor, the semantical condition is still met and hence the constructed model is indeed a model for the logic. Adapting the proofs for the base case accordingly, we then obtain the analogue of Thm. 23:

Theorem 27. *Proof search in the annotated systems produces either a derivation or a saturated minimal nested sequent yielding a countermodel.* □

6 Implementation

A prototype implementation of proof search and countermodel construction in the basic system $\mathcal{N}^a_{\mathsf{M}}$ is available under `http://subsell.logic.at/bprover/nnProver/`. The core of the program is written in SWI Prolog. It recursively performs the backwards proof search of Def. 19, at every step either returning a labelled tree representing a derivation, or a saturated nested sequent. The result is converted into a Latex file containing either the derivation or the countermodel, the latter in the form of a tikz picture. The webinterface automatically typesets this file to produce a pdf containing the derivation or countermodel.

7 Conclusion

In this article, we presented the calculus $\mathcal{N}_{\mathsf{M}}$, complete with a syntactic cut elimination result, countermodel construction, an implementation and some extensions. This seems to be the first sequent-style calculus for the logic biM aka Brown's ability logic. Its main interest, however, lies in the fact that it provides the key for properly treating monotone non-normal modal logics in the nested sequent framework in that the inclusion of the modality $[]$ enables a formula interpretation and facilitates direct countermodel construction. As such it should serve as a foundation both for obtaining nested sequent calculi for extensions of monotone modal logic, and for a more detailed proof-theoretic analysis of normal modal logics making use of a more fine-grained analysis of the successor states in terms of the neighbourhood function.

In line with this, it would be very interesting to extend $\mathcal{N}_{\mathsf{M}}$ to modularly capture other axioms for $\langle]$ and $[]$, in particular those of the normal modal cube [3] and the modal tesseract [14]. Further, we are planning to adapt the countermodel construction to the logics of [5] to provide certificates for the underivability statements used in the non-monotonic calculus considered there.

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