Hypersequent Rules with Restricted Contexts for Propositional Modal Logics

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Abstract

As part of a general research programme into the expressive power of different generalisations of the sequent framework we investigate hypersequent calculi given by rules of the newly introduced format of hypersequent rules with context restrictions. The introduced rule format is used to prove uniform syntactic cut elimination, decidability and complexity results. We also introduce transformations between hypersequent rules of this format and Hilbert axioms, entailing a result about the limits of such rules. As case studies, we apply our methods to several modal logics and obtain e.g. a complexity-optimal decision procedure for the logic S5 and new calculi for the logic K4.2 as well as combinations of modal logics in the form of simply dependent bimodal logics.

Keywords: Structural Proof Theory, Hypersequents, Modal Logic, Hilbert Axioms, Cut Elimination, Decidability

1. Introduction

The automatic construction of reasoning systems and decision procedures from specifications for various logics is an important emerging area in the field of automated reasoning. Results in this area provide general decision procedures and complexity results applicable to specific logics in the spirit of Logic Engineering [48], and also yield deeper insights into strengths, weaknesses, and fundamental properties of different types of calculi used for reasoning systems. However, also from the perspective of producing such systems for specific logics investigating the connections between specifications and different frameworks is important, since this allows to choose the most efficient framework for the logic at hand. Beginnings of this research programme with its shift of focus from the investigation of calculi for specific logics towards general procedures and results for wide classes of logics were carried out e.g. in [35], where connections between Hilbert-style axioms for modal (tense) logics and display calculi where established, or in [16], where the connection between axioms for substructural logics and structural rules in a sequent or hypersequent framework was investigated.

Here we contribute to this research programme and investigate the framework of *hypersequent calculi* for extensions of classical propositional logic. Hypersequents offer one of the simplest generalisations of the original sequent framework and capture several logics for which no cut-free sequent or tableau system seems to exist. The prime example of such a logic of course is the modal logic S5, which so far has several different cut-free hypersequent calculi, but no cut-free sequent calculus. On the specification side, we take the logics to be given by a set of *Hilbert axioms*. This allows for a (almost) purely syntactic treatment independent of a basic semantics, and thus in principle also allows us to capture *non-normal* modal logics. The objective of

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our investigation is twofold. On the one hand we aim for correspondence results stating which specifications (i.e., Hilbert axioms) correspond to which rules in a hypersequent calculus. Such results would yield a method to construct hypersequent calculi for a certain class of specifications and conversely characterise those logics which can *not* be captured in a hypersequent calculus. On the other hand we would like to develop a general decision procedure where we can "plug in" the constructed calculi to obtain a decision procedure for the original logic.

Of course the kind of correspondence results and general decision procedures envisaged in this project demand general results about hypersequent calculi, which in turn necessitates a clarification of which kind of calculi we consider. To this aim we introduce the format of *hypersequent rules with context restrictions* which is general enough to capture many existing calculi, e.g. for modal logics $\mathsf{S5}$ [4, 54] and $\mathsf{S4.3}$ [31] as well as for modal logics without symmetry given by simple frame properties [37]. We obtain sufficient conditions for (syntactic) cut elimination, decidability, and complexity results for such systems. The results apply e.g. to the calculi for extensions of K or $\mathsf{K4}$ from [37]. We also show a correspondence between rules of our format and axioms of a certain syntactical form (Def. 5.25). This yields general decidability and complexity results for modal logics axiomatised this way, and as a byproduct finite axiomatisations for modal logics given by certain simple frame properties. Moreover, we can use the correspondence result to show limitative results about which logics can not be captured by hypersequent rules of the given format. As application we consider a number of case studies including a closer analysis of a recent calculus for $\mathsf{S5}$ together with a complexity-optimal decision procedure based on this calculus. Furthermore, we investigate a number of convergent or connected modal logics and construct a simple and apparently new hypersequent calculus for the logic $\mathsf{K4.2}$. A further case study treats combinations of modal logics in the form of *simply dependent bimodal logics* [24].

Related Work. This work is a continuation of previous work done in the *sequent framework* [38, 40, 41]. In particular, in [40] we considered the format of sequent rules with context restrictions on which the format of hypersequent rules with context restrictions is based and constructed rules from axioms. Limits of this rule format were shown in [41], where it is shown that the logic $\mathsf{S5}$ cannot be captured using a certain format of sequent rules. Perhaps the first result linking rules with axioms of a specific format in modal logics is [35], in which translations between *primitive* axioms for modal tense logics and structural rules of a *display calculus* in the style of [7] are given. The construction of the rules was later generalised in [21] to a wide class of logics including also intermediate and substructural logics. In contrast to these methods, the methods presented in this article do not rely on the existence of a residuum for the modal connectives and apply also to non-normal modal logics. The connection between Hilbert axioms and structural rules in the sequent or hypersequent framework was explored in [16, 17], which are among the basis for the present article. Subsequent work extended the construction of rules from axioms to logical rules and *intermediate logics* [19] or *paraconsistent logics* [18]. While the algorithm for the construction of rules from axioms given in [18] could in principle be used to treat modal axioms as well, the format of the axioms is very much geared towards axioms for paraconsistent logics, and the resulting rules are in the sequent framework instead of the hypersequent framework. The general proof of cut elimination given in Sec. 4 of the present article is based on the cut elimination procedure for modal extensions of the fuzzy logic MTL given in [20]. While for the propositional part of these logics cut elimination can be shown by applying general criteria such as *substitutivity* and *reductivity* [44], no general criteria for the modal part are given in [20].

Systematic constructions of rules specifically for *modal logics* from *semantic* specifications instead of Hilbert axioms were given e.g. in [58, 47]. In these works semantic characterisations in terms of *frame conditions* were taken as the starting point, and the resulting rules were structural rules in a *labelled sequent calculus*. While this automatically gives cut-free calculi, termination of proof search needs to be shown for every constructed calculus separately and there seem to be no general complexity results. Most relevant for the present article is the recent [37], where the rules for cut-free hypersequent calculi for modal logics are constructed from *simple frame conditions* in a systematic way. While there is some overlap between the logics which can be treated using this approach and the one of the present article, the characterisations of the logics (frame conditions vs. axioms) as well as the methods (semantic vs. syntactic) are different, and each of the approaches captures some logics which are not treated by the other. More details regarding this will be given in Sec. 6.1. Cut-free calculi for some standard modal and tense logics are also given in a systematic fashion in the framework of *nested sequents* or *tree-hypersequents*, see e.g. [33, 12, 52] or the recent [43] for modular calculi for intuitionistic modal logics. However, the focus in this area seems to be more on good calculi for the standard modal logics instead of procedures for the systematic construction of such calculi from axioms. A methodology for constructing and identifying sequent calculi satisfying the *bounded proof property* (a weaker version of the subformula property) was given in [8, 9] and later extended to the hypersequent framework in [10]. Since the constructed calculi are not expected to satisfy cut elimination, a much more general conversion of axioms into rules suffices in these works.

Plan of the paper. After fixing some notation in Sec. 2 we will extract the format of a hypersequent rule with context restrictions from some standard rules in Sec. 3. The following Sec. 4 then presents some general criteria for cut elimination and as an application of cut-free calculi in Sec. 4.1 a method of making the contraction rules admissible and a general decidability and complexity result. The connection between Hilbert axioms and hypersequent rules with context restrictions is explored in Sec. 5, including a general perspective on the formula interpretation of a hypersequent as well as translations from rules to axioms in Sec. 5.2 and vice versa in Sec. 5.3 for normal modal logics, followed by an extension to non-normal modal logics and restrictions in Sec. 5.4 and an application of these methods to show some limits of the rule format in Sec. 5.5. Finally, Sec. 6 contains a number of case studies, including modal logics given by simple frame properties in Sec. 6.1, modal logic $\mathsf{S5}$ in Sec. 6.2, convergent and connected modal logics in Sec. 6.3 and simply dependent bimodal logics in Sec. 6.4.

This article is a revised and substantially extended version of [39] containing more details and explanations, examples, full proofs and a number of new results. In particular: the complexity bound of Thm. 4.26 has been lowered from double exponential time to exponential space; Sec. 5.4 concerning non-normal modalities and context restrictions and Sec. 5.5 presenting a limitative result are new; the case studies have been extended to include Sec. 6.2 on $\mathsf{S5}$ containing a complexity optimal decision procedure, Sec. 6.3 with a discussion of convergent and connective modal logics containing a new hypersequent calculus for the logic $\mathsf{K4.2}$, and Sec. 6.4 containing a result about combining modal logics to simply dependent bimodal logics entailing an exponential time lower bound for the general decision procedure. The section on the calculus for the Logic of Uniform Deontic Frames from [55] is no longer included, since unfortunately there is a problem with the soundness of the rules.

2. Preliminaries and Notation

In the following we write $\mathbb{N}$ for $\{0, 1, 2, \dots\}$. We take $\mathcal{V}$ to be a countable set of propositional variables. The set of *boolean connectives* is $\Lambda_\mathsf{B} := \{\wedge, \vee, \to\}$. For a set $\Lambda \subseteq \Lambda_\mathsf{U} \cup \Lambda_\mathsf{B}$ with Λ_U a set of unary connectives the set $\mathcal{F}(\Lambda)$ of *formulae over* Λ is defined by $\mathcal{F}(\Lambda) \ni \varphi ::= p \mid \bot \mid \heartsuit\varphi \mid \varphi \circ \varphi$ with $p \in \mathcal{V}, \heartsuit \in \Lambda \cap \Lambda_\mathsf{U}$ and $\circ \in \Lambda \cap \Lambda_\mathsf{B}$. The connectives $\leftrightarrow$ and $\neg$ are introduced as abbreviations as usual. Connectives in Λ_U are called *modalities*. The set $\{\Box\} \cup \Lambda_\mathsf{B}$ is denoted $\Lambda_\Box$. We sometimes abbreviate $\neg\Box\neg$ to $\Diamond$ and use the *strong box* notation $\boxplus\varphi$ for $\varphi \wedge \Box\varphi$. For $F \subseteq \mathcal{F}(\Lambda)$ we write $\Lambda(F)$ for $\{\heartsuit\varphi : \heartsuit \in \Lambda \setminus \Lambda_\mathsf{B} \text{ and } \varphi \in F\} \cup \{\varphi \circ \psi : \circ \in \Lambda \cap \Lambda_\mathsf{B} \text{ and } \varphi, \psi \in F\}$. The *modal rank* of a formula φ, denoted $\mathsf{mrk}(\varphi)$, is the maximum nesting depth of modalities in φ, and its *complexity* $|\varphi|$ is the number of symbols occurring in it. Sequences $\varphi_1, \dots, \varphi_n$ of formulae are written $\vec{\varphi}$, and $||\vec{\varphi}||$ denotes the length of $\vec{\varphi}$. Similarly $*\varphi_1, \dots, *\varphi_n$ is written $*\vec{\varphi}$ for $* \in \Lambda$.

A *multiset* Γ over a set F of formulae is a function $F \to \mathbb{N}$ with finite support, and we write $\varphi \in \Gamma$ for $\Gamma(\varphi) > 0$. The *union* of multisets Γ and Δ is denoted by Γ, Δ and defined by $(\Gamma, \Delta)(\varphi) := \Gamma(\varphi) + \Delta(\varphi)$. We also write $\bigsqcup_{i=1}^n \Gamma_n$ for $\Gamma_1, \dots, \Gamma_n$ and φ for the multiset containing only one occurrence of φ. We write H for the hypersequent version of a standard context-sharing sequent calculus for classical logic [41] with the standard external and internal weakening and contraction rules [4], see Table 2. The standard modal rules of $\mathcal{R}_\mathsf{K}$, $\mathcal{R}_\mathsf{KT}$ and $\mathcal{R}_\mathsf{K4}$ are given in Table 3.

A Λ-*logic* is a set $\mathcal{L}$ of formulae over Λ closed under *modus ponens* (if $\varphi \in \mathcal{L}$ and $\varphi \to \psi \in \mathcal{L}$, then $\psi \in \mathcal{L}$) and *uniform substitution* (if $\varphi \in \mathcal{L}$, then $\varphi\sigma \in \mathcal{L}$ for every substitution $\sigma : \mathcal{V} \to \mathcal{F}(\Lambda)$) and containing classical propositional logic. A modality $\heartsuit$ of a logic $\mathcal{L}$ is *congruential* (also called *classical*) if $\mathcal{L}$ is closed under the rule of *congruence* (if $\varphi \leftrightarrow \psi \in \mathcal{L}$, then $\heartsuit\varphi \leftrightarrow \heartsuit\psi \in \mathcal{L}$) for $\heartsuit$, and *monotone* if $\mathcal{L}$ is closed under the rule of *monotonicity* (if $\varphi \to \psi \in \mathcal{L}$, then $\heartsuit\varphi \to \heartsuit\psi \in \mathcal{L}$). The modality $\heartsuit$ is *normal* if $\mathcal{L}$ contains the axiom $\heartsuit(p \to q) \land \heartsuit p \to \heartsuit q$ and is closed under the rule of *necessitation* (if $\varphi \in \mathcal{L}$, then $\heartsuit\varphi \in \mathcal{L}$). If every modality of a logic is congruential, resp. monotone, resp. normal we also call the logic itself congruential, resp. monotone, resp. normal. For a set $\mathcal{A}$ of formulae, $\mathcal{L}_{\mathcal{A}}$ is the smallest congruential Λ-logic containing $\mathcal{A}$. We call the formulae in $\mathcal{A}$ *axioms* of the logic $\mathcal{L}_{\mathcal{A}}$. For a Λ-logic $\mathcal{L}$ and $\varphi \in \mathcal{F}(\Lambda)$ we write $\mathcal{L} \oplus \varphi$ for the smallest congruential Λ-logic $\mathcal{L}'$ with $\mathcal{L} \cup \{\varphi\} \subseteq \mathcal{L}'$. We also write $\models_{\mathcal{L}} \varphi$ for $\varphi \in \mathcal{L}$. For the standard notions of modal logic see [11, 15]. In some results we use basic notions of complexity theory, see e.g. [49].

3. Hypersequent Rules with Restrictions

Our first goal is to identify a general format of rules in the hypersequent framework. Since this format should capture as many existing hypersequent calculi for modal logics as possible, we will extract its features from existing calculi such as the one for modal logic S5. This will allow us to define the general notions of applications of such rules, derivations and derivability using sets of rules. As a first consequence of the definition of the rule format we will obtain admissibility of weakening. Let us start with the basic notions.

The *hypersequent framework*, first employed in the area of modal logics in [45, 46, 53, 4], can be seen as one of the simplest extensions of the original *sequent framework* introduced by Gentzen [27]. While the main object of the latter are *sequents*, i.e., structures build from formulae in a specific way, the main idea in the hypersequent framework is to consider sequents not only in isolation, but also in the context of other sequents. This opens up the possibility of interaction between different sequents and adds another layer of structure (and with it expressivity) to the framework. Formally the definition is as follows.

Definition 3.1. Let $F \subseteq \mathcal{F}(\Lambda)$ be a set of formulae over a set Λ of connectives. A *sequent* over F is a pair of multisets over F, written as $\Gamma \Rightarrow \Delta$. We write $\mathcal{S}(F)$ for the set of all sequents over F. A *hypersequent* over F is a multiset $\mathcal{G}$ of sequents over F, written as $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$. Each of the sequents $\Gamma_i \Rightarrow \Delta_i$ is called a *component* of the hypersequent $\mathcal{G}$.

Perhaps the prime example of a modal logic for which this additional structure seems to be necessary is that of modal logic S5. While this logic can be captured using a sequent calculus with the *analytic cut rule* [57], so far no satisfactory fully analytic (i.e., cut-free) sequent calculus for it has been discovered. Moreover, it can be shown that under some restrictions to the format of the rules no sequent calculus for S5 can exist [41]. In one of the first proposed hypersequent calculi for this logic [4] the additional hypersequent structure is interpreted by

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \quad \mapsto \quad \Box(\bigwedge \Gamma_1 \to \bigvee \Delta_1) \lor \cdots \lor \Box(\bigwedge \Gamma_n \to \bigvee \Delta_n)$$

and the modal rules are given as follows, where $\mathcal{G}$ is an arbitrary *side hypersequent*:

$$\frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Box\varphi \Rightarrow \Delta}\ \mathsf{T} \qquad \frac{\mathcal{G} \mid \Box\Gamma \Rightarrow \varphi}{\mathcal{G} \mid \Box\Gamma \Rightarrow \Box\varphi}\ 4 \qquad \frac{\mathcal{G} \mid \Box\Gamma, \Sigma \Rightarrow \Box\Delta, \Pi}{\mathcal{G} \mid \Box\Gamma \Rightarrow \Box\Delta \mid \Sigma \Rightarrow \Pi}\ \mathsf{MS}$$

We will set the interpretation of the hypersequents aside for the moment (different interpretations will be considered in Sec. 5) and first concentrate on extracting the general format of rules. The above rule set for S5 illustrates a number of features common to most hypersequent calculi: usually at most one layer of logical connectives is introduced in the conclusion; only one component in the premisses is *active*, i.e., not part of the side hypersequent; the rules allow for an unrestricted side hypersequent; and each premiss may copy the whole or a syntactically restricted part of the context of each of the active components in the conclusion. Of the three rules the rule MS is the only rule which genuinely makes use of the hypersequent framework, since it involves more than one active component in the conclusion. In order to cast these observations into an abstract rule format we make use of the following notion, originally introduced in the sequent framework.

Definition 3.2 (Context restrictions [40, 41]). For $F \subseteq \mathcal{F}(\Lambda)$ the set of *context restrictions over* F is $\mathfrak{C}(F) := \{\langle F_1, F_2\rangle : F_1, F_2 \subseteq F\}$. If the set F is clear from the context we also write $\mathfrak{C}$ for $\mathfrak{C}(F)$. For a sequent $\Gamma \Rightarrow \Delta$ and a context restriction $\mathcal{C} = \langle F_1, F_2 \rangle$ the *restriction of* $\Gamma \Rightarrow \Delta$ *according to* $\mathcal{C}$ is the sequent $(\Gamma \Rightarrow \Delta)\lceil_{\mathcal{C}}$ defined as $\Gamma\lceil_{F_1} \Rightarrow \Delta\lceil_{F_2}$ where for a multiset Σ and $F \subseteq \mathcal{F}(\Lambda)$ the multiset $\Sigma\lceil_F$ contains exactly those formulae from Σ which are substitution instances of formulae in F (respecting multiplicities).

Example 3.3. The following context restrictions will feature extensively in the rest of the article.

1. Since no formula is a substitution instance of a formula in the empty set, the context restriction $\mathcal{C}_\emptyset := \langle \emptyset, \emptyset \rangle$ intuitively deletes the whole context: For every sequent $\Gamma \Rightarrow \Delta$ the sequent $(\Gamma \Rightarrow \Delta)\lceil_{\mathcal{C}_\emptyset}$ is the empty sequent $\Rightarrow$.
2. Since every formula is a substitution instance of the formula p, the context restriction $\mathcal{C}_{\mathsf{id}} := \langle \{p\}, \{p\} \rangle$ intuitively copies the whole context: For every sequent $\Gamma \Rightarrow \Delta$ the sequent $(\Gamma \Rightarrow \Delta)\lceil_{\mathcal{C}_{\mathsf{id}}}$ is $\Gamma \Rightarrow \Delta$.
3. Since only formulae of the form $\Box A$ are substitution instances of $\Box p$, the context restriction $\mathcal{C}_\Box := \langle \{\Box p\}, \emptyset \rangle$ copies only the boxed formulae on the left side of the context. E.g., we have $(\Box A, \Box B, \Box A, \neg C \Rightarrow \Box C)\lceil_{\mathcal{C}_\Box} = \Box A, \Box B, \Box A \Rightarrow$.

This allows us to formulate the restrictions on the contexts in the rules 4 and MS considered above. Since the active part of the premiss might contain parts of several active components of the conclusion (as in the rule MS), in every premiss we need one restriction for each of these. Together with the other features identified above we obtain the following notion.

Definition 3.4 (Rule with restrictions). An *m-premiss hypersequent rule with context restrictions*, written as

$$\frac{\big(\, \Gamma_1 \Rightarrow \Delta_1 \ ; \ \langle F_1^1, G_1^1 \rangle, \dots, \langle F_1^n, G_1^n \rangle \,\big) \qquad \dots \qquad \big(\, \Gamma_m \Rightarrow \Delta_m \ ; \ \langle F_m^1, G_m^1 \rangle, \dots, \langle F_m^n, G_m^n \rangle \,\big)}{\Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_n \Rightarrow \Pi_n}$$

is given by a natural number $n > 0$ together with:

- a sequence $\Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_n \Rightarrow \Pi_n$ called *principal part* consisting of sequents $\Sigma_i \Rightarrow \Pi_i \in \mathcal{S}(\Lambda(\mathcal{V}))$ of modalised variables
- a set of *premisses*, where each premiss $\big(\, \Gamma_i \Rightarrow \Delta_i; \langle F_i^1, G_i^1 \rangle, \dots, \langle F_i^n, G_i^n \rangle \,\big)$ consists of a sequent $\Gamma_i \Rightarrow \Delta_i \in \mathcal{S}(\mathcal{V})$ of propositional variables and a sequence $\langle F_i^1, G_i^1 \rangle, \dots, \langle F_i^n, G_i^n \rangle$ of context restrictions, one for each component of the principal part,

subject to the *variable condition*: every variable occurs at most once in the principal part and it occurs in the principal part whenever it occurs in the premisses.

While for technical reasons we take the principal part of a rule to be a sequence instead of a hypersequent, we stipulate that sets of rules are closed under permutation of the components in the principal part. Thus for all practical purposes the principal part can be seen as a hypersequent. Since the precise names of the variables occurring in a rule are not relevant we also stipulate that sets of rules are closed under renaming of the variables. For space reasons we may sometimes write the premisses of such a rule in set notation as $\{(\Gamma_i \Rightarrow \Delta_i \ ; \ \vec{\mathcal{C}_i}) \ : \ i \leq m\}$ where $\vec{\mathcal{C}_i}$ is the sequence $\langle F_i^1, G_i^1 \rangle, \dots, \langle F_i^n, G_i^n \rangle$ of context restrictions.

Before considering examples of rules with restrictions, note that the formulation of rules given in the previous definition does not make use of metavariables for formulae or sequents as tacitly used in the description of the rules for S5 above. We obtain such a more standard formulation by considering *applications* of hypersequent rules in our sense, given by substituting formulae for the variables occurring in the principal part and the sequents in the premisses of the rule and adding a side hypersequent as well as a context to each component of the principal part. The context restrictions associated with a premiss then determine which part of the context sequents from each active component in the conclusion of such an application is copied into the premiss. Formally:

Table 1: Examples of hypersequent rules with context restrictions and their applications

$\wedge_{\mathsf{L}}$	$\dfrac{(p, q \Rightarrow\ ;\ \mathcal{C}_{\mathsf{id}})}{p \wedge q \Rightarrow}$	$\dfrac{\mathcal{G} \mid \Gamma, \varphi, \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \wedge \psi \Rightarrow \Delta}$
K_n	$\dfrac{(p_1, \dots, p_n \Rightarrow q\ ;\ \mathcal{C}_\emptyset)}{\Box p_1, \dots, \Box p_n \Rightarrow \Box q}$	$\dfrac{\mathcal{G} \mid \varphi_1, \dots, \varphi_n \Rightarrow \psi}{\mathcal{G} \mid \Gamma, \Box\varphi_1, \dots, \Box\varphi_n \Rightarrow \Box\psi, \Delta}$
$\mathsf{4}_n$	$\dfrac{(p_1, \dots, p_n \Rightarrow q\ ;\ \mathcal{C}_\Box)}{\Box p_1, \dots, \Box p_n \Rightarrow q}$	$\dfrac{\mathcal{G} \mid \Box\Gamma, \varphi_1, \dots, \varphi_n \Rightarrow \psi}{\mathcal{G} \mid \Sigma, \Box\Gamma, \Box\varphi_1, \dots, \Box\varphi_n \Rightarrow \Box\psi, \Delta}$ (no formula in Σ is boxed)
MS	$\dfrac{(\ \Rightarrow\ ;\ \langle\{\Box p\}, \{\Box p\}\rangle, \mathcal{C}_{\mathsf{id}})}{\Rightarrow \mid \Rightarrow}$	$\dfrac{\mathcal{G} \mid \Box\Gamma, \Omega \Rightarrow \Box\Delta, \Xi}{\mathcal{G} \mid \Sigma, \Box\Gamma \Rightarrow \Box\Delta, \Pi \mid \Omega \Rightarrow \Xi}$ (no formula in $\Sigma \sqcup \Pi$ is boxed)
$\mathsf{5}$	$\dfrac{(p \Rightarrow\ ;\ \mathcal{C}_\emptyset, \mathcal{C}_{\mathsf{id}})}{\Box p \Rightarrow \mid \Rightarrow}$	$\dfrac{\mathcal{G} \mid \varphi, \Omega \Rightarrow \Xi}{\mathcal{G} \mid \Gamma, \Box\varphi \Rightarrow \Delta \mid \Omega \Rightarrow \Xi}$

Definition 3.5 (Rule application). An *application* of the m-premiss rule

$$\frac{\big\{(\Gamma_i \Rightarrow \Delta_i\ ;\ \langle F_i^1, G_i^1\rangle, \dots, \langle F_i^n, G_i^n\rangle)\ :\ i \le m\big\}}{\Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_n \Rightarrow \Pi_n}$$

is given by a substitution $\sigma : \mathcal{V} \to \mathcal{F}(\Lambda)$, a *side hypersequent* $\mathcal{G}$ and a sequence $\Omega_1 \Rightarrow \Upsilon_1 \mid \cdots \mid \Omega_n \Rightarrow \Upsilon_n$ of *context sequents* with $\Omega_i \Rightarrow \Upsilon_i \in \mathcal{S}(\mathcal{F}(\Lambda))$ for $i \le n$. It is written as

$$\frac{\Big\{\ \mathcal{G} \mid \Omega_1{\restriction_{F_i^1}}, \dots, \Omega_n{\restriction_{F_i^n}}, \Gamma_i\sigma \Rightarrow \Delta_i\sigma, \Upsilon_1{\restriction_{G_i^1}}, \dots, \Upsilon_n{\restriction_{G_i^n}}\ :\ i \le m\ \Big\}}{\mathcal{G} \mid \Omega_1, \Sigma_1\sigma \Rightarrow \Pi_1\sigma, \Upsilon_1 \mid \cdots \mid \Omega_n, \Sigma_n\sigma \Rightarrow \Pi_n\sigma, \Upsilon_n}\ .$$

Rules and applications of rules are also written inline using "/" to separate premisses and conclusion.

Example 3.6. The format of hypersequent rules with context restrictions encompasses most rules commonly used in hypersequent calculi for modal logics including the above mentioned rules for $\mathsf{S5}$, but also the standard hypersequent versions of the logical rules for the propositional connectives. Some examples of hypersequent rules with context restrictions together with exemplary applications are given in Table 1. The side conditions for the applications of the rules $\mathsf{4}_n$ and MS stem from the fact that the restrictions $\mathcal{C}_\Box$ resp. $\langle\{\Box p\}, \{\Box p\}\rangle$ copy all the boxed formulae from the respective sides of the context sequent into the premiss. The rule $\mathsf{5}$ is from a different hypersequent calculus for $\mathsf{S5}$ given in [54] (it is called $(\Box L)$ there).

The rules given in the previous example resp. Table 1 will serve as running examples throughout the article. In particular, the rules MS and $\mathsf{5}$ will be used to reconstruct and analyse different possibilities for a hypersequent calculus for the logic $\mathsf{S5}$.

We base our calculi on the standard hypersequent rules for the propositional connectives together with *initial hypersequents* or *axioms* ($\mathcal{A}$), the *cut rule* (Cut) and the standard *structural rules* of *external weakening* (EW), *internal weakening* (IW), *external contraction* (EC) and *internal contraction* (ICL and ICR) as given in Table 2 and write H for the set of these rules without the cut rule Cut and HCut for the set including this rule. While axioms, the cut rule and the structural rules are not rules with context restrictions, the notion of an application of these rules is the standard one. The notions of derivation and derivability now are defined in the usual way:

Definition 3.7 (Derivation, derivability). Let $\mathcal{R}$ be a set of hypersequent rules with context restrictions or structural rules, let $\mathcal{H}$ be a hypersequent and S a set of hypersequents. A *derivation of* $\mathcal{H}$ *in* $\mathcal{R}$ *from* S is a finite labelled tree with leaves labelled with *initial hypersequents* $\mathcal{G} \mid \Gamma, \varphi \Rightarrow \varphi, \Delta$ or elements of S and all other nodes labelled with a hypersequent which follows from the labels of its children by an application of a rule in $\mathcal{R}$. A hypersequent is *derivable in* $\mathcal{R}$ *from* S if there is a derivation of it. If S is empty and the rule

Table 2: The propositional and structural rules of H

$$\frac{(p \Rightarrow\ ;\ \mathcal{C}_{\mathsf{id}}) \quad (q \Rightarrow\ ;\ \mathcal{C}_{\mathsf{id}})}{p \vee q \Rightarrow} \vee_{\mathsf{L}} \qquad \frac{(p, q \Rightarrow\ ;\ \mathcal{C}_{\mathsf{id}})}{p \wedge q \Rightarrow} \wedge_{\mathsf{L}} \qquad \frac{(q \Rightarrow\ ;\ \mathcal{C}_{\mathsf{id}}) \quad (\ \Rightarrow p\ ;\ \mathcal{C}_{\mathsf{id}})}{p \to q \Rightarrow} \to_{\mathsf{L}} \qquad \frac{}{\bot \Rightarrow} \bot_{\mathsf{L}}$$

$$\frac{(\ \Rightarrow p, q\ ;\ \mathcal{C}_{\mathsf{id}})}{\Rightarrow p \vee q} \vee_{\mathsf{R}} \qquad \frac{(\ \Rightarrow p\ ;\ \mathcal{C}_{\mathsf{id}}) \quad (\ \Rightarrow q\ ;\ \mathcal{C}_{\mathsf{id}})}{\Rightarrow p \wedge q} \wedge_{\mathsf{R}} \qquad \frac{(p \Rightarrow q\ ;\ \mathcal{C}_{\mathsf{id}})}{\Rightarrow p \to q} \to_{\mathsf{R}}$$

$$\frac{}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta, \varphi} \mathcal{A} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi} \mathsf{IW} \qquad \frac{\mathcal{G} \mid \Gamma, \varphi, \varphi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta} \mathsf{ICL} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \mathsf{EC}$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi \quad \mathcal{G} \mid \Sigma, \varphi \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi} \mathsf{Cut} \qquad \frac{\mathcal{G}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \mathsf{EW} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi, \varphi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi} \mathsf{ICR}$$

Table 3: The standard modal rule sets

$$\frac{(p_1, \dots, p_n \Rightarrow q; \mathcal{C}_\emptyset)}{\Box p_1, \dots, \Box p_n \Rightarrow \Box q} \mathsf{K}_n \qquad \frac{(p_1, \dots, p_n \Rightarrow\ ; \mathcal{C}_\emptyset)}{\Box p_1, \dots, \Box p_n \Rightarrow} \mathsf{D}_n \qquad \frac{(p_1, \dots, p_n \Rightarrow\ ; \mathcal{C}_{\mathsf{id}})}{\Box p_1, \dots, \Box p_n \Rightarrow} \mathsf{T}_n \qquad \frac{(p_1, \dots, p_n \Rightarrow q; \mathcal{C}_\Box)}{\Box p_1, \dots, \Box p_n \Rightarrow \Box q} \mathsf{4}_n$$

$$\begin{array}{lll} \mathcal{R}_{\mathsf{K}} := \{\mathsf{K}_n : n \geq 0\} & \mathcal{R}_{\mathsf{KT}} := \mathcal{R}_{\mathsf{K}} \cup \{\mathsf{T}_n : n \geq 1\} & \mathcal{R}_{\mathsf{KD4}} := \mathcal{R}_{\mathsf{KD}} \cup \mathcal{R}_{\mathsf{K4}} \\ \mathcal{R}_{\mathsf{KD}} := \mathcal{R}_{\mathsf{K}} \cup \{\mathsf{D}_n : n \geq 1\} & \mathcal{R}_{\mathsf{K4}} := \mathcal{R}_{\mathsf{K}} \cup \{\mathsf{4}_n : n \geq 0\} & \mathcal{R}_{\mathsf{KT4}} := \mathcal{R}_{\mathsf{KT}} \cup \mathcal{R}_{\mathsf{K4}} \end{array}$$

set clear from the context we simply talk about a *derivation* and say that a hypersequent is *derivable.* The *depth of a derivation* is the depth of the underlying tree, i.e., the maximum number of nodes in a branch minus one. Finally, a hypersequent rule with context restrictions R is *derivable in* $\mathcal{R}$ if for all its applications the conclusion is derivable in $\mathcal{R}$ from the premisses and *admissible* if whenever the premisses are derivable in $\mathcal{R}$, then so is the conclusion.

By the definition of rule applications we now immediately obtain admissibility of weakening in the systems where these rules are not taken as primitive.

Lemma 3.8 (Admissibility of weakening)**.** *The external and internal weakening rules are depth-preserving admissible: If hypersequents $\mathcal{G}$ resp. $\mathcal{G} \mid \Gamma \Rightarrow \Delta$ are derivable with derivations of depth at most n not using* IW, EW*, then so are the hypersequents $\mathcal{G} \mid \Sigma \Rightarrow \Pi$ resp. $\mathcal{G} \mid \Gamma, \Omega \Rightarrow \Delta, \Xi$.*

Proof. By a standard induction on the depth of the derivation. ☐

In the following we are also going to consider extensions of the standard modal logics $\mathsf{K}, \mathsf{KD}, \mathsf{KT}, \mathsf{K4}$ and combinations. The hypersequent versions of the standard sequent rules for these logics (see e.g. [52, 59]) are given in Table 3. Note that the rules T_n permit to treat multiple formulae at the same time. While an application of the rule T_n could be replaced with n applications of the rule T_1, for the sake of uniformity of the rule sets we adopt this presentation.

Remark 3.9 (Multiple active components in a premiss). While most calculi make use of rules with only one active component per premiss, there are calculi which use rules with more than one active component. E.g., in the calculus for $\mathsf{S4.3}$ from [30] the possible choices of the principal formula of the standard right rule for $\Box$ in sequent calculi for $\mathsf{S4}$ are collected in a single rule to obtain a confluent calculus. The resulting rule $(\Rightarrow \Box)$ has applications as shown below left.

$$\frac{\mathcal{G} \mid \Box\Gamma \Rightarrow \varphi_1 \mid \cdots \mid \Box\Gamma \Rightarrow \varphi_n}{\mathcal{G} \mid \Sigma, \Box\Gamma \Rightarrow \Pi, \Box\varphi_1, \dots \Box\varphi_n} (\Rightarrow \Box) \qquad \frac{\mathcal{G} \mid \Box\Gamma \Rightarrow \varphi_1 \quad \dots \quad \mathcal{G} \mid \Box\Gamma \Rightarrow \varphi_n}{\mathcal{G} \mid \Sigma, \Box\Gamma \Rightarrow \Pi, \Box\varphi_1, \dots, \Box\varphi_n}$$

However, using external weakening and contraction this rule can be seen to be equivalent to the rule with applications as shown above right, where the active components in the premiss are distributed over different premisses. The latter rule in turn can be simplified to the rule with restrictions $(\Rightarrow p; \mathcal{C}_\Box)/ \Rightarrow \Box p$.

In general, consider a rule R with principal part $\mathcal{H}$ and premisses $\mathcal{P} \cup \{P\}$, where the premiss P has active components $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$. In the presence of external weakening and contraction this rule is interderivable with the set $\{R_1, \dots, R_n\}$ of rules, where for $1 \leq i \leq n$ the rule R_i differs from R only in that instead of premiss P with active components $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ it contains a premiss with the single active component $\Gamma_i \Rightarrow \Delta_i$. The derivation of R_i using R is shown below left and that of R using the R_i below right (writing double lines for repeated applications of the same rule).

$$\dfrac{\dfrac{\mathcal{G} \mid \Gamma_i \Rightarrow \Delta_i}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n}\,\mathsf{EW} \qquad \mathcal{P}}{\mathcal{G} \mid \mathcal{H}}\,R \qquad\qquad \begin{array}{c} \dfrac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \qquad \mathcal{P}}{\mathcal{G} \mid \mathcal{H} \mid \Gamma_2 \Rightarrow \Delta_2 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n}\,R_1 \\ \vdots \\ \dfrac{\dfrac{\mathcal{G} \mid \mathcal{H} \mid \cdots \mid \mathcal{H} \mid \Gamma_n \Rightarrow \Delta_n \qquad \mathcal{P}}{\mathcal{G} \mid \mathcal{H} \mid \cdots \mid \mathcal{H} \mid \mathcal{H}}\,R_n}{\mathcal{G} \mid \mathcal{H}}\,\mathsf{EC} \end{array}$$

By iterating this process every rule with more than one active component in a premiss can be seen to be equivalent (modulo the structural rules) to a rule with at most one active component per premiss. Thus in terms of expressivity the stipulation that every premiss of a rule with restrictions contains only one active component is not a restriction.

4. Cut Elimination and Applications

Now that we have identified a general rule format, the next step is to identify criteria for cut elimination in such calculi and prove a general cut elimination result. For this we will go through a suitable cut elimination proof and extract conditions which ensure that this particular proof goes through. After this in Sec. 4.1 we will consider slight modifications of the resulting cut-free calculi which ensure admissibility of the contraction rules and lead to general decidability and complexity results.

We obtain sufficient criteria for cut elimination by generalising the cut elimination proof in [20]. The cut elimination strategy is to permute a cut into the premisses of the last applied rule on the left until the cut formula is principal in the last applied rule. Then the cut is permuted into the premisses on the right until it is principal here as well, in which case it is reduced to cuts on formulae of smaller complexity. To state the condition used to reduce principal cuts we use the notion of a *cut between rules*, where intuitively a new rule is constructed from two rules by cutting their conclusions on a formula $\heartsuit p$ and eliminating p from the premisses by cutting on p in all possible ways (compare [38, 40] for the sequent case). In addition, for permuting the cut into the context on the right we need a condition on the context restrictions which ensures that whenever the cut formula satisfies a context restriction, then so does the whole left premiss of the cut, i.e., that we can permute *mixed cuts* upwards. To make these notions precise, define the *union* of two sequences $\vec{\mathcal{C}} = \langle F_1, G_1\rangle, \dots, \langle F_n, G_n\rangle$ and $\vec{\mathcal{D}} = \langle F'_1, G'_1\rangle, \dots, \langle F'_n, G'_n\rangle$ of restrictions component-wise as the sequence of restrictions $\vec{\mathcal{C}} \cup \vec{\mathcal{D}} := \langle F_1 \cup F'_1, G_1 \cup G'_1\rangle, \dots, \langle F_n \cup F'_n, G_n \cup G'_n\rangle$.

Definition 4.1 (Cuts between rules, principal-cut closed, mixed-cut permuting). For sets $\mathcal{P}_1, \mathcal{P}_2$ of premisses and rules $R_1 = \mathcal{P}_1/\Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_{n-1} \Rightarrow \Pi_{n-1} \mid \Sigma_n \Rightarrow \Pi_n, \heartsuit p$ and $R_2 = \mathcal{P}_2/\heartsuit p, \Omega_1 \Rightarrow \Theta_1 \mid \Omega_2 \Rightarrow \Theta_2 \mid \cdots \mid \Omega_k \Rightarrow \Theta_k$ the *cut between* R_1 *and* R_2 on (the displayed occurrence of) $\heartsuit p$ is the rule $\mathsf{cut}(R_1, R_2, \heartsuit p)$ given by

$$\dfrac{\Big\{ (\Gamma, \Gamma' \Rightarrow \Delta, \Delta' \,;\, \vec{\mathcal{C}} \cup \vec{\mathcal{D}}) \;:\; (\Gamma \Rightarrow \Delta, p \,;\, \vec{\mathcal{C}}), (p, \Gamma' \Rightarrow \Delta' \,;\, \vec{\mathcal{D}}) \in \mathcal{P} \Big\} \quad \cup \quad \Big\{ (\Gamma \Rightarrow \Delta \,;\, \vec{\mathcal{C}}) \in \mathcal{P} \;:\; p \notin \Gamma, \Delta \Big\}}{\Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_{n-1} \Rightarrow \Pi_{n-1} \mid \Sigma_n, \Omega_1 \Rightarrow \Pi_n, \Theta_1 \mid \Omega_2 \Rightarrow \Theta_2 \mid \dots \Omega_k \Rightarrow \Theta_k}$$

where the set $\mathcal{P}$ contains the premisses of R_1 and R_2 combined in the form

$$\mathcal{P} := \Big\{ (\Gamma \Rightarrow \Delta \ ; \ \vec{\mathcal{C}}, \mathcal{C}_\emptyset, \overset{(k-1)\text{-times}}{\dots}, \mathcal{C}_\emptyset) \ : \ (\Gamma \Rightarrow \Delta \ ; \ \vec{\mathcal{C}}) \in \mathcal{P}_1 \Big\}$$
$$\cup \Big\{ (\Gamma \Rightarrow \Delta \ ; \ \mathcal{C}_\emptyset, \overset{(n-1)\text{-times}}{\dots}, \mathcal{C}_\emptyset, \vec{\mathcal{D}}) \ : \ (\Gamma \Rightarrow \Delta \ ; \ \vec{\mathcal{C}}) \in \mathcal{P}_2 \Big\} \ .$$

A set $\mathcal{R}$ of rules is *principal-cut closed* if it is closed under the addition of cuts between rules. It is *mixed-cut permuting* if for all $R_1, R_2 \in \mathcal{R}$: if $\Gamma \Rightarrow \Delta, \heartsuit p$ is a component of the principal part of R_1 and $(\heartsuit p \Rightarrow\ ){\restriction_{\mathcal{C}}} = \ \heartsuit p \Rightarrow$ for a restriction $\mathcal{C}$ of R_2, then $(\Gamma \Rightarrow \Delta){\restriction_{\mathcal{C}}} = \ \Gamma \Rightarrow \Delta$ and $(\Sigma \Rightarrow \Pi){\restriction_{\mathcal{D}}}{\restriction_{\mathcal{C}}} = \ (\Sigma \Rightarrow \Pi){\restriction_{\mathcal{D}}}$ for every restriction $\mathcal{D}$ for this component in R_1 and for every sequent $\Sigma \Rightarrow \Pi$.

Example 4.2. 1. The cut between the rules K_n and K_{m+1} as shown below left and middle on the formula $\Box q$ is the rule $\mathsf{cut}(\mathsf{K}_n, \mathsf{K}_{m+1}, \Box q)$ given below right.

$$\frac{(p_1, \dots, p_n \Rightarrow q \ ; \ \mathcal{C}_\emptyset)}{\Box p_1, \dots, \Box p_n \Rightarrow \Box q}\ \mathsf{K}_n \qquad \frac{(q, q_1, \dots, q_m \Rightarrow r) \ ; \ \mathcal{C}_\emptyset)}{\Box q, \Box q_1, \dots, \Box q_m \Rightarrow \Box r}\ \mathsf{K}_{m+1} \qquad \frac{(p_1, \dots, p_n, q_1, \dots, q_m \Rightarrow r \ ; \ \mathcal{C}_\emptyset)}{\Box p_1, \dots, \Box p_n, \Box q_1, \dots, \Box q_m \Rightarrow \Box r}$$

Since modulo renaming of the variables this is the rule K_{n+m}, the standard rule set $\mathcal{R}_\mathsf{K} = \{\mathsf{K}_n : n \in \mathbb{N}\}$ for modal logic K from Table 3 is principal-cut closed. In contrast, the rule set consisting only of the single rule K_2 is not principal-cut closed, since it does not contain the rule K_3.

2. The cut between the rules K_n and $\mathsf{5}$ as shown below left and middle on the formula $\Box q$ is the rule $\mathsf{cut}(\mathsf{K}_n, \mathsf{5}, \Box q)$ given below right. We also call this rule $\mathsf{5}_n$.

$$\frac{(p_1, \dots, p_n \Rightarrow q \ ; \ \mathcal{C}_\emptyset)}{\Box p_1, \dots, \Box p_n \Rightarrow \Box q}\ \mathsf{K}_n \qquad \frac{(q \Rightarrow \ ; \ \mathcal{C}_\emptyset, \mathcal{C}_{\mathsf{id}})}{\Box q \Rightarrow \mid \Rightarrow}\ \mathsf{5} \qquad \frac{(p_1, \dots, p_n \Rightarrow \ ; \ \mathcal{C}_\emptyset, \mathcal{C}_{\mathsf{id}})}{\Box p_1, \dots, \Box p_n \Rightarrow \mid \Rightarrow}\ \mathsf{5}_n$$

Applications of this rule have the form $\mathcal{G} \mid \varphi_1, \dots, \varphi_n, \Gamma \Rightarrow \Delta \ / \ \mathcal{G} \mid \Sigma, \Box\varphi_1, \dots, \Box\varphi_n \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta$. It is straightforward to see that the rule set $\mathcal{R}_{\mathsf{KT5}} := \mathcal{R}_{\mathsf{KT}} \cup \{\mathsf{5}_n : n \geq 1\}$ is principal-cut closed. Note that this produces precisely the multi-set-based versions of the rules for modal logic $\mathsf{S5}$ from [37].

3. The cut between the rules $\mathsf{4}_n$ and $\mathsf{5}$ as shown below left and middle on $\Box q$ is the rule $\mathsf{cut}(\mathsf{4}_n, \mathsf{5}, \Box q)$ given below right. We also call this rule $\mathsf{45}_n$.

$$\frac{(p_1, \dots, p_n \Rightarrow q \ ; \ \mathcal{C}_\Box)}{\Box p_1, \dots, \Box p_n \Rightarrow \Box q}\ \mathsf{4}_n \qquad \frac{(q \Rightarrow \ ; \ \mathcal{C}_\emptyset, \mathcal{C}_{\mathsf{id}})}{\Box q \Rightarrow \mid \Rightarrow}\ \mathsf{5} \qquad \frac{(p_1, \dots, p_n \Rightarrow \ ; \ \mathcal{C}_\Box, \mathcal{C}_{\mathsf{id}})}{\Box p_1, \dots, \Box p_n \Rightarrow \mid \Rightarrow}\ \mathsf{45}_n$$

Applications of this rule have the form $\mathcal{G} \mid \Box\Sigma, \varphi_1, \dots, \varphi_n, \Gamma \Rightarrow \Delta \ / \ \mathcal{G} \mid \Omega, \Box\Sigma, \Box\varphi_1, \dots, \Box\varphi_n \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta$, where no formula in Ω is boxed. Again, the rule set $\mathcal{R}_{\mathsf{KT45}} := \mathcal{R}_{\mathsf{KT4}} \cup \{\mathsf{45}_n : n \geq 0\}$ is principal-cut closed. Note that starting with the rule $\mathsf{4}_0$ this produces the rule $\mathsf{45}_0$, which is precisely the version of the modal splitting rule for modal logic $\mathsf{S5}$ from [36].

Example 4.3. The rule sets $\mathcal{R}_\mathsf{K}$, $\mathcal{R}_{\mathsf{KT5}}$ and $\mathcal{R}_{\mathsf{KT45}}$ from the previous example are also mixed-cut permuting. As one of the more interesting cases consider the rules $\mathsf{4}_n = (\vec{p} \Rightarrow q \ ; \ \mathcal{C}_\Box) \ / \ \Box\vec{p} \Rightarrow \Box q$ and $\mathsf{45}_m = (\vec{r} \Rightarrow \ ; \ \mathcal{C}_\Box, \mathcal{C}_{\mathsf{id}}) \ / \ \Box\vec{r} \Rightarrow \mid \Rightarrow$ from $\mathcal{R}_{\mathsf{KT45}}$. The sequent $\Box\vec{p} \Rightarrow \Box q$ is part of the principal part of $\mathsf{4}_n$ and we have $(\Box q \Rightarrow\ ){\restriction_{\mathcal{C}_\Box}} = \ \Box q \Rightarrow$ with the restriction $\mathcal{C}_\Box$ occurring in $\mathsf{45}_m$. Thus we need to check first that $(\Box\vec{p} \Rightarrow\ ){\restriction_{\mathcal{C}_\Box}} = \ \Box\vec{p} \Rightarrow$, which obviously is the case. Furthermore, the only restriction of $\mathsf{4}_n$ which refers to the component $\Box\vec{p} \Rightarrow \Box q$ of the principal part is the restriction $\mathcal{C}_\Box$, so we also need to check that for every sequent $\Sigma \Rightarrow \Pi$ we have $(\Sigma \Rightarrow \Pi){\restriction_{\mathcal{C}_\Box}}{\restriction_{\mathcal{C}_\Box}} = \ (\Sigma \Rightarrow \Pi){\restriction_{\mathcal{C}_\Box}}$ which again is the case. Together these two properties ensure that a cut on $\Box q$ occurring in the principal part of $\mathsf{4}_n$ and as part of the context in the left component of $\mathsf{45}_m$ can be permuted into the premiss of the rule $\mathsf{45}_m$.

As a non-example, consider the rule set containing the rules $\mathcal{R}_{\mathsf{K4}}$ and the single additional rule $\{(\Rightarrow q; \langle\{\Box p\}, \{\Box p\}\rangle)\} / \ \Rightarrow \Box q$. It is not mixed-cut permuting since the principal part of the additional rule contains the component $\Rightarrow \Box q$ and for the restriction $\mathcal{C}_\Box$ occurring in the rule $\mathsf{4}_n$ we have $(\Box q \Rightarrow\ ){\restriction_{\mathcal{C}_\Box}} = \Box q \Rightarrow$, but e.g. for the sequent $\Box\Sigma \Rightarrow \Box\Pi$ we have $(\Box\Sigma \Rightarrow \Box\Pi){\restriction_{\langle\{\Box p\},\{\Box p\}\rangle}}{\restriction_{\mathcal{C}_\Box}} = (\Box\Sigma \Rightarrow \Box\Pi){\restriction_{\mathcal{C}_\Box}} = \Box\Sigma \Rightarrow$, which is not the same as $(\Box\Sigma \Rightarrow \Box\Pi){\restriction_{\langle\{\Box p\},\{\Box p\}\rangle}} = \Box\Sigma \Rightarrow \Box\Pi$.

In the sequent case, the analogue of the principal-cut closure condition for rules with one principal formula is known as *reductivity* [22] or *coherence* [5], and it corresponds to Belnap's condition $C8$ for display calculi [7]. The two properties of Def. 4.1 ensure that we can eliminate topmost instances of a restricted version of multicut, where the cut formula occurs only once in the left premiss (and is principal in the last applied rule there), but several times in several components in the right premiss, by induction on the depth of the derivation of the right premiss and the maximal complexity of a cut formula occurring in the whole derivation. Allowing the cut formula to occur more than once on the right is necessary due to the internal and external contraction rules. The fact that several instances of the cut formula in the right premiss of such a restricted multicut can be principal also is the reason why we take the cuts between rules of a principal-cut closed rule set to be *in* the rule set and not just derivable: we need to be able to replace iterated cuts by a single rule from the rule set. To give the formal argument we use the following standard definition.

Definition 4.4 (Cut-rank). The *cut-rank* of a derivation $\mathcal{D}$ is the maximal complexity of the cut formulae occurring in $\mathcal{D}$ and is denoted by $\rho(\mathcal{D})$.

In the following we abbreviate m occurrences $\varphi, \ldots, \varphi$ of formulae resp. $\Gamma, \ldots, \Gamma$ of multisets resp. $\mathcal{G} \mid \cdots \mid \mathcal{G}$ of hypersequents to φ^m resp. Γ^m resp. $\mathcal{G}^m$.

Lemma 4.5 (Shift right)**.** *Let* $\mathsf{H}\mathcal{R}$ *be principal-cut closed and mixed-cut permuting. Assume in* $\mathsf{H}\mathcal{R}\mathsf{Cut}$ *we have derivations*

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots\end{matrix}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \varphi}\ R \qquad \textit{and} \qquad \begin{matrix}\mathcal{D}_2\\ \vdots\\ \mathcal{H} \mid \Sigma_1, \varphi^{m_1} \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_n, \varphi^{m_n} \Rightarrow \Pi_n\end{matrix}$$

such that φ *is principal in the application of* R *and* $\rho(\mathcal{D}_1) < |\varphi| > \rho(\mathcal{D}_2)$*. Then there is a derivation* $\mathcal{D}$ *in* $\mathsf{H}\mathcal{R}\mathsf{Cut}$ *of the hypersequent*

$$\mathcal{G} \mid \mathcal{H} \mid \Gamma, \Sigma_1 \Rightarrow \Delta, \Pi_1 \mid \cdots \mid \Gamma, \Sigma_n \Rightarrow \Delta, \Pi_n$$

with $\rho(\mathcal{D}) < |\varphi|$.

Proof. By induction on the depth of $\mathcal{D}_2$. The idea is to permute the (multi-)cut into the premisses of the last applied rule in $\mathcal{D}_2$. If the last applied rule in $\mathcal{D}_2$ was one of $\mathsf{ICL}, \mathsf{ICR}, \mathsf{IW}, \mathsf{EC}, \mathsf{EW}, \mathsf{Cut}, \mathcal{A}$, then the cut is permuted into its premisses or replaced with applications of IW, EW. Otherwise, w.l.o.g. assume that every formula in the component $\Gamma \Rightarrow \Delta, \varphi$ of the conclusion of the rule R occurs in at least one of its premisses (formulae for which this is not the case can be omitted now and added back later using IW). Let Q be the last applied rule in $\mathcal{D}_2$. Taking ℓ as the number of premisses of this rule, for $j \leq \ell$ the j-th premiss can be written as

$$\mathcal{H}_j \mid \Omega_{1,j}, \varphi^{s_{1,j}} \Rightarrow \Theta_{1,j} \mid \cdots \mid \Omega_{k_j,j}, \varphi^{s_{k_j,j}} \Rightarrow \Theta_{k_j,j}$$

where the displayed $\varphi^{i,j}$ are the occurrences of φ propagated in the context of the application of Q. Using the induction hypothesis we have for $j \leq \ell$ derivations $\mathcal{E}_j$ of

$$\mathcal{I}_j := \mathcal{G} \mid \mathcal{H}_j \mid \Gamma, \Omega_{1,j} \Rightarrow \Delta, \Theta_{1,j} \mid \cdots \mid \Gamma, \Omega_{k_j,j} \Rightarrow \Delta, \Theta_{k_j,j}$$

with $\rho(\mathcal{E}_j) < |\varphi|$. Since the rule set $\mathsf{H}\mathcal{R}$ is mixed-cut permuting, the formulae in $\Gamma \Rightarrow \Delta$ satisfy the context restrictions associated with the different premisses, and we can apply the rule Q (possibly followed by IW and / or contractions) to the premisses $\mathcal{I}_j$ to obtain the conclusion (with $m'_i \leq m_i$)

$$\mathcal{G} \mid \mathcal{H} \mid \Gamma, \varphi^{m'_1}, \Sigma_1 \Rightarrow \Delta, \Pi_1 \mid \cdots \mid \Gamma, \varphi^{m'_n}, \Sigma_n \Rightarrow \Delta, \Pi_n$$

where the remaining displayed occurrences of φ are principal in the application of Q. Furthermore, we know that φ is principal in the application of the rule R, and hence by the principal-cut closure condition (or the standard transformations in the propositional case) there is a (possibly trivial) rule $Q' \in \mathsf{H}\mathcal{R}$ which allows to derive the conclusion

$$\mathcal{G}^2 \mid \mathcal{H} \mid \Gamma^2, \varphi^{m'_1 - 1}, \Sigma_1 \Rightarrow \Delta^2, \Pi_1 \mid \Sigma_2, \varphi^{m'_2} \Rightarrow \Pi_2 \mid \cdots \mid \Sigma_n, \varphi^{m_n} \Rightarrow \Pi_n$$

from premisses obtained by applying Cut to the premisses of R and the premisses $\mathcal{I}_j$. Since φ was principal in the conclusion of R, these newly introduced cuts are on proper subformulae of φ and hence on formulae of complexity strictly less than $|\varphi|$. Moreover, from the definition of principal-cut closure we obtain that the remaining displayed occurrences of φ are principal also in the application of Q'. Thus, repeating the argument for every single displayed occurrence of φ we obtain a rule $Q'' \in \mathsf{H}\mathcal{R}$ (again possibly trivial) which allows to derive the conclusion

$$\mathcal{G}^{m_1'+\cdots+m_n'} \mid \mathcal{H} \mid \Gamma^{m_1'}, \Sigma_1 \Rightarrow \Delta^{m_1'}, \Pi_1 \mid \ldots \Gamma^{m_n'}, \Sigma_n \Rightarrow \Delta^{m_n'}, \Pi_n$$

from premisses obtained by applying Cut to the premisses of R and the premisses $\mathcal{I}_j$ on formulae of complexity strictly less than $|\varphi|$. Finally, applying internal and external contractions yields the desired derivation $\mathcal{D}$ of $\mathcal{G} \mid \mathcal{H} \mid \Gamma, \Sigma_1 \Rightarrow \Delta, \Pi_1 \mid \cdots \mid \Gamma, \Sigma_n \Rightarrow \Delta, \Pi_n$ with cut rank strictly less than $|\varphi|$. $\square$

To get to a stage where we can apply the previous lemma we need to ensure that not more than one instance of the cut formula is principal in the last applied rule of the derivation of the left premiss of the cut. The following two conditions will guarantee that this is the case.

Definition 4.6. A rule set $\mathcal{R}$ is *right-contraction closed* if applications of internal contraction right (ICR in Table 2) to the conclusion of a rule are derived by internal contractions on the premisses of that rule followed by one rule from $\mathcal{R}$. It is *single-component right* if the principal part of no rule contains $\Gamma \Rightarrow \Delta, \heartsuit p \mid \Sigma \Rightarrow \Pi, \heartsuit q$ for $\heartsuit \in \Lambda$ and $p, q \in \mathcal{V}$.

Example 4.7.
1. The standard modal rule sets from Table 3 as well as the rule sets H and $\mathcal{R}_{\mathsf{KT5}}$ and $\mathcal{R}_{\mathsf{KT45}}$ from Ex. 4.2 are trivially right-contraction closed since the right sides of the sequents in the principal parts contain at most one formula.
2. The rule set containing the single rule $\{(\Rightarrow p_1; \mathcal{C}_\Box), (\Rightarrow p_2; \mathcal{C}_\Box)\}/ \Rightarrow \Box p_1, \Box p_2$ from Remark 3.9 is not right-contraction closed, since the result of applying internal contraction right to its conclusion is not derivable using internal contraction followed by the rule itself. It becomes derivable if we add the rule $\mathsf{4}_0$ to the rule set, and hence the rule set containing these two rules is right-contraction closed.
3. The rule sets H as well as $\mathcal{R}_{\mathsf{K}}$ and $\mathcal{R}_{\mathsf{K4}}$ are trivially single-component right since their principal parts contain only one component.
4. The rule sets $\mathcal{R}_{\mathsf{KT5}}$ and $\mathcal{R}_{\mathsf{KT45}}$ from Ex. 4.2 are single-component right since no component of the principal part in the rules $\mathsf{5}_n$ resp. $\mathsf{45}_n$ introduces a boxed formula on the right hand side.
5. The rule $\{(\Rightarrow q; \mathcal{C}_\Box, \mathcal{C}_\emptyset), (\Rightarrow p; \mathcal{C}_\emptyset, \mathcal{C}_\Box)\}/ \Rightarrow \Box p \mid \Rightarrow \Box q$ is not single-component right.

The notion of right-contraction closed rule sets is the hypersequent version of the restriction of the notion of *contraction-closed* rule sets from [40] resp. rule sets satisfying the *closure condition* from [47] to the right hand side only. Obviously, if the rule set is single-component right, then the cut formula is not principal in more than one component in the left premiss of a cut. Furthermore, right-contraction closure prevents the cut formula from occurring twice in a single component of the principal part:

Lemma 4.8. *Let $\mathcal{R}$ be right-contraction closed and single-component right. Then whenever there is a derivation $\mathcal{D}$ of a hypersequent $\mathcal{G}$ in $\mathsf{HCut}\mathcal{R}$, then there is a derivation $\mathcal{D}'$ of $\mathcal{G}$ in $\mathsf{HCut}\mathcal{R}$ with $\rho(\mathcal{D}') \leq \rho(\mathcal{D})$ in which in every application of a rule from $\mathcal{R}$ the right hand sides of the principal part are fully contracted, i.e., the right contraction rule ICR cannot be applied to the principal part.*

Proof. We show by induction on n: Suppose there is a derivation of $\mathcal{G}$ in $\mathsf{HCut}\mathcal{R}$ with the property (P_n): whenever the principal part of a rule application contains a component $\Gamma \Rightarrow \Delta, \varphi, \varphi$, then φ has complexity at most n. Then there is a derivation of $\mathcal{G}$ in $\mathsf{HCut}\mathcal{R}$ where the principal part of no rule application contains such a component (i.e., the right hand sides of the principal parts are fully contracted).

So suppose we have a derivation with property (P_{n+1}). Pick a topmost rule application with principal part containing a component $\Gamma \Rightarrow \Delta, \varphi, \varphi$ and φ of complexity $n+1$. Using right-contraction closure (possibly repeatedly) this is replaced by contractions on the premisses of this application, an application of a rule from $\mathcal{R}$ which does not contain such a component and applications of Weakening to add back missing copies of φ. Since φ was part of the principal part, the newly introduced contractions are on formulae of complexity at most n. Continuing in this fashion we replace all problematic rule applications. The resulting derivation has property (P_n) and we are done using the induction hypothesis. $\square$

Finally, we impose a further restriction which ensures that cuts with cut formula contextual on the left can be permuted into the premisses on the left.

Definition 4.9. A rule is *right-substitutive* if all restrictions occurring in it have the form $\langle\{p\},\{p\}\rangle$ or $\langle F,\emptyset\rangle$ for some $F \subseteq \mathcal{F}(\Lambda)$.

Example 4.10. While the logical rules of H as well as the rules $\mathsf{5}$ from Ex. 3.6 and the rules $\mathsf{5}_n$ and $\mathsf{45}_n$ from Ex. 4.2.2 resp. Ex. 4.2.3 are clearly right-substitutive, the modal splitting rule MS from Ex. 3.6 is not since it contains the restriction $\langle\{\Box p\},\{\Box p\}\rangle$.

The notion of a right-substitutive rule is an adaption of the notion of a *substitutive* rule from e.g. [20] to our framework. Using the above mentioned restrictions we have:

Lemma 4.11 (Shift Left)**.** *Let $\mathcal{R}$ be right-substitutive, single-component right and right-contraction closed. Assume in $\mathsf{H}\mathcal{R}\mathsf{Cut}$ we have derivations*

$$\begin{array}{c} \mathcal{D}_1 \\ \vdots \\ \mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1, \varphi^{m_1} \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n, \varphi^{m_n} \end{array} \qquad \textit{and} \qquad \begin{array}{c} \mathcal{D}_2 \\ \vdots \\ \mathcal{H} \mid \varphi, \Sigma \Rightarrow \Pi \end{array}$$

with $\rho(\mathcal{D}_1) < |\varphi|$ and $\rho(\mathcal{D}_2) < |\varphi|$. Then there is a derivation $\mathcal{D}$ in $\mathsf{H}\mathcal{R}\mathsf{Cut}$ of the hypersequent

$$\mathcal{G} \mid \mathcal{H} \mid \Gamma_1, \Sigma \Rightarrow \Delta_1, \Pi \mid \cdots \mid \Gamma_n, \Sigma \Rightarrow \Delta_n, \Pi$$

with $\rho(\mathcal{D}) < |\varphi|$.

Proof. By induction on the depth of $\mathcal{D}_1$. We actually show a slightly stronger statement, namely that whenever the principal parts of every rule application in $\mathcal{D}_1$ and $\mathcal{D}_2$ are fully contracted on the right, then we can find a derivation $\mathcal{D}$ with the properties stated in the lemma in which this is again the case. Using Lemma 4.8 we may assume that the original derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ are of this form.

If the last applied rule in $\mathcal{D}_1$ was one of $\mathsf{ICL}, \mathsf{ICR}, \mathsf{EC}, \mathsf{IW}, \mathsf{EW}, \mathsf{Cut}, \mathcal{A}$, then the cut is permuted into the premisses of this rule or replaced by applications of IW, EW. Then we are done using Lemma 4.8.

Otherwise, let Q be the last applied rule in $\mathcal{D}_1$. Again, taking ℓ to be the number of its premisses, for $j \leq \ell$ its j-th premiss can be written as

$$\mathcal{G}_j \mid \Omega_{1,j} \Rightarrow \Theta_{1,j}, \varphi^{s_{1,j}} \mid \cdots \mid \Omega_{k_j,j} \Rightarrow \Theta_{k_j,j}, \varphi^{s_{k_j,j}}$$

where the displayed $\varphi^{i,j}$ are propagated in the context from the conclusion. Using the induction hypothesis again we have for $j \leq \ell$ derivations $\mathcal{E}_j$ of

$$\mathcal{I}_j := \mathcal{G}_j \mid \mathcal{H} \mid \Omega_{1,j}, \Sigma \Rightarrow \Theta_{1,j}, \Pi \mid \Omega_{k_j,j}, \Sigma \Rightarrow \Theta_{k_j,j}, \Pi$$

with $\rho(\mathcal{E}_j) < |\varphi|$. Since the rule Q is right-substitutive, we can apply it to these premisses to obtain the conclusion

$$\mathcal{G} \mid \mathcal{H} \mid \Gamma_1, \Sigma \Rightarrow \Delta_1, \Pi, \varphi^{m_1'} \mid \cdots \mid \Gamma_n, \Sigma \Rightarrow \Delta_n, \Pi, \varphi^{m_n'}$$

where the remaining displayed $\varphi^{m_i'}$ are principal in the application of Q. Since Q is single-component right we furthermore have $m_i' \neq 0$ for at most one $i \leq n$. By right-contraction closure together with Lemma 4.8 and w.l.o.g. taking the i with $m_i' > 0$ to be n we obtain the derivation

$$\dfrac{\begin{array}{c} \mathcal{D}_1' \\ \vdots \end{array}}{\mathcal{G} \mid \mathcal{H} \mid \Gamma_1, \Sigma \Rightarrow \Delta_1, \Pi \mid \cdots \mid \Gamma_{n-1}, \Sigma \Rightarrow \Delta_{n-1}, \Pi \mid \Gamma_n, \Sigma \Rightarrow \Delta_n, \Pi, \varphi}\ Q'$$

with $\rho(\mathcal{D}'_1) < |\varphi|$. Moreover, from the proof of Lemma 4.8 we obtain that the displayed occurrence of φ is principal in the application of the rule Q'. Now using the Shift Right Lemma 4.5 we eliminate the remaining cut with the hypersequent $\mathcal{H} \mid \varphi, \Sigma \Rightarrow \Pi$ on the last occurrence of φ to obtain a derivation $\mathcal{D}'$ of

$$\mathcal{G} \mid \mathcal{H} \mid \Gamma_1, \Sigma \Rightarrow \Delta_1, \Pi \mid \cdots \mid \Gamma_n, \Sigma \Rightarrow \Delta_n, \Pi$$

with $\rho(\mathcal{D}') < |\varphi|$. Finally, using Lemma 4.8 we turn this into a derivation $\mathcal{D}$ with $\rho(\mathcal{D}) < |\varphi|$ in which the principal parts of all applications of rules are fully contracted on the right. □

Theorem 4.12 (Cut elimination). *Let $\mathsf{H}\mathcal{R}$ be right-substitutive, single-component right, right-contraction closed, principal-cut closed and mixed-cut permuting. Then for every hypersequent $\mathcal{G}$ we have:*

$$\vdash_{\mathsf{H}\mathcal{R}\mathsf{Cut}} \mathcal{G} \text{ iff } \vdash_{\mathsf{H}\mathcal{R}} \mathcal{G} \ .$$

Proof. For a derivation $\mathcal{D}$ let $\#\rho(\mathcal{D})$ be the number of applications of cut on a cut formula φ with $|\varphi| = \rho(\mathcal{D})$. Let $\mathcal{D}$ be a derivation of $\mathcal{G}$ in $\mathsf{H}\mathcal{R}\mathsf{Cut}$. The proof is by induction on the tuple $(\rho(\mathcal{D}), \#\rho(\mathcal{D}))$ in the lexicographic ordering. Topmost cuts with maximal rank are eliminated using Lemma 4.11, thus reducing $\rho(\mathcal{D})$ or preserving $\rho(\mathcal{D})$ while reducing $\#\rho(\mathcal{D})$. □

Corollary 4.13. *The hypersequent calculi $\mathsf{H}, \mathsf{H}\mathcal{R}_{\mathsf{K}}, \mathsf{H}\mathcal{R}_{\mathsf{K4}}, \mathsf{H}\mathcal{R}_{\mathsf{KT}}, \mathsf{H}\mathcal{R}_{\mathsf{KT5}}$ and $\mathsf{H}\mathcal{R}_{\mathsf{KT45}}$ admit cut elimination.*

Proof. Inspection of the rules together with Examples 4.2, 4.3, 4.7, and 4.10 shows that these rule sets satisfy the conditions of Thm. 4.12. □

The fact that despite not being right-substitutive the rule set for the logic $\mathsf{S5}$ containing the modal splitting rule nevertheless does admit cut elimination via a different proof [4] illustrates that the given criteria for cut elimination are sufficient but not necessary. However, as we will see, they seem to capture a reasonably large class of logics. Compared to other general methods of cut elimination for hypersequents such as the one used in [54] this method has the advantage that it allows for more than one principal formula and for restrictions on the context.

Thm. 4.12 together with the following Lemma also provides the basis of the extension of the method of *cut elimination by saturation* from e.g. [38, 40] to the hypersequent framework. In this method cut-free hypersequent calculi are constructed from given rules by saturating the rule set under the addition of cuts between rules (Def. 4.1) and *contractions of rules*, i.e. the result of contracting two principal formulae and the corresponding variables in the premisses (compare [40, Def. 12]). While the resulting rule set by construction will be principal-cut closed and contraction closed, we will still need to check that the remaining conditions of Thm. 4.12 are satisfied. Soundness of the additional rules is ensured by the following Lemma.

Lemma 4.14 (Soundness of cuts between rules). *Let R_1, R_2 be hypersequent rules with context restrictions. Then the rule $\mathsf{cut}(R_1, R_2, \heartsuit p)$ is a derivable rule in $\mathsf{H}R_1R_2\mathsf{Cut}$.*

Proof. Suppose we have two rules

$$R_1 = \frac{\Big\{(\Gamma_i \Rightarrow \Delta_i, p; \vec{\mathcal{C}_i}) \ : \ i \le m\Big\} \cup \mathcal{P}_1}{\Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_{n-1} \Rightarrow \Pi_{n-1} \mid \Sigma_n \Rightarrow \Pi_n, \heartsuit p}$$

$$R_2 = \frac{\Big\{(p, \Omega_j \Rightarrow \Psi_j; \vec{\mathcal{D}_j}) \ : \ j \le \ell\Big\} \cup \mathcal{P}_2}{\heartsuit p, \Upsilon_1 \Rightarrow \Xi_1 \mid \Upsilon_2 \Rightarrow \Xi_2 \mid \cdots \mid \Upsilon_k \Rightarrow \Xi_k}$$

where p does not occur in $\mathcal{P}_1, \mathcal{P}_2$. Furthermore, writing $\vec{\mathcal{C}_i} \odot \vec{\mathcal{D}_j}$ for $(\vec{\mathcal{C}_i}, \overbrace{\mathcal{C}_\emptyset, \ldots, \mathcal{C}_\emptyset}^{k-1 \text{ times}}) \cup (\overbrace{\mathcal{C}_\emptyset, \ldots, \mathcal{C}_\emptyset}^{n-1 \text{ times}}, \vec{\mathcal{D}_j})$ suppose we have an application of the cut between these rules on $\heartsuit p$, i.e., the rule

$$\frac{\Big\{(\Gamma_i, \Omega_j \Rightarrow \Delta_i, \Psi_j; \vec{\mathcal{C}_i} \odot \vec{\mathcal{D}_j}) \ : \ i \le m, j \le \ell\Big\} \cup \mathcal{P}_1 \cup \mathcal{P}_2}{\Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_{n-1} \Rightarrow \Pi_{n-1} \mid \Sigma_n, \Upsilon_1 \Rightarrow \Pi_n, \Xi_1 \mid \Upsilon_2 \Rightarrow \Xi_2 \mid \cdots \mid \Upsilon_k \Rightarrow \Xi_k} \ \mathsf{cut}(R_1, R_2, \heartsuit p)$$

given by a substitution σ, a side hypersequent $\mathcal{G}$ and $n+k-1$ contexts $\Theta_r \Rightarrow \Phi_r$. Thus we have the premisses from $\mathcal{P}_1, \mathcal{P}_2$ not including p (with context) and the premisses

$$\mathcal{G} \mid \vec{\Theta}\restriction_{\vec{\mathcal{C}}_i \odot \vec{\mathcal{D}}_j} \Gamma_i, \Omega_j \Rightarrow \Delta_i, \Psi_j, \vec{\Phi}\restriction_{\vec{\mathcal{C}}_i \odot \vec{\mathcal{D}}_j}$$

for $i \leq n, j \leq \ell$. Now setting

$$\chi := \bigvee_{i \leq n} (\bigwedge \vec{\Theta}\restriction_{\vec{\mathcal{C}}_i \odot \mathcal{C}^k_\emptyset} \wedge \bigwedge \Gamma_i \wedge \neg \bigvee \vec{\Phi}\restriction_{\vec{\mathcal{C}}_i \odot \mathcal{C}^k_\emptyset} \wedge \neg \bigvee \Delta_i)$$

we can derive the hypersequents

$$\mathcal{G} \mid \vec{\Theta}\restriction_{\vec{\mathcal{C}}_i \odot \mathcal{C}^k_\emptyset}, \Gamma_i \Rightarrow \vec{\Phi}\restriction_{\vec{\mathcal{C}}_i \odot \mathcal{C}^k_\emptyset}, \Delta_i, \chi$$

from axioms using propositional logic and the hypersequents

$$\mathcal{G} \mid \vec{\Theta}\restriction_{\mathcal{C}^n_\emptyset \odot \vec{\mathcal{D}}_j}, \chi, \Omega_j \Rightarrow \Psi_j, \vec{\Phi}\restriction_{\mathcal{C}^n_\emptyset \odot \vec{\mathcal{D}}_j}$$

from the premisses of $\mathsf{cut}(R_1, R_2, \heartsuit p)$. Now applications of R_1 and R_2 give the hypersequents

$$\mathcal{G} \mid \Theta_1, \Sigma_1 \Rightarrow \Phi_1, \Pi_1 \mid \cdots \mid \Theta_n, \Sigma_n \Rightarrow \Phi_n, \Pi_n, \heartsuit\chi$$

and

$$\mathcal{G} \mid \Theta_n, \heartsuit\chi, \Upsilon_1 \Rightarrow \Phi_n, \Xi_1 \mid \cdots \mid \Theta_{n+\ell}, \Upsilon_\ell \Rightarrow \Phi_{n+\ell}, \Xi_\ell$$

Finally, an application of Cut together with external and internal contractions gives the desired conclusion. □

Example 4.15. 1. Since the rules $\mathsf{5}_n$ from Ex. 4.2.2 are the result of cuts between the rules K_n and the rule $\mathsf{5}$, by Lemma 4.14 they are derivable in $\mathsf{H}\mathcal{R}_{\mathsf{KT}}\mathsf{5Cut}$ and hence together with Cor. 4.13 and the fact that the rule $\mathsf{5}$ is subsumed by the rule $\mathsf{5}_1$ we have

$$\vdash_{\mathsf{H}\mathcal{R}_{\mathsf{KT}}\mathsf{5Cut}} \mathcal{G} \quad \text{iff} \quad \vdash_{\mathsf{H}\mathcal{R}_{\mathsf{KT5}}} \mathcal{G}$$

for every hypersequent $\mathcal{G}$.

2. Similarly, for the rule set $\mathcal{R}_{\mathsf{KT45}}$ constructed by cutting the rules $\mathsf{K4}_n$ with $\mathsf{5}$ we have

$$\vdash_{\mathsf{H}\mathcal{R}_{\mathsf{KT4}}\mathsf{5Cut}} \mathcal{G} \quad \text{iff} \quad \vdash_{\mathsf{H}\mathcal{R}_{\mathsf{KT45}}} \mathcal{G}$$

for every hypersequent $\mathcal{G}$.

Example 4.16. A further simplification step in the process of cut elimination by saturation permits the omission of rules which are derivable from other rules from the rule set. A small computation shows that the rules $\mathsf{K}_n, \mathsf{T}_n$ and $\mathsf{5}_n$ indeed are derivable using the two rules K_0 and $\mathsf{5}$ together with the structural rules. Moreover, the rule set $\mathsf{HK}_0\mathsf{5}$ satisfies all the conditions of Thm. 4.12. Hence we have:

$$\vdash_{\mathsf{H}\mathcal{R}_{\mathsf{KT5}}\mathsf{Cut}} \mathcal{G} \quad \text{iff} \quad \vdash_{\mathsf{HK}_0\mathsf{5}} \mathcal{G}$$

for every hypersequent $\mathcal{G}$. Note that this gives precisely the modal rules for the logic $\mathsf{S5}$ from [54].

Further examples of this procedure will be considered in Sec. 6.

4.1. Applications: Decision Procedures and Complexity Bounds

Now that we have established criteria for cut elimination it is time to consider some applications of these results. One of the main advantages of cut-free hypersequent calculi of our format is that they satisfy the *subformula property*: every formula occurring in a (cut-free) derivation of a hypersequent $\mathcal{H}$ must be a subformula of a formula occurring in $\mathcal{H}$. Indeed, the subformula property is a key ingredient of decision procedures based on *proof search*, i.e., on inspecting all possible derivations of a given hypersequent, since it greatly reduces the search space of possible derivations. However, the cut rule is not the only problematic rule in this respect: in the presence of the external resp. internal contraction rules it is not clear that we can limit the size of the hypersequents occurring in a possible derivation, since they might contain arbitrarily many copies of a single component resp. formula. Thus in order to render our calculi suitable for proof search techniques we need to modify them slightly so as to take the sting out of the contraction rules. The main idea is to use a trick introduced by Kleene for the $\mathsf{G3}$ systems in [34] and modify the notion of a rule application such that the relevant bits of the principal part are copied into the active components of the premisses, and in addition the whole principal part is copied into each premiss as well. The first modification allows to push internal contractions between context formulae and active formulae into the premisses and thus we obtain admissibility of internal contractions, while the second one allows to permute external contractions between components of the principal part and the side hypersequent upwards. Formally:

Definition 4.17 (Kleene's Trick). A *modified application* of a hypersequent rule $R = \{(\Gamma_i \Rightarrow \Delta_i; \vec{\mathcal{C}_i}) : i \in \mathcal{P}\}/\Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_n \Rightarrow \Pi_n$ is given by a side hypersequent $\mathcal{G}$, a substitution $\sigma : \mathcal{V} \to \mathcal{F}$ and contexts $\Theta_1 \Rightarrow \Omega_1 \mid \cdots \mid \Theta_n \Rightarrow \Omega_n$ and written as

$$\frac{\left\{\mathcal{G} \mid \mathcal{H} \mid \Gamma_i\sigma, \bigsqcup_{j\leq n}(\Sigma_j\sigma, \Theta_j)\restriction_{\mathcal{C}_i^j} \Rightarrow \Delta_i\sigma, \bigsqcup_{j\leq n}(\Pi_j\sigma, \Omega_j)\restriction_{\mathcal{C}_i^j} \,:\, i \in \mathcal{P}\right\}}{\mathcal{G} \mid \Sigma_1\sigma, \Theta_1 \Rightarrow \Pi_1\sigma, \Omega_1 \mid \cdots \mid \Sigma_n\sigma, \Theta_n \Rightarrow \Pi_n\sigma, \Omega_n}\ R^*$$

with $\mathcal{H} = \Sigma_1\sigma, \Theta_1 \Rightarrow \Pi_1\sigma, \Omega_1 \mid \cdots \mid \Sigma_n\sigma, \Theta_n \Rightarrow \Pi_n\sigma, \Omega_n$. Modified applications of the logical rules for the propositional connectives are defined analogously. For a rule set $\mathcal{R}$ we say that a hypersequent is *derivable in* $\mathsf{H}^*\mathcal{R}^*$ if it is derivable using modified applications of propositional rules and rules in $\mathcal{R}$ instead of applications and without using the internal and external structural rules of Weakening (IW, EW) and the internal structural rules of Contraction ($\mathsf{ICL}, \mathsf{ICR}$).

Example 4.18. A modified application of the rule $\mathsf{45}_n = \{(\vec{p} \Rightarrow ; \mathcal{C}_\Box, \mathcal{C}_{\mathsf{id}})\}/\Box\vec{p} \Rightarrow \mid \Rightarrow$ from Ex. 4.2.3 is of the form

$$\frac{\mathcal{G} \mid \Sigma, \Box\Gamma, \Box\varphi_1, \ldots, \Box\varphi_n \Rightarrow \Pi \mid \Omega \Rightarrow \Xi \mid \Box\Gamma, \Box\varphi_1, \ldots, \Box\varphi_n, \varphi_1, \ldots, \varphi_n, \Omega \Rightarrow \Xi}{\mathcal{G} \mid \Sigma, \Box\Gamma, \Box\varphi_1, \ldots, \Box\varphi_n \Rightarrow \Pi \mid \Omega \Rightarrow \Xi}$$

Note that the notion of derivability using modified rule sets does allow applications of the external Contraction rule EC. While this could be strengthened to a notion of derivability not using any structural rules, we will see below that for the purpose of showing decidability results the current formulation suffices (Lemma 4.24). In a modified application of a rule in addition to the context formulae all principal formulae satisfying the corresponding restriction are copied into the premiss, and all components of the principal part are copied to deal with external contractions between components of the principal part and the side hypersequent. If internal contractions can be permuted with rules this yields admissibility of internal contraction. For this we consider the full version of right-contraction closure (Def. 4.6), essentially an adaption of the notion of contraction-closed rule sets [50, 40] resp. the *closure condition* [47] from the sequent to the hypersequent framework.

Definition 4.19. A rule set $\mathcal{R}$ is *contraction closed* if for every rule $R \in \mathcal{R}$ with principal part $\mathcal{G} \mid \Gamma \Rightarrow \Delta, \heartsuit p, \heartsuit q$ (resp. $\mathcal{G} \mid \Gamma, \heartsuit p, \heartsuit q \Rightarrow \Delta$) there is a rule $R' \in \mathcal{R}$ with principal part $\mathcal{G} \mid \Gamma \Rightarrow \Delta, \heartsuit p$ (resp. $\Gamma, \heartsuit p \Rightarrow \Delta$) whose premisses are derivable from those of R by renaming q to p and contractions.

Example 4.20. 1. Renaming q to p in the premiss of the rule $\mathsf{K}_{n+2} = \{(p, q, \vec{r} \Rightarrow s; \mathcal{C}_\emptyset)\}/\Box p, \Box q, \vec{\Box r} \Rightarrow \Box s$ and applying contraction yields the premiss of the rule $\mathsf{K}_{n+1} = \{(p, \vec{r} \Rightarrow s; \mathcal{C}_\emptyset)\}/\Box p, \vec{\Box r} \Rightarrow \Box s$. Together with the fact that the rule set $\mathcal{R}_\mathsf{K}$ is trivially right-contraction closed (see Ex. 4.7.1) this gives contraction closure of $\mathcal{R}_\mathsf{K}$.

2. Similarly, the rule sets $\mathsf{H}\mathcal{R}_{\mathsf{K4}}$, $\mathsf{H}\mathcal{R}_{\mathsf{KT}}$ and $\mathsf{H}\mathcal{R}_{\mathsf{KT4}}\{\mathsf{5}_n : n \in \mathbb{N}\}$ are contraction closed.
3. The rule set consisting only of the rule $\mathsf{K}_2 = \{(p, q \Rightarrow r; \mathcal{C}_\emptyset)\}/\Box p, \Box q \Rightarrow \Box r$ does not contain the rule K_1. Hence it is not contraction closed (see point 1 above).

Lemma 4.21 (Admissibility of internal contraction)**.** *For contraction closed $\mathcal{R}$, internal contraction is admissible in $\mathsf{H}^*\mathcal{R}^*$.*

Proof. By simultaneous double induction on the complexity of φ and the depth of the derivation we show: whenever $\vdash_{\mathsf{H}^*\mathcal{R}^*} \mathcal{G} \mid \varphi, \varphi, \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \varphi, \varphi, \Gamma_n \Rightarrow \Delta_n$, then $\vdash_{\mathsf{H}^*\mathcal{R}^*} \mathcal{G} \mid \varphi, \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \varphi, \Gamma_n \Rightarrow \Delta_n$ and analogously for φ on the right. Contractions between context and principal formulae are dealt with by modified rule applications and the inner induction hypothesis (on the depth of the derivation), those between principal formulae using contraction closure and the outer induction hypothesis (on the complexity). $\Box$

To deal with external contractions where both components are part of the principal part we could formulate an external version of contraction closure. However, inspection of the common hypersequent calculi reveals that usually the principal parts of the rules in a rule set contain only a bounded number of components. This already suffices to bound the number of copies of each component in a hypersequent occurring in a derivation.

Definition 4.22. A rule set $\mathcal{R}$ is *bounded component* if there is $c \in \mathbb{N}$ such that the principal part of every rule in $\mathcal{R}$ has at most c components.

Example 4.23. 1. A quick inspection of the rules shows that all of the rule sets considered in this article (apart from the one given below) are bounded component, and often the bound is even as low as 2.

2. Natural examples of rule sets which are not bounded component seem to be hard to find in the literature. A somewhat artificial example of such a rule set is given by the set of rules $\bigcup_{k\geq 2} \mathsf{btw}_k$, where btw_k is the set of rules

$$\left\{ \frac{\left\{(p_1^i, \dots, p_{n_i}^i, p_1^j, \dots, p_{n_j}^j \Rightarrow ; \mathcal{C}_\emptyset, \dots, \mathcal{C}_\emptyset) : 1 \leq i < j \leq k\right\}}{\Box p_1^1, \dots, \Box p_{n_1}^1 \Rightarrow \mid \cdots \mid \Box p_1^k, \dots \Box p_{n_k}^k \Rightarrow} \quad : \quad n_1 \geq 1, \dots, n_k \geq 1 \right\}$$

from [37]. Since the rule set includes the rule set btw_k for every $k \geq 2$ and the principal part of every rule in btw_k has k components, the rule set $\bigcup_{k\geq 2} \mathsf{btw}_k$ is not bounded component. However, since every rule in btw_{2+n} for $n \geq 1$ is derivable using a rule in btw_2 and external weakening, this rule set could be replaced with the set btw_2, which is clearly bounded component.

Lemma 4.24. *Let $\mathcal{R}$ be a bounded component set of rules with bound c and let $\mathcal{G}$ be a hypersequent with k components. If $\mathcal{G}$ is derivable in $\mathsf{H}^*\mathcal{R}^*$, then there is a derivation of $\mathcal{G}$ in $\mathsf{H}^*\mathcal{R}^*$ in which every hypersequent contains at most* $\max\{c, k\}$ *copies of every sequent.*

Proof. By induction on the depth of the derivation. If the depth is 0, then since the axiom rule is local we immediately obtain the statement. If the depth is $n+1$, then consider the last modified rule application, say R^* and assume that we have $\max\{c, k\} + 1$ copies of the sequent $\Gamma \Rightarrow \Delta$. Since modified rule applications copy the whole principal part into the premisses, every premiss contains at least $\max\{c, k\} + 1$ copies of this sequent. Using the induction hypothesis we thus obtain derivations of the premisses in which the sequent occurs at most $\max\{c, k\}$ times. Finally, since $\mathcal{R}$ is bounded component, at most c copies of the sequent were in the principal part of this rule application, and thus we may still apply the rule to obtain the desired conclusion. $\Box$

The previous two lemmata together with cut elimination allow us to bound the size of the hypersequents occurring in a derivation. But to search for a derivation of a hypersequent we also need to be able to recognise a derivation, and in particular to recognise whether one hypersequent follows from other hypersequents via a rule application. To be able to do so (and still obtain a reasonable complexity bound) we slightly adjust the corresponding notion from [40] to the hypersequent framework and restrict the rule sets in such a way that we can handle them automatically.

Definition 4.25. A rule set $\mathcal{R}$ is *pspace-tractable* if there is an encoding $\ulcorner . \urcorner$ of applications of rules from $\mathcal{R}$ of size polynomial in the size of the conclusion such that given a hypersequent $\mathcal{G}$ and an encoding $\ulcorner R \urcorner$ of a rule application it is decidable in space polynomial in the size of $\mathcal{G}$ whether $\mathcal{G}$ is the conclusion of R and it is decidable in space polynomial in the size of $\ulcorner R \urcorner$ whether $\mathcal{G}$ is a premiss of the rule application R.

A little thought shows that all of the rules sets considered in this article are indeed pspace-tractable in this sense. In fact, the polynomial space bounds in the above definition are quite generous and in most examples can be lowered to polynomial time. However, since the complexity of recognising a rule application is not the main source of complexity in the proof search procedure we will use the more general definition. This allows us to perform backwards proof search for such rule sets.

Theorem 4.26 (Decidability and complexity)**.** *Let $\mathcal{R}$ be a contraction closed, bounded component and pspace-tractable set of rules. Then derivability in $\mathsf{H}\mathcal{R}$ is decidable in exponential space.*

Proof. Using Weakening and Contraction as well as Lemmas 3.8 and 4.21 derivability in $\mathsf{H}\mathcal{R}$ is equivalent to derivability in $\mathsf{H}^*\mathcal{R}^*$. Moreover, Lemma 4.21 allows us to equivalently work with hypersequents build from *set-set* sequents, i.e., pairs of sets of formulae instead of pairs of multisets of formulae. Now assume the input consists of the hypersequent $\mathcal{G}$ with size n. Using the fact that modified applications of rules copy the whole conclusion into the premisses, we may w.l.o.g. assume that for every rule application all premisses properly extend the conclusion. Using Lemma 4.24 we may furthermore assume that every hypersequent occurring in a derivation of $\mathcal{G}$ contains at most $\max\{c, n\} =: k$ copies of each component. Since all the rules have the subformula property and there are at most n subformulae of formulae occurring in $\mathcal{G}$, the number of relevant (set-set) sequents thus is bounded by $2^n \cdot 2^n = 2^{n^2}$, and the number of relevant hypersequents is bounded by $k \cdot 2^{n^2}$. Since moreover the size of the relevant (set-set) sequents is bounded by $2n^2$, the size of the relevant hypersequents is bounded by $k \cdot 2^{n^2} \cdot 2n^2$. We implement backwards proof search on an alternating exponential time machine (see [13, 49]), by existentially guessing an encoding of the last rule application, universally guessing its premisses and recursively checking that these are derivable and properly extend the conclusion. Since the rule set is pspace-tractable, the size of the encoding of the rule application is bounded by a polynomial p in the size of its conclusion, and hence the size of the encodings of rule applications is bounded by $p(k \cdot 2^{n^2} \cdot 2n^2)$. Moreover, since PSPACE is the same as alternating polynomial time [13, 49], checking whether a hypersequent is the conclusion resp. premiss of a rule application can be done in alternating polynomial time, where the time is bounded by polynomials q_1 and q_2 in the size of the hypersequent resp. encoding of the rule application. Thus the time needed for each of these checks is bounded by $q_1(k \cdot 2^{n^2} \cdot 2n^2)$ resp. $q_2(p(k \cdot 2^{n^2} \cdot 2n^2))$. Since the length of the branches in the search tree is bounded by the number $k \cdot 2^{n^2}$ of relevant hypersequents, we thus obtain an alternating exponential time algorithm, which by [13] can be transformed into an exponential space algorithm. $\square$

The previous theorem straightforwardly yields $\mathsf{EXPSPACE}$ upper bounds for essentially all the logics considered as examples in this article. In view of the fact that these logics typically are decidable in PSPACE or even coNP the exponential space bound might seem a bit excessive. It should be clear, however, that logic-tailored decision procedures can produce much better complexity bounds than general decision procedures (indeed, an example will be given in Sec. 6.2). At the moment it is not yet clear whether the $\mathsf{EXPSPACE}$ upper bound for logics given in the format considered here can be improved in general, but the example in Sec. 6.4 will show that it cannot be lowered below $\mathsf{EXPTIME}$.

5. Axioms and Rules

So far we considered the basic rules of a hypersequent calculus as given. Now we investigate the connection between hypersequent rules with context restrictions and Hilbert-style axiomatisations in a mainly syntactical way. In order to do so we first take a closer look at the *interpretation* or *formula translation* of a hypersequent in a general way and consider which notions of soundness this gives rise to and what we need to ensure soundness of the standard rules (Sec. 5.1). Then we will consider translations from rules to axioms (Sec. 5.2) and from axioms to rules (Sec. 5.3) for normal modal logics and simpler context restrictions, before giving the general method (Sec. 5.4). Finally, we will apply the translation to show a limitative result about the rule format (Sec. 5.5).

5.1. Interpretations

In contrast to more semantical approaches such as [37] for this syntactic approach we need to interpret hypersequents as formulae. While the standard interpretation for modal logics is the already mentioned interpretation of a hypersequent $\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$ as the formula $\Box(\bigwedge \Gamma_1 \to \bigvee \Delta_1) \vee \cdots \vee \Box(\bigwedge \Gamma_n \Rightarrow \Delta_n)$ from [4], some other interpretations have been suggested as well, although mainly in the context of hypersequent calculi for substructural and intermediate logics (see e.g. [16]). For this reason we consider interpretations in an abstract way and view them as a family of formulae, one for each number of components in a hypersequent. Of course, since we plan to construct extensions of a hypersequent calculus with all the structural rules, we need to make sure that such an interpretation is compatible with the structural rules. Formally:

Definition 5.1 (Interpretation). An *interpretation* for a Λ-logic $\mathcal{L}$ is a sequence $\iota = (\iota_n(p_1, \ldots, p_n))_{n\geq 1}$ of formulae in $\mathcal{F}(\Lambda)$ which *respects the structural rules*, i.e. for all $n \geq 1$:

1. ι respects (external) exchange: $\models_{\mathcal{L}} \iota_n(\vec{\varphi}, \psi, \chi, \vec{\xi})$ iff $\models_{\mathcal{L}} \iota_n(\vec{\varphi}, \chi, \psi, \vec{\xi})$
2. ι respects external Weakening: if $\models_{\mathcal{L}} \iota_n(\vec{\varphi})$, then $\models_{\mathcal{L}} \iota_{n+1}(\vec{\varphi}, \psi)$
3. ι respects external Contraction: if $\models_{\mathcal{L}} \iota_{n+1}(\vec{\varphi}, \psi, \psi)$, then $\models_{\mathcal{L}} \iota_n(\vec{\varphi}, \psi)$
4. ι respects Cut: if $\models_{\mathcal{L}} \iota_n(\vec{\varphi}, \psi \to \chi)$ and $\models_{\mathcal{L}} \iota_m(\chi \to \xi, \vec{\zeta})$, then we have $\models_{\mathcal{L}} \iota_{n+m-1}(\vec{\varphi}, \psi \to \xi, \vec{\zeta})$.

The interpretation is *regular for* $\mathcal{L}$ if for all $\varphi \in \mathcal{F}$ we have $\models_{\mathcal{L}} \varphi$ iff $\models_{\mathcal{L}} \iota_1(\varphi)$.

An interpretation $\iota = (\iota_n)_{n\geq 1}$ for a logic induces a map ι from hypersequents to formulae defined by

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \quad \mapsto \quad \iota_n(\bigwedge \Gamma_1 \to \bigvee \Delta_1, \ldots, \bigwedge \Gamma_n \to \bigvee \Delta_n) \ .$$

Example 5.2.
1. The above mentioned standard interpretation for normal $\Lambda_\Box$-logics from [4] is $\iota_\Box$ given by $\iota^\Box_n(p_1, \ldots, p_n) = \bigvee_{i=1}^n \Box p_i$. It is regular for a normal logic iff $\Box\varphi/\varphi$ is admissible, in particular if the axiom (T) $\Box p \to p$ is contained in the logic. It is not regular for e.g. the logic KB of symmetric frames.
2. The interpretation $\iota_\boxplus$ for normal $\Lambda_\Box$-logics is given by the formulae $\iota^\boxplus_n(\varphi_1, \ldots, \varphi_n) = \bigvee_{i=1}^n (\varphi_i \wedge \Box\varphi_i)$. It is an interpretation by normality of $\Box$ and obviously regular.

While in the spirit of [23] it should be possible to obtain general results on the form of interpretations for a logic from the fact that they respect the structural rules, for the present purpose the general properties given in Def. 5.1 are enough.

Depending on whether we involve the interpretation we obtain different notions of soundness. Regular interpretations link these notions and imply soundness of the propositional rules. Recall that a modal logic is congruential if it is closed under the rules $A \leftrightarrow B / \heartsuit A \leftrightarrow \heartsuit B$ for every modality $\heartsuit$.

Definition 5.3 (Soundness, hssp). Let $\mathcal{R}$ be a set of rules and ι an interpretation for the logic $\mathcal{L}$. Then $\mathcal{R}$ is *hypersequent soundness preserving* (briefly: *hssp*) for $(\mathcal{L}, \iota)$ if for every application of a rule from $\mathcal{R}$ with n premisses $\mathcal{H}_1, \ldots, \mathcal{H}_n$ and conclusion $\mathcal{G}$: if $\models_{\mathcal{L}} \iota(\mathcal{H}_k)$ for all $k \leq n$, then $\models_{\mathcal{L}} \iota(\mathcal{G})$. The calculus $\mathsf{H}\mathcal{R}$ is *sound* for $\mathcal{L}$, if $\vdash_{\mathsf{H}\mathcal{R}} \Rightarrow \varphi$ implies $\models_{\mathcal{L}} \varphi$, and *complete* for $\mathcal{L}$, if $\models_{\mathcal{L}} \varphi$ implies $\vdash_{\mathsf{H}\mathcal{R}} \Rightarrow \varphi$.

Proposition 5.4. *Let $\mathcal{L}$ be a congruential logic.*

1. *If ι is a regular interpretation for $\mathcal{L}$, then* H *is hssp for $(\mathcal{L}, \iota)$.*
2. *If $\mathcal{R}$ is hssp for $(\mathcal{L}, \iota)$ and ι is a regular interpretation for $\mathcal{L}$, then $\mathcal{R}$ is sound for $\mathcal{L}$.*

Proof. 1. Using the fact that $\mathcal{L}$ includes all propositional tautologies, all the modalities have congruence and thus $\models_{\mathcal{L}} \iota_n(\vec{\varphi}, \psi)$ iff $\models_{\mathcal{L}} \iota_n(\vec{\varphi}, \top \to \psi)$ and the properties of a regular interpretation.

2. By induction on the depth of a derivation we have: $\vdash_{\mathcal{R}} \mathcal{H}$ implies $\models_{\mathcal{L}} \iota(\mathcal{H})$. Now regularity of ι gives the statement. $\square$

The interpretation $\iota_\Box$ is regular not only for reflexive normal $\Lambda_\Box$-logics, but also e.g. for normal $\Lambda_\Box$-logics given by a class of Kripke frames closed under the addition of a (reflexive) predecessor to every world (a simpler version of the notion of extensibility from [32]), or more generally the addition of a suitable predecessor to certain problematic worlds:

Definition 5.5. A class F of frames is *extensible* if whenever for a frame $\mathfrak{F} = (W, R)$ we have $\mathfrak{F} \in \mathsf{F}$ then also $\mathfrak{F}^\circ \in \mathsf{F}$ where $\mathfrak{F}^\circ = (W \cup \{x\}, R \cup \{(x, y) : y \in W \cup \{x\}\})$ with $x \notin W$. A class F of Kripke-frames is *p-extensible* if for every frame $\mathfrak{F} = (W, R) \in \mathsf{F}$ and world $w \in W$ with $R^{-1}[\{w\}] = \emptyset$ there is a frame $\mathfrak{F}_w = (W \cup \{z\}, R^w)$ such that: $\mathfrak{F}_w \in \mathsf{F}$, $z \notin W$, $R^w \restriction_{W \times W} = R$, $z \notin R^w[W]$, and $zR^w w$.

Here as usual we write $R[S]$ for $\{y \in W : \exists x \in S\, xRy\}$ and R^{-1} for the inverse relation of R. Thus a class of frames is p-extensible if every frame in the class which includes a world not accessible from within the frame can be extended to a frame in the class by adding a predecessor to this world not accessible from within the original frame. Obviously, every extensible class of frames is also p-extensible.

Example 5.6. 1. The classes of serial and transitive frames are extensible, since adding a (reflexive) world from which every world is accessible again yields a serial resp. transitive frame.

2. The class of *euclidean* frames, i.e., frames satisfying the property $\forall x \forall y \forall z (xRy \wedge xRz \to yRz)$ is neither extensible nor p-extensible. To see this, consider the euclidean frame $(\{a, b\}, \{(a, b), (b, b)\})$. Every euclidean frame $\mathfrak{F}_a$ in the sense of the above definition would need to satisfy $aR^a a$ by euclideanity, in contradiction to the stipulation that $R^a \restriction_{\{a,b\} \times \{a,b\}} = R$ and the fact that aRa does not hold in the original frame. The same frame and reasoning shows that the class of *shift reflexive* frames, i.e., frames with the property $\forall x \forall y (xRy \to yRy)$ is not p-extensible.

Lemma 5.7 (Regularity). *If* F *is a p-extensible class of Kripke-frames, then the rule $\Box\varphi/\varphi$ is admissible in the logic given by* F *and hence $\iota_\Box$ is a regular interpretation for this logic.*

Proof. Let F be a p-extensible class of frames and suppose that $\neg\varphi$ is satisfiable in the frame $\mathfrak{F} \in \mathsf{F}$. Then for some world w of $\mathfrak{F}$ and valuation σ we have $\mathfrak{F}, w, \sigma \not\models \varphi$. Thus for the additional world z in $\mathfrak{F}_w$ we have $\mathfrak{F}_w, z, \sigma' \not\models \Box\varphi$ with $\sigma' = \sigma$ on the worlds of $\mathfrak{F}$ and arbitrary on the new world z. Since $\mathfrak{F}_w \in \mathsf{F}$ we thus have $\mathsf{F} \not\Vdash \Box\varphi$. Hence $\Box\varphi/\varphi$ is admissible. Since the logic given by F is normal, $\iota_\Box$ indeed is an interpretation for it, and regularity follows from admissibility of $\Box\varphi/\varphi$. $\square$

Corollary 5.8. *If $\mathcal{L}$ is a normal $\Lambda_\Box$-logic defined by an extensible class of frames, then $\iota_\Box$ is a regular interpretation for $\mathcal{L}$. In particular, $\iota_\Box$ is a regular interpretation for* $\mathsf{K}, \mathsf{KD}, \mathsf{K4}$ *and* $\mathsf{KD4}$. $\square$

A rather curious fact about hypersequent calculi for normal modal logics with the interpretation $\iota_\Box$ or $\iota_\boxplus$ is that the standard rules $\mathcal{R}_\mathsf{K}$ for modal logic K formulated with a side hypersequent are not always hssp for extensions of this logic. Intuitively this is due to the fact that the rules operate inside the context given by the side hypersequent, which makes Hilbert-style rules obtained by replacing their premisses resp. conclusions with the corresponding formula translations only admissible rules instead of derivable rules in the Hilbert-system for K. The situation here is maybe similar to that for first-order Gödel-Dummett logic, where using the communication rule and the standard quantifier rules it is possible to derive the quantifier shift axiom $\forall x (A(x) \vee B) \to (\forall x A(x) \vee B)$ with x not free in B, which is not valid in the extension of standard intuitionistic logic with the linearity axiom $(A \to B) \vee (B \to A)$ corresponding to the communication rule [6].

Example 5.9. The logics $\mathsf{K.2}$ and $\mathsf{KT.2}$ are given by adding the axiom $(.2) = \Diamond\Box p \to \Box\Diamond p$ to K resp. KT. Equivalently, $\mathsf{K.2}$ (resp. $\mathsf{KT.2}$) is the logic of *convergent* (resp. convergent and reflexive) frames, i.e., (reflexive) frames satisfying the property $\forall x \forall y \forall z(xRy \wedge xRz \to \exists w(yRw \wedge zRw))$, see e.g. [29] and also Sec. 6.3. The rule K_0 is not hssp for $(\mathsf{K.2}, \iota_\Box)$ or $(\mathsf{KT.2}, \iota_\Box)$ as can be seen by considering the hypersequent $\mathcal{H} = \Box\neg p \Rightarrow \mid \Rightarrow \neg\Box p$: The interpretation of this is $\iota_\Box(\mathcal{H}) = \Box\neg\Box\neg p \vee \Box\neg\Box p \equiv \Diamond\Box p \to \Box\Diamond p$ and thus a theorem of $\mathsf{K.2}$. But applying the rule K_0 with side hypersequent $\Box\neg p \Rightarrow$ we obtain $\Box\neg p \Rightarrow \mid \Rightarrow \Box\neg\Box p$ which has the translation $\Box\Diamond p \vee \Box\Box\Diamond\neg p$. The negation of this is satisfiable in the convergent and reflexive frame $(\{a, b, c\}, \{(a, a), (a, b), (b, b), (b, c), (c, c)\})$ under the valuation making p true at c only.

Example 5.10. Similarly to the last example, setting $\mathcal{L} := \mathsf{K} \oplus (\boxplus\Diamond\neg p \vee \boxplus\Diamond p)$ and using the same hypersequent as before together with the model based on $(\{a, b, c\}, \{(a, b), (b, b), (b, c), (c, c)\})$ where p holds only in c we obtain that the rule K_0 is not hssp for $(\mathcal{L}, \iota_\boxplus)$.

Of course this raises the question for which normal modal logics and interpretations the rules $\mathcal{R}_\mathsf{K}$ are hssp. The following proposition gives a handy characterisation.

Proposition 5.11 (Soundness of the standard rules)**.** *If $\mathcal{L}$ is a normal $\Lambda_\Box$-logic and ι a regular interpretation for $\mathcal{L}$, then $\mathcal{R}_\mathsf{K}$ is hssp for $(\mathcal{L}, \iota)$ iff K_0 is hssp for $(\mathcal{L}, \iota)$.*

Proof. Since $\mathsf{K}_0 \in \mathcal{R}_\mathsf{K}$, the "only if" direction is trivial. For the other direction, while not rules with restrictions in our sense, the rules with applications

$$\frac{\varphi_1, \dots, \varphi_n \Rightarrow \psi}{\mathcal{G} \mid \Gamma, \Box\varphi_1, \dots, \Box\varphi_n \Rightarrow \Box\psi, \Delta}\ \mathsf{K}'_n$$

(i.e., which in particular do *not* copy the side hypersequent into the premiss) are always sound for normal modal logics. But given that the rule K_0 is hssp for $(\mathcal{L}, \iota)$, the rule K_n is derivable using $\mathsf{K}'_{n+1}, \mathsf{K}_0$, the propositional rules and Cut:

$$\dfrac{\dfrac{\dfrac{\mathcal{G} \mid \varphi_1, \dots, \varphi_n \Rightarrow \psi}{\mathcal{G} \mid \Rightarrow \bigwedge_{i \le n} \varphi_i \to \psi}\ \wedge_\mathsf{L}, \to_\mathsf{R}}{\mathcal{G} \mid \Rightarrow \Box(\bigwedge_{i \le n} \varphi_i \to \psi)}\ \mathsf{K}_0 \qquad \dfrac{\dfrac{}{\bigwedge_{i \le n} \varphi_i \to \psi, \varphi_1, \dots, \varphi_n \Rightarrow \psi}\ \mathsf{Prop}}{\Box(\bigwedge_{i \le n} \varphi_i \to \psi), \Box\varphi_1, \dots, \Box\varphi_n \Rightarrow \Box\psi}\ \mathsf{K}'_{n+1}}{\mathcal{G} \mid \Gamma, \Box\varphi_1, \dots, \Box\varphi_n \Rightarrow \Box\psi, \Delta}\ \mathsf{Cut}$$

Thus the rules $\mathcal{R}_\mathsf{K}$ are hssp for $(\mathcal{L}, \iota)$ as well. ☐

Corollary 5.12. *If $\mathcal{L}$ is a transitive or extensible normal modal logic, then $\mathcal{R}_\mathsf{K}$ is hssp for $(\mathcal{L}, \iota_\Box)$ as well as for $(\mathcal{L}, \iota_\boxplus)$.*

Proof. By the previous Proposition we only need to show that K_0 is hssp. But this is equivalent to showing that for every $n \in \mathbb{N}$ the rule $\bigvee_{i \le n} \Box\varphi_i \vee \Box\psi / \bigvee_{i \le n} \Box\varphi_i \vee \Box\Box\psi$ (resp. $\bigvee_{i \le n} \boxplus\varphi_i \vee \boxplus\psi / \bigvee_{i \le n} \boxplus\varphi_i \vee \boxplus\Box\psi$ in the case of $\iota_\boxplus$) is admissible in $\mathcal{L}$. If the logic is transitive, then it contains the axiom $\Box\psi \to \Box\Box\psi$ and the claim follows immediately. In the extensible case, for the interpretation $\iota_\boxplus$ the negations of the premiss and conclusion of this rule are equivalent to

$$\bigwedge_{i \le n} (\neg\varphi_i \vee \Diamond\neg\varphi_i) \wedge (\neg\psi \vee \Diamond\neg\psi) \quad \text{and} \quad \bigwedge_{i \le n} (\neg\varphi_i \vee \Diamond\neg\varphi_i) \wedge (\Diamond\neg\psi \vee \Diamond\Diamond\neg\psi)$$

If the latter is satisfiable on a frame $\mathfrak{F}$ for an extensible logic, then since the additional world in the frame $\mathfrak{F}^\circ$ sees every other world, the former is satisfiable in the extension $\mathfrak{F}^\circ$. The reasoning for $\iota_\Box$ is the same. ☐

Corollary 5.13. *The rules $\mathcal{R}_\mathsf{K}$ are hssp for $(\mathsf{KD}, \iota), (\mathsf{KT}, \iota), (\mathsf{K4}, \iota), (\mathsf{KD4}, \iota), (\mathsf{S4}, \iota)$ with ι the interpretation $\iota_\Box$ or $\iota_\boxplus$.* ☐

While the criteria given in Cor. 5.12 are only sufficient and not necessary, they seem to suggest that regarding modal logics the hypersequent formalism with a reasonable standard formulation of the modal rules and a standard formula translation is suitable mainly for the treatment of transitive or extensible logics.

5.2. From Rules to Axioms

Now that we have fixed the notion of an interpretation we turn to the connections between rules and axioms. We start with the often less investigated direction from rules to axioms. The objective is to give a general translation of rules with context restrictions into axioms of a Hilbert style system. The obvious approach to this problem of course would be to translate an application of a rule with restrictions such as

$$\frac{\mathcal{P}_1 \quad \dots \quad \mathcal{P}_n}{\mathcal{H}}$$

with premisses $\mathcal{P}_1, \dots, \mathcal{P}_n$ and conclusion $\mathcal{H}$ into the implication

$$\iota(\mathcal{P}_1) \wedge \dots \wedge \iota(\mathcal{P}_n) \to \iota(\mathcal{H})$$

While this approach works fine in settings where the only allowed restriction is $\mathcal{C}_{\mathsf{id}}$, e.g. for the hypersequent systems for intermediate logics different from bd_k in [19], it breaks down already in settings where additionally the restriction $\mathcal{C}_\emptyset$ can occur. Perhaps the simplest counterexample is provided by the rules $\mathcal{R}_{\mathsf{K}}$ for modal logic K, and in particular the modal necessitation rule K_0 $\{(\ \Rightarrow p; \mathcal{C}_\emptyset)\}/\ \Rightarrow \Box p$. The result of translating the application

$$\frac{\Rightarrow \varphi}{\Rightarrow \Box\varphi}$$

of this rule using the above method (under the formula interpretation $\iota_\Box$) would be the axiom $\Box\varphi \to \Box\Box\varphi$, i.e., the axiom for transitivity. As this axiom is not a theorem of K it clearly cannot be equivalent to the rule application.

The construction we use instead is the one applied in [56, 41, 38] to translate certain sequent rules into axioms, slightly adapted to the hypersequent framework. The main idea of the construction is to turn the conclusion of a rule into a formula using the interpretation, and then to inject the information contained in the premisses into this formula via a suitable substitution constructed from the premisses. To illustrate the method, in this section we only consider the normal modality $\Box$ and restrict the context restrictions to $\{\mathcal{C}_\emptyset, \mathcal{C}_{\mathsf{id}}, \mathcal{C}_\Box\}$. The generalisation to monotone or antitone n-ary modalities and arbitrary context restrictions will be considered in Sec. 5.4. For technical reasons we need the premisses to contain at least one non-modalised propositional variable or unrestricted context formula, and thus we need to assume the following for every premiss $(\Gamma \Rightarrow \Delta; \vec{\mathcal{C}})$:

$$\text{If } \mathcal{C}_{\mathsf{id}} \notin \vec{\mathcal{C}} \text{ then } \Gamma \sqcup \Delta \neq \emptyset \tag{1}$$

For the rest of this section we fix a rule R with this property. In presence of HCut we may assume furthermore w.l.o.g. that the restriction $\mathcal{C}_{\mathsf{id}}$ does not occur in R: If it does occur we simply convert R into a rule of this format by introducing a *dummy modality* $\cdot$ satisfying $\cdot\varphi \leftrightarrow \varphi$ for all formulae and replacing every restriction $\mathcal{C}_{\mathsf{id}}$ by the sequent $\Rightarrow s$ for a fresh variable s in the premisses and by $\Rightarrow \cdot s$ in the corresponding component in the principal part. By Lemma 4.14 the resulting rule is equivalent to the original one modulo $\mathsf{H}\mathcal{R}_{\mathsf{dm}}\mathsf{Cut}$ where $\mathcal{R}_{\mathsf{dm}} = \{(p \Rightarrow\ ; \mathcal{C}_{\mathsf{id}})/ \cdot p \Rightarrow\ ,\ (\ \Rightarrow p; \mathcal{C}_{\mathsf{id}})/\ \Rightarrow \cdot p\}$ states equivalence of p and $\cdot p$. Together with property (1) this means that $\Gamma, \Delta \neq \emptyset$ for every premiss $(\Gamma \Rightarrow \Delta; \vec{\mathcal{C}})$.

Example 5.14. In order to remove the restriction $\mathcal{C}_{\mathsf{id}}$ from the premiss of the rule $R_5 := (\ \Rightarrow\ ; \mathcal{C}_\Box, \mathcal{C}_{\mathsf{id}})/ \Rightarrow\ \mid\ \Rightarrow$, we introduce a fresh variable s in the premiss and the formula $\cdot s$ in the conclusion to obtain the rule $(\ \Rightarrow s; \mathcal{C}_\Box, \mathcal{C}_\emptyset)/\ \Rightarrow\ \mid\ \Rightarrow \cdot s$.

Since the number of context formulae might vary, a rule can not be translated into a formula directly. This is avoided by fixing the number of context formulae. For normal modalities and the limited restrictions $\{\mathcal{C}_\emptyset, \mathcal{C}_{\mathsf{id}}, \mathcal{C}_\Box\}$ considered in this section this gives:

Definition 5.15 (Canonical protorule). The *canonical protorule* for a rule $R = \{(\Gamma_i \Rightarrow \Delta_i; \mathcal{C}_i^1, \dots, \mathcal{C}_i^n) : i \leq m\}/\Sigma_1 \Rightarrow \Pi_1 \mid \dots \mid \Sigma_n \Rightarrow \Pi_n$ is given by the context sequents $\Omega_1 \Rightarrow\ \mid \dots \mid \Omega_n \Rightarrow$ with $\Omega_j = \Box p_j$ if $\mathcal{C}_i^j = \mathcal{C}_\Box$ for some i and empty otherwise, using fresh variables $\vec{p}$. An *application* of the canonical protorule for R given by $\mathcal{G}$ and σ is the same as the application of R given by $\mathcal{G}, \sigma$ and the above contexts.

Example 5.16. 1. The canonical protorule for the rule 4_n from Table 3 is given by the context $\Box p \Rightarrow$ and has applications $\mathcal{G} \mid \Box\chi, \vec{\varphi} \Rightarrow \psi / \mathcal{G} \mid \Box\chi, \vec{\Box\varphi} \Rightarrow \Box\psi$.

2. The canonical protorule for the version of the rule R_5 from Example 5.14 with the dummy modality $(\Rightarrow s; \mathcal{C}_\Box, \mathcal{C}_\emptyset)/ \Rightarrow \cdot s$ is given by the contexts $\Box p \Rightarrow \mid \Rightarrow$ and has applications $\mathcal{G} \mid \Box\varphi \Rightarrow \psi / \mathcal{G} \mid \Box\varphi \Rightarrow \mid \Rightarrow \cdot\psi$.

Using the rules for normal modal logics and HCut it is straightforward to see that the canonical protorule is enough:

Lemma 5.17. *R and its canonical protorule are interderivable in $\mathsf{H}\mathcal{R}_\mathsf{K}\mathsf{Cut}$.*

Proof. Using Cut and the fact that the sequents $\Box\varphi_1, \dots, \Box\varphi_n \Rightarrow \Box \bigwedge_{i\leq n} \varphi_i$ and $\Box \bigwedge_{i\leq n} \varphi_i \Rightarrow \bigwedge_{i\leq n} \Box\varphi_i$ are derivable in $\mathsf{H}\mathcal{R}_\mathsf{K}$. $\Box$

Now suppose we have an interpretation $\iota = \{\iota_n : n \geq 1\}$ and that

$$R = \{(\Gamma_i \Rightarrow \Delta_i; \vec{\mathcal{C}_i}) : i \leq m\}/\Sigma_1 \Rightarrow \Pi_1 \mid \dots \mid \Sigma_n \Rightarrow \Pi_n$$

with $\mathcal{C}_i^j = \langle F_i^j, G_i^j \rangle$. The canonical protorule $\hat{R}$ for R is given by the contexts $\Omega_1 \Rightarrow \mid \dots \mid \Omega_n \Rightarrow$. We first construct a formula corresponding to its premisses as

$$\varphi_{\mathsf{prem}} := \bigwedge\nolimits_{i\leq m} \left(\bigwedge(\Omega_1{\restriction}_{F_i^1}, \dots, \Omega_n{\restriction}_{F_i^n}, \Gamma_i) \to \bigvee \Delta_i\right) \quad . \tag{2}$$

The idea then is to construct a substitution out of this formula, which we then can use to transfer the information contained in it to a formula corresponding to the conclusion of the rule. For this we adapt the notion of a *projective formula*, originally introduced by Ghilardi in the context of unification [28].

Definition 5.18 (Projectivity [28]). A formula φ is a *projective formula* if there is a substitution θ with

1. $\vdash_{\mathsf{HK_1Cut}} \Rightarrow \varphi\theta$; and
2. the rule $\mathcal{G} \mid \Rightarrow \varphi / \mathcal{G} \mid \Rightarrow p \leftrightarrow p\theta$ is a derivable rule in $\mathsf{HK_1Cut}$ for every variable p.

We then say that the substitution θ *witnesses projectivity of* φ.

Now for the formula φ_{prem} from (2) above, we define the substitution θ_R by

$$\theta_R(x) := \begin{cases} \varphi_{\mathsf{prem}} \land x & x \in \Gamma_i \text{ for some } i \leq m \\ \varphi_{\mathsf{prem}} \to x & x \in \Delta_i \text{ for some } i \leq m \\ x & \text{otherwise.} \end{cases}$$

Since by monotonicity w.l.o.g. no variable occurs both in antecedent and succedent of a premiss, θ_R is well-defined. Moreover, straightforward (mostly) propositional reasoning gives:

Lemma 5.19. *The substitution θ_R witnesses projectivity of φ_{prem}.* $\Box$

Using the properties of a projective formula we can now use the substitution θ_R to construct a hypersequent from the conclusion of $\hat{R}$ and show equivalence of $\hat{R}$ to the *ground hypersequent* obtained from this, i.e. the set of hypersequents constructed from this single hypersequent by closing under uniform substitution:

Lemma 5.20 (Translation to ground hypersequent)**.** *The canonical protorule $\hat{R}$ above is interderivable over* $\mathsf{HK_1Cut}$ *with the ground hypersequent* $\mathcal{H}_R := \; \Rightarrow (\bigwedge(\Omega_1, \Sigma_1) \to \bigvee \Pi_1)\, \theta_R \mid \dots \mid \Rightarrow (\bigwedge(\Omega_n, \Sigma_n) \to \bigvee \Pi_n)\, \theta_R$.

Proof. By Lemma 5.19 we have $\vdash_{\mathsf{HK_1Cut}} \Rightarrow \varphi_{\mathsf{prem}}\theta_R$ and thus $\vdash_{\mathsf{HK_1Cut}} \Rightarrow \varphi_{\mathsf{prem}}\theta_R\sigma$ for every substitution σ. Now inverting the propositional rules using Cut and an application of $\hat{R}$ give $\mathcal{H}_R\sigma$. For the other direction, Lemma 5.19 implies derivability of $\mathcal{G} \mid \Rightarrow \varphi_{\mathsf{prem}} / \mathcal{G} \mid \Rightarrow \psi \leftrightarrow \psi\theta_R$ in $\mathsf{HK_1Cut}$ (by induction on the complexity of ψ). Hence for every substitution σ we have derivability of $\mathcal{G} \mid \Rightarrow \varphi_{\mathsf{prem}}\sigma / \mathcal{G} \mid \chi_i\theta_R\sigma \Rightarrow \chi_i\sigma$ with $\chi_i = \bigwedge(\Omega_i, \Sigma_i) \to \bigvee \Delta_i$. From the premisses of an application of $\hat{R}$ we obtain $\mathcal{G} \mid \Rightarrow \varphi_{\mathsf{prem}}\sigma$ and thus $\mathcal{G} \mid \chi_i\theta_R\sigma \Rightarrow \chi_i\sigma$, and cutting these and the ground hypersequent $\mathcal{H}_R\sigma$ followed by invertibility of the propositional rules and external Contraction yield the conclusion of this application. $\Box$

The ground hypersequent $\mathcal{H}_R$ now is converted into a formula in the obvious way using the interpretation.

Theorem 5.21 (Soundness)**.** *If* $\mathsf{H}\mathcal{R}_{\mathsf{K}}\mathsf{Cut}R$ *is hssp for* $(\mathcal{L}, \iota)$*, then* $\iota(\mathcal{H}_R) \in \mathcal{L}$*.*

Proof. Since $\mathcal{H}_R$ is derivable in $\mathsf{H}\mathcal{R}_{\mathsf{K}}\mathsf{Cut}R$ by Lemma 5.20 and $\mathsf{H}\mathcal{R}_{\mathsf{K}}\mathsf{Cut}R$ is hssp for $(\mathcal{L}, \iota)$, the former is hssp for $(\mathcal{L}, \iota)$ as well (as a zero-premiss rule). Thus $\iota(\mathcal{H}_R) \in \mathcal{L}$. $\square$

Theorem 5.22 (Completeness)**.** *If for sets* $\mathcal{A}$ *of axioms and* $\mathcal{R}$ *of rules* $\mathsf{HCut}\mathcal{R}$ *is complete for* $\mathcal{L}_\mathcal{A}$ *and the rule* $\Rightarrow \varphi_1 \mid \cdots \mid \Rightarrow \varphi_n / \ \Rightarrow \iota_n(\varphi_1, \ldots, \varphi_n)$ *is derivable in* $\mathsf{HCut}\mathcal{R}$*, then* $\mathsf{HCut}\mathcal{R}R$ *is complete for* $\mathcal{L}_\mathcal{A} \oplus \iota(\mathcal{H}_R)$*.*

Proof. By Lemma 5.20 the ground hypersequent $\mathcal{H}_R$ is derivable in $\mathsf{HCut}\mathcal{R}R$, and thus the axiom $\iota(\mathcal{H}_R)$ is derivable in $\mathsf{HCut}\mathcal{R}R$ as well. Simulating modus ponens by Cut we thus obtain completeness of this calculus for $\mathcal{L}_\mathcal{A} \oplus \iota(\mathcal{H}_R)$. $\square$

Example 5.23.

1. The premiss of the canonical protorule for R_5 from Ex. 5.16.2 is turned into $\varphi = \Box p \to s$. Then with θ defined by $\theta(p) = p$ and $\theta(s) = \varphi \to s$ we obtain $\mathcal{H} = \Rightarrow \neg\Box p\theta \mid \Rightarrow \cdot s\theta = \Rightarrow \neg\Box p \mid \Rightarrow \cdot(\varphi \to s)$. Thus R_5 is equivalent under $\iota_\Box$ to the axiom $\iota_\Box(\mathcal{H}) = \Box\neg\Box p \vee \Box \cdot ((\Box p \to s) \to s)$ which modulo propositional reasoning and monotonicity is easily seen to be equivalent (as an axiom) to $\Box\neg\Box p \vee \Box \cdot \Box p$. By idempotency of $\cdot$ this is equivalent to $\Box\neg\Box p \vee \Box\Box p$.
2. The premiss of the canonical protorule $p \Rightarrow s/\Box p \Rightarrow \mid \Rightarrow \cdot s$ for the rule $\mathsf{5}$ is turned into $\varphi = p \to s$, and with $\theta(p) = (p \to s) \wedge q$ and $\theta(s) = (p \to s) \to s$ we obtain $\mathcal{H} = \Rightarrow \neg\Box p\theta \mid \Rightarrow \cdot s\theta = \Rightarrow \neg\Box((p \to s) \wedge p) \mid \Rightarrow \cdot((p \to s) \to s)$. Hence the rule $\mathsf{5}$ is equivalent under $\iota_\Box$ to the axiom $\Box\neg\Box((p \to s) \wedge p) \vee \Box((p \to s) \to s)$. Moreover, substituting p for s, validity of this axiom implies validity of the axiom $(\mathsf{5}) = \Box\neg\Box p \vee \Box p$, and by monotonicity of $\Box$ validity of the latter implies validity of the former. Hence the rule $\mathsf{5}$ is equivalent in a precise sense to the axiom $(\mathsf{5})$.

Crucially, Thm. 5.21 also implies that rules stay hssp in extensions of a logic, provided the rule K_1 is hssp in the extension:

Corollary 5.24. *If* $\mathcal{L}_1 \subseteq \mathcal{L}_2$*, and* ι *is an interpretation for* $\mathcal{L}_1, \mathcal{L}_2$*, and* $\mathsf{HK}_1\mathsf{Cut}$ *is hssp for* $(\mathcal{L}_1, \iota)$ *and* $(\mathcal{L}_2, \iota)$*, then if* R *is hssp for* $(\mathcal{L}_1, \iota)$ *it is also hssp for* $(\mathcal{L}_2, \iota)$*.*

Proof. Since R and $\mathcal{H}_R$ are interderivable over $\mathsf{HK}_1\mathsf{Cut}$ and $\iota(\mathcal{H}_R) \in \mathcal{L}_1 \subseteq \mathcal{L}_2$. $\square$

5.3. From Axioms to Rules

The translation from axioms to rules proceeds similar to that for sequent rules in [41, 38], but uses the interpretation to peel away one layer of the formula first. The idea is to treat some subformulae of an axiom as *context formulae* and translate the axiom into a protorule (i.e. a rule with a fixed number of context formulae). To simplify presentation we assume monotonicity of the modalities, i.e., we take the logics to be closed under the monotonicity rules $A \to B/\heartsuit A \to \heartsuit B$. We now define the class of translatable axioms and then give the actual translation procedure.

Definition 5.25 (Hypertranslatable). Let $C_\ell, C_r \subseteq \mathcal{F}(\Lambda)$ and $V \subseteq \mathcal{V}$. The class of *translatable clauses* for (C_ℓ, V, C_r) is defined by the following grammar (starting variable S):

$$
\begin{aligned}
S &::= L \to R \\
L &::= L \wedge L \mid \heartsuit P_r \mid \psi_\ell \mid \top \mid \bot \qquad R ::= R \vee R \mid \heartsuit P_\ell \mid \psi_r \mid \top \mid \bot \\
P_r &::= P_r \vee P_r \mid P_r \wedge P_r \mid P_\ell \to P_r \mid \psi_r \mid p \mid \bot \mid \top \\
P_\ell &::= P_\ell \vee P_\ell \mid P_\ell \wedge P_\ell \mid P_r \to P_\ell \mid \psi_\ell \mid p \mid \bot \mid \top
\end{aligned}
$$

where $\heartsuit \in \Lambda_\mathsf{U}, p \in V$ and $\psi_i \in C_i$ for $i \in \{\ell, r\}$. A formula is *hypertranslatable for an interpretation* $\iota = \{\iota_n : n \geq 1\}$ if has the form $\iota_n(\chi_1, \ldots, \chi_n)$ with χ_i a translatable clause for (C_ℓ, V, C_r) where no distinct formulae in $C_\ell \cup V \cup C_r$ share a variable, and every formula in $C_\ell \cup C_r$ occurs in the χ_i exactly once not in the scope of a modality and at least once in the scope of a modality.

The intuition behind this definition is the following. The interpretation ι provides the outermost structural layer of hypertranslatable axioms, with translatable clauses inside. This is used to convert an axiom into a ground hypersequent. The next structural layer is given by the variables S, L, R in the grammar above and is used to convert this ground hypersequent into one which only contains propositional variables, modalised formulae, formulae from C_ℓ on the left hand side and formulae from C_r on the right hand side (called the *ground hypersequent stage* below). The formulae in C_ℓ (resp. C_r) will play the role of context formulae on the left (resp. right). Then the formulae under the modalities are moved into the premisses, thereby moving from the left hand side to the right hand side and vice versa (*shaping the conclusion*). The innermost structural layer of hypertranslatable axioms, given by the variables P_r and P_ℓ is then used to eliminate propositional connectives from these premisses (*resolving propositional logic*). A little thought shows that hypersequents $\mathcal{G} \mid \Rightarrow \varphi$ (resp. $\mathcal{G} \mid \varphi \Rightarrow$) with φ generated by taking P_r (resp. P_ℓ) as starting variable in the above grammar can be decomposed using invertibility of the propositional rules into sets of hypersequents $\mathcal{G} \mid \Gamma \Rightarrow \Delta$ with $\Gamma \subseteq C_\ell \cup V$ and $\Delta \subseteq C_r \cup V$. The global conditions on the formulae in C_ℓ, C_r ensure that the context formulae behave in the right way. The translation procedure then also brings the resulting rules into a nicer shape by eliminating propositional variables occurring in the premisses but not the conclusion (*cleaning the premisses*) and by replacing the context formulae with context restrictions (*introducing context restrictions*).

We now fix a logic $\mathcal{L}$, an interpretation $\iota = \{\iota_n : n \geq 1\}$ and a hypertranslatable formula φ for ι and consider the stages of the translation in detail.

Ground hypersequent stage. We have $\varphi = \iota_n(\varphi_1, \dots, \varphi_n)$ where $\varphi_i = \bigwedge \vec{\psi^i} \wedge \bigwedge \vec{\chi^i} \to \bigvee \vec{\xi^i} \vee \bigvee \vec{\zeta^i}$ with context formulae $\chi^i_j \in C_\ell$, $\zeta^i_j \in C_r$ and formulae ψ^i_j (resp. ξ^i_j) of the form $\heartsuit \delta_j$ with δ_j generated by the above grammar with starting variable P_r (resp. P_ℓ). This is turned into the ground hypersequent $\mathcal{H}_\varphi := \vec{\psi^1}, \vec{\chi^1} \Rightarrow \vec{\xi^1}, \vec{\zeta^1} \mid \cdots \mid \vec{\psi^n}, \vec{\chi^n} \Rightarrow \vec{\xi^n}, \vec{\zeta^n}$ which by HCut is hssp for $(\mathcal{L}, \iota)$.

Shaping the conclusion. We replace each $\psi^i_j = \heartsuit \delta^i_j$ with $\heartsuit p^i_j$ where $p^i_j \in \mathcal{V}$ is fresh and add the premiss $p^i_j \Rightarrow \delta^i_j$. Analogously we replace $\xi^i_j = \heartsuit \gamma^i_j$ with $\heartsuit q^i_j$ and add the premiss $\gamma^i_j \Rightarrow q^i_j$. By monotonicity (i.e., the rule K_1 for $\Box$-modalities) and Cut this is equivalent to $\mathcal{H}_\varphi$.

Resolving propositional logic. Using invertibility of the propositional rules we replace each of these premisses by a number of sequents $\Gamma \Rightarrow \Delta$ with $\Gamma \subseteq C_\ell \cup V$ and $\Delta \subseteq C_r \cup V$. In presence of HCut this gives an equivalent rule.

Cleaning the premisses. To ensure that every variable occurring in the premisses of the rule also occurs in the conclusion we eliminate the variables from V from the premisses by successively cutting the premisses on all variables in V as in Def.4.1 disregarding context restrictions. Reasoning as in Lemma 4.14 the resulting rule is seen to be equivalent to the original rule (compare also [16]).

Introducing context restrictions. The global condition on the context formulae in Def. 5.25 guarantees that every formula in $C_\ell \cup C_r$ occurs exactly once in the conclusion and at least once in the premisses. Moreover, it occurs always on the same side of the sequent. Thus we now have a rule with a fixed number of context formulae. Provided the context formulae are *normal* in the sense that formulae in C_ℓ distribute over $\wedge$ and those in C_r over $\vee$ we may replace them with context restrictions by turning a premiss $\chi_1, \dots, \chi_m, \Gamma \Rightarrow \Delta, \zeta_1, \dots, \zeta_k$ with context formulae χ_j and ζ_j occurring in the i_j-th component of the conclusion into the premiss with restriction $(\Gamma \Rightarrow \Delta; \vec{\mathcal{C}})$ where $\mathcal{C}^i = \langle \{\chi_j : i_j = i\}; \{\zeta_j : i_j = i\} \rangle$ and deleting all context formulae from the conclusion. Call the resulting rule R_φ.

Since all steps in the above construction yield rules interderivable with the original ones using HCut and monotonicity, the rules in H are hssp by Prop. 5.4 and soundness of additional rules is preserved by Cor. 5.24, we obtain soundness and completeness provided the monotonicity rule K_1 stays hssp.

Proposition 5.26 (Soundness and Completeness)**.** *Let ι be a regular interpretation for $\mathcal{L}$ and let* $\mathsf{HCut}\mathcal{R}$ *be hssp and complete for $(\mathcal{L}, \iota)$ with the rule $\Rightarrow p_1 \mid \cdots \mid \Rightarrow p_n / \Rightarrow \iota_n(\vec{p})$ derivable in $\mathcal{R}$. If φ is* *hypertranslatable for ι with normal context formulae (C_ℓ, C_r), the rules $\mathcal{R}_\mathsf{K}$ are hssp for $(\mathcal{L} \oplus \varphi, \iota)$, and ι is a regular interpretation for $\mathcal{L} \oplus \varphi$ then $\mathsf{HCut}\mathcal{R}R_\varphi$ is sound and complete for $(\mathcal{L} \oplus \varphi, \iota)$.*

Example 5.27.
1. Using $\iota_\Box$ the axiom $\Box\neg\Box p \vee \Box \cdot \Box p$ from Ex. 5.23 is converted into the ground hypersequent $\Box p \Rightarrow \mid \Rightarrow \cdot\Box p$. Taking $\Box p$ to be in C_ℓ we introduce a fresh variable q and the corresponding premiss to obtain $\Box p \Rightarrow q/\Box p \Rightarrow \mid \Rightarrow \cdot q$. Using normality of $\Box$ (for $\mathcal{R}_\mathsf{K}$) the formula $\Box p$ is now replaced with the context restriction $\langle\{\Box p\}, \emptyset\rangle = \mathcal{C}_\Box$ resulting in the rule $(\Rightarrow q; \mathcal{C}_\Box, \mathcal{C}_\emptyset)/ \Rightarrow \mid \Rightarrow \cdot q$. Now a cut with the left rule for $\cdot$ gives the rule R_5.
2. Again using $\iota_\Box$ the axiom $(\mathsf{5})$ $\Box\neg\Box p \vee \Box \cdot p$ is converted into the ground hypersequent $\Box p \Rightarrow \mid \Rightarrow \cdot p$. Taking C_ℓ and C_r to be empty we introduce two fresh variables r, s for the two occurrences of p together with the corresponding premisses to obtain $\{r \Rightarrow p,\ p \Rightarrow s\}/\Box r \Rightarrow \mid \Rightarrow \cdot s$. Now cleaning the premisses by cutting on the variable p yields the rule $r \Rightarrow s/\Box r \Rightarrow \mid \Rightarrow \cdot s$ and a cut with the left rule for $\cdot$ gives the rule $\mathsf{5} = \{(q \Rightarrow ; \mathcal{C}_\emptyset, \mathcal{C}_\mathsf{id})\}/\Box q \Rightarrow \mid \Rightarrow$ from Ex. 4.2.3.

5.4. Non-normal restrictions

In the more general case, where the connectives and context formulae are not normal, the method still goes through, but now a single axiom corresponds to a rule with a fixed number of context formulae (compare [38, 40, 41] in the sequent framework). Thus instead of considering the canonical protorule of Def. 5.15 we need to consider a set of such protorules containing one instance for every number of context formulae. Formally this is defined as follows.

Definition 5.28 (Protorule). A *protorule* for a rule $R = \{(\Gamma_i \Rightarrow \Delta_i; \vec{\mathcal{C}_i}) : 1 \leq i \leq m\}$ / $\Sigma_1 \Rightarrow \Pi_1 \mid \cdots \mid \Sigma_n \Rightarrow \Pi_n$ is given by context sequents $\Theta_j \Rightarrow \Omega_j$ for $1 \leq j \leq n$ such that

1. no variable occurs more than once in $\bigsqcup_{j \leq n}(\Theta_j, \Omega_j)$; and
2. no variable occurs both in $\bigsqcup_{j \leq n}(\Theta_j, \Omega_j)$ and in R.

An *application* of this protorule is given by a substitution σ and a side hypersequent $\mathcal{G}$ and is the same as the application of R given by $\sigma, \mathcal{G}$ and the context sequents $\Theta_i\sigma \Rightarrow \Omega_i\sigma, (i \leq m)$.

Example 5.29. In order to analyse the modalised splitting rule MS from Ex. 3.6 we again use the dummy modality $\cdot$ to replace the restriction $\mathcal{C}_\mathsf{id}$ by a variable r in the premiss and $\cdot r$ in the conclusion. A protorule for this rule then is given e.g. by the context sequents $\Box p \Rightarrow \Box q_1, \Box q_2 \mid \Rightarrow$. Its applications have the form $\mathcal{G} \mid \Box\chi \Rightarrow \varphi, \Box\psi_1, \Box\psi_2/\mathcal{G} \mid \Box\chi \Rightarrow \Box\psi_1, \Box\psi_2 \mid \Rightarrow \cdot\varphi$.

Now in the translation of rules into axioms from Sec. 5.2 we replace the canonical protorule by the *canonical set of protorules* for it where for every context formula and every $n \in \mathbb{N}$ we introduce n instances of the context formula with fresh variables each.

Example 5.30. The canonical set of protorules for the rule MS is the set of protorules for MS given by the set of context sequents $\Box p_1, \ldots, \Box p_m \Rightarrow \Box q_1, \ldots, \Box q_n \mid \Rightarrow$ for $m, n \in \mathbb{N}$.

Now equivalence of the original rule and the canonical set of protorules for it is obvious and we may simply translate every protorule in this set into an axiom. Since the substitution constructed in the translation relied on the fact that no variable occurs both in the antecedent and succedent of a premiss, we still need to stipulate monotonicity (or antitonicity) of the connectives in each component. Thus writing Mon for the set containing for every modality $\heartsuit$ the *monotonicity* or the *antitonicity rules*

$$\frac{(p \Rightarrow q; \mathcal{C}_\emptyset)}{\heartsuit p \Rightarrow \heartsuit q}\ \mathsf{Mon}_\heartsuit \qquad \text{resp.} \qquad \frac{(p \Rightarrow q; \mathcal{C}_\emptyset)}{\heartsuit q \Rightarrow \heartsuit p}\ \mathsf{Ant}_\heartsuit$$

depending on whether $\heartsuit$ is monotone or antitone, we have equivalence over $\mathsf{HMonCut}$ of the original rule R with the set of axioms resulting from translating all the protorules in the canonical set of protorules for R.

Similarly, translating a single hypertranslatable formula yields a protorule with exactly one instance of each context formula in the conclusion. We obtain a proper rule if we start with a *set* of hypertranslatable formulae constructed from a single such formula by uniformly replacing each context formula with a conjunction resp. disjunction of instances of this formula.

Definition 5.31 (ω-sets). Let the formula φ be hypertranslatable for ι with context formulae in $C_\ell = \{\psi_1, \dots, \psi_n\}$ and $C_r = \{\chi_1, \dots, \chi_m\}$. The *$\omega$-set for φ* is the set $\{\varphi_{i_1,\dots,i_n,j_1,\dots,j_m} : i_k, j_k \in \mathbb{N}\}$, where the formula $\varphi_{i_1,\dots,i_n,j_1,\dots,j_m}$ is obtained from φ by uniformly replacing the context formulae ψ_k (resp. χ_k) by $\bigwedge_{s=1}^{i_k} \varphi_k^s$ (resp. $\bigvee_{s=1}^{j_k} \chi_k^s$) for φ_k^s an instance of φ_k with fresh variables and analogously for χ_k.

It is straightforward to see that translating all protorules in the canonical set of protorules for a rule yields an ω-set for an axiom and vice versa. Thus in the general case with monotone or antitone connectives we see that hypersequent rules with context restrictions correspond to ω-sets for hypertranslatable axioms.

Proposition 5.32 (Correspondence between rules and ω-sets). *Let $\mathcal{R}$ be a rule set containing the monotonicity or antitonicity rules for every connective such that $\mathsf{HCut}\mathcal{R}$ is hssp and complete for $(\mathcal{L}_\mathcal{A}, \iota)$ with ι regular for $\mathcal{L}_\mathcal{A}$ and such that the rule $\Rightarrow \varphi_1 \mid \cdots \mid \Rightarrow \varphi_n / \Rightarrow \iota_n(\varphi_1, \dots, \varphi_n)$ is derivable in $\mathsf{HCut}\mathcal{R}$.*

1. *If R is a rule with restrictions and $\mathcal{B}$ is the set of translations of protorules in the canonical set of protorules for R, then $\mathcal{B}$ is an ω-set for a hypertranslatable axiom for ι, and $\mathsf{HCut}\mathcal{R}R$ is hssp and complete for $(\mathcal{L}_{\mathcal{AB}}, \iota)$ provided the monotonicity resp. antitonicity rules are hssp for $(\mathcal{L}_{\mathcal{AB}}, \iota)$.*
2. *If $\mathcal{B}$ is an ω-set for a hypertranslatable axiom for ι such that ι is a regular interpretation for $\mathcal{L}_{\mathcal{AB}}$, the monotonicity resp. antitonicity rules are hssp for $(\mathcal{L}_{\mathcal{AB}}, \iota)$, and R is the translation of $\mathcal{B}$ into a rule, then $\mathsf{HCut}\mathcal{R}R$ is hssp and complete for $(\mathcal{L}_{\mathcal{AB}}, \iota)$.* $\square$

Example 5.33. The right context formula $\Box p$ in the modalised splitting rule $\{(\Rightarrow ; \langle\{\Box p\}, \{\Box p\}\rangle, \mathcal{C}_{\mathsf{id}})\}/ \Rightarrow \mid \Rightarrow$ from Ex. 3.6 is not normal since $\Box$ does not distribute over disjunctions. Its canonical set of protorules (after replacing $\mathcal{C}_{\mathsf{id}}$ by $\mathcal{C}_\emptyset$ using the dummy modality $\cdot$) is

$$\left\{\frac{\Box p_1, \dots, \Box p_n \Rightarrow \Box q_1, \dots, \Box q_m, r}{\Box p_1, \dots, \Box p_n \Rightarrow \Box q_1, \dots \Box q_m \mid \Rightarrow \cdot r} : n, m \in \mathbb{N}\right\}$$

and the corresponding set of axioms under the standard interpretation $\iota_\Box$ is the ω-set for the axiom $\Box(\Box p \Rightarrow \Box q) \vee \Box((\Box p \to \Box q \vee r) \to r)$. Using equivalence of this axiom to $\Box(\Box p \to \Box q) \vee \Box\neg(\Box p \to \Box q)$ we thus obtain correspondence of the original rule with the ω-set

$$\left\{\Box(\bigwedge_{i=1}^{n} \Box p_i \to \bigvee_{j=1}^{m} \Box q_j) \vee \Box\neg(\bigwedge_{i=1}^{n} \Box p_i \to \bigvee_{j=1}^{m} \Box q_j) : n, m \in \mathbb{N}\right\}$$

for the latter axiom. Moreover, translating this ω-set back into a rule yields exactly the rule MS.

5.5. Applications: Limitative Results

Apart from the more constructive motivation of producing hypersequent calculi from Hilbert-style axiomatisations, the correspondence between axioms and rules also provides the means to show *limitative* results stating which logics cannot be captured by hypersequent rules of a specific format. Following [38, 41] the main idea is to show that the translations of rules of a certain format have a particular syntactic shape, and that formulae of this shape cannot modally define a given (modally definable) class of Kripke-frames and hence cannot axiomatise the logic of this class of frames. We need the following simple Lemma.

Lemma 5.34. *Let F be a modally definable class of Kripke-frames and $\mathcal{A}$ a set of modal formulae. If $\mathcal{L}_{\mathsf{K}\mathcal{A}}$ is the logic of F, then $\mathcal{A}$ modally defines F.*

Proof. Since the formulae in $\mathcal{A}$ are valid in every frame in F and the modal formulae defining F are derivable in $\mathcal{L}_{\mathsf{K}\mathcal{A}}$. $\square$

Given a modally definable class of Kripke-frames we may now try to exhibit two frames which cannot be distinguished by modal formulae of the syntactic shape of translations of hypersequent rules of a certain format. While the format of hypersequent rules with context restrictions already gives a restriction on the format of the corresponding axioms, we consider a further (mild) restriction on the rule format.

$$\begin{tikzcd} & y_1 \arrow[r] & y_2 \arrow[r] & y_3 \arrow[dr] & \\ x \arrow[ur] \arrow[r] & w_1 \arrow[r] & w_2 \arrow[rr] & & z \end{tikzcd} \qquad \begin{tikzcd} & \tilde{y}_1 \arrow[r] & \tilde{y}_2 \arrow[r] & \tilde{y}_3 \arrow[r] & \tilde{z}_2 \\ \tilde{x} \arrow[ur] \arrow[r] & \tilde{w}_1 \arrow[r] & \tilde{w}_2 \arrow[r] & \tilde{z}_1 & \end{tikzcd}$$

$$\mathfrak{F}_1 \qquad\qquad\qquad \mathfrak{F}_2$$

Figure 1: The frames used in the proof of Thm. 5.37

Definition 5.35. A hypersequent rule with context restrictions has *shallow restrictions* if all formulae occurring in its context restrictions have modal nesting depth at most 1.

Obviously, every rule set involving only the restrictions $\mathcal{C}_\emptyset, \mathcal{C}_{\mathsf{id}}, \mathcal{C}_\Box$ has shallow restrictions. The following (relatively simple) example illustrates the method.

Definition 5.36. A Kripke-frame is *3-transitive* if it satisfies

$$\forall x \forall y_1 \forall y_2 \forall y_3 \forall z (xRy_1 \wedge y_1Ry_2 \wedge y_2Ry_3 \wedge y_3Rz \to \exists w_1 \exists w_2 (xRw_1 \wedge w_1Rw_2 \wedge w_2Rz)) \ .$$

The logic given by the class of 3-transitive Kripke-frames is denoted by $\mathcal{L}_{3\mathrm{tr}}$.

It is not too hard to see that $\mathcal{L}_{3\mathrm{tr}}$ is modally defined by the formula $\Box\Box\Box p \to \Box\Box\Box\Box p$. While the class of 3-transitive Kripke-frames is extensible, and thus the interpretation $\iota_\Box$ is regular for it, and the rules $\mathcal{R}_{\mathsf{K}}$ are hssp for $(\mathcal{L}_{3\mathrm{tr}}, \iota_\Box)$ and $(\mathcal{L}_{3\mathrm{tr}}, \iota_\boxplus)$, the logic $\mathcal{L}_{3\mathrm{tr}}$ nevertheless cannot be captured using hypersequent rules with shallow restrictions satisfying the restriction on the premisses necessary for the translation from rules to axioms:

Theorem 5.37 (3-transitivity)**.** *There is no set of rules with shallow restrictions such that every premiss either contains a restriction $\mathcal{C}_{\mathsf{id}}$ or a variable, which is sound and complete for $(\mathcal{L}_{3\mathrm{tr}}, \iota_\Box)$ or $(\mathcal{L}_{3\mathrm{tr}}, \iota_\boxplus)$.*

Proof. Since the translation from rules to axioms of Sec. 5.2 resp. 5.4 replaces variables in the conclusion of a rule by formulae with modal nesting depth at most one, the translations of the rules under $\iota_\Box$ or $\iota_\boxplus$ have modal nesting depth at most 3. But formulae of this format cannot distinguish the two frames $\mathfrak{F}_1$ and $\mathfrak{F}_2$ of Fig. 1: From a valuation witnessing satisfiability of the negation of such a formula in one of the frames it is possible to construct a valuation witnessing satisfiability of the same formula in the other frame. E.g., if for φ with modal nesting depth ≤ 3 we have $\mathfrak{F}_2, \tilde{\sigma}, \tilde{x} \Vdash \neg\varphi$, then setting $\sigma(v) = \tilde{\sigma}(\tilde{v})$ for $v \neq z$ and $\sigma(z) = \tilde{\sigma}(\tilde{z}_1)$ we have $\mathfrak{F}_1, \sigma, x \Vdash \neg\varphi$. The remaining cases are similar. Hence no set $\mathcal{A}$ of formulae with modal nesting depth ≤ 3 modally defines the class of 3-transitive frames, and thus using Lemma 5.34 no such $\mathcal{A}$ axiomatises the logic $\mathcal{L}_{3\mathrm{tr}}$. $\Box$

6. Case Studies

Apart from being useful for showing the limits of expressivity of a specific format of hypersequent rules as shown above, the methods developed in the previous section are also useful for constructing and investigating hypersequent calculi for specific logics. We will now consider a number of examples of logics and calculi which can be treated this way, starting with a recently presented class of semantically given calculi, followed by a closer look at calculi for the logic $\mathsf{S5}$, logics for convergent and connected frames, and finally certain combinations of different modal logics.

6.1. Logics for Simple Frame Properties

An interesting class of examples are the rules constructed from *simple* frame properties for normal modal logics [37]. We apply our methods to these calculi to obtain cut elimination and complexity results and results on the translation of such rules into axioms. A *simple* frame property is a formula $\forall w_1 \ldots \forall w_n \exists u \varphi_S$ in the frame language, with $\varphi_S = \bigvee_{\langle S_R, S_= \rangle \in S} (\bigwedge_{i \in S_R} w_i R u \wedge \bigwedge_{i \in S_=} w_i = u)$ for some non-empty *description* S consisting of a set of tuples $\langle S_R, S_= \rangle$ with $S_R, S_= \subseteq \{1, \ldots, n\}$ and $S_R \cup S_= \neq \emptyset$. We identify a simple frame property with its description. In [37] hypersequent rules corresponding to simple frame properties based on $\mathsf{K}, \mathsf{K4}$ and KB are given and cut admissibility for the calculi based on K or $\mathsf{K4}$ is shown via the semantics. Here we consider the rules based on K and $\mathsf{K4}$ (those for KB do not fit our rule format). The set of *hypersequent rules induced by S for $\mathcal{R}_\mathsf{K}$* is $\mathcal{R}_S := \{R_{k_1,\dots,k_n} : k_i \geq 0 \text{ for } i \leq n\}$ with

$$R_{k_1,\dots,k_n} := \frac{\left\{(\bigsqcup_{j\in S_R} p_1^j, \dots, p_{k_j}^j \Rightarrow ; \mathcal{C}^1_{\langle S_R, S_=\rangle}, \dots, \mathcal{C}^n_{\langle S_R, S_=\rangle}) \ : \ \langle S_R, S_=\rangle \in S\right\}}{\Box p_1^1, \dots, \Box p_{k_1}^1 \Rightarrow | \cdots | \Box p_1^n, \dots, \Box p_{k_n}^n \Rightarrow}$$

where $\mathcal{C}^j_{\langle S_R,S_=\rangle} = \mathcal{C}_{\mathsf{id}}$ for $j \in S_=$ and $\mathcal{C}_\emptyset$ otherwise. The set of *hypersequent rules induced by S for $\mathcal{R}_{\mathsf{K4}}$* is the set $\mathcal{R}^4_S := \{R^4_{k_1,\dots,k_n} : k_i \geq 0\}$ with $R^4_{k_1,\dots,k_n}$ the rule $R_{k_1,\dots,k_n}$ with $\mathcal{C}^j_{\langle S_R,S_=\rangle} = \mathcal{C}_{\mathsf{id}}$ for $j \in S_=$ and $\mathcal{C}_\Box$ for $j \in S_R \smallsetminus S_=$ and $\mathcal{C}_\emptyset$ otherwise. Inspection of the rule sets constructed in this way shows that together with $\mathsf{H}\mathcal{R}_\mathsf{K}$ (resp. $\mathsf{H}\mathcal{R}_{\mathsf{K4}}$) they satisfy all conditions given in Thm. 4.12. Thus we obtain a purely syntactic analogue to the semantic cut admissibility proof in [37] with an additional complexity bound:

Corollary 6.1 (Cut elimination, complexity)**.** *If $\mathcal{R}$ is a set of rules induced by simple frame properties for $\mathcal{R}_\mathsf{K}$ (resp. $\mathcal{R}_{\mathsf{K4}}$), then $\mathsf{H}\mathcal{R}_\mathsf{K}\mathcal{R}$ (resp. $\mathsf{H}\mathcal{R}_{\mathsf{K4}}\mathcal{R}$) has cut elimination and an* $\mathsf{EXPSPACE}$*-decision procedure.* □

Using the translation from rules to axioms we furthermore obtain finite axiomatisations from the so constructed rules, provided we have a regular interpretation and the rules are hssp for this interpretation. While $\iota_\boxplus$ is always regular, the interpretation $\iota_\Box$ gives cleaner axioms. Sometimes regularity of $\iota_\Box$ can be read off the frame properties directly: if $S_R \neq \emptyset \neq S_=$ for all $\langle S_R, S_=\rangle \in S$ for one property S, then the logic is reflexive, and if $S_= = \emptyset$ for all $\langle S_R, S_=\rangle \in S$ for every S, then the logic is extensible (Def. 5.5). Under certain conditions we may also adjust the original soundness proof to our setting:

Proposition 6.2 (Soundness [37])**.** *If S is a simple frame property and $\mathcal{L}_S$ resp. $\mathcal{L}^4_S$ are the logics of the class of frames (resp. transitive frames) with this property, then:*

1. *$\mathcal{R}^4_S$ is hssp for $(\mathcal{L}^4_S, \iota_\Box)$ and $(\mathcal{L}^4_S, \iota_\boxplus)$*
2. *if $\mathcal{L}_S$ is extensible or if $S_= \neq \emptyset$ for all $\langle S_R, S_=\rangle \in S$, then $\mathcal{R}_S$ is hssp for $(\mathcal{L}_S, \iota_\Box)$ and $(\mathcal{L}_S, \iota_\boxplus)$.*

Proof. We show the statement for $\iota_\boxplus$, the case for $\iota_\Box$ is similar but easier. We show that if we have a model refuting the interpretation of the conclusion of an application of an induced rule, then there is also a refuting model for the interpretation of one of the premisses. So suppose there is a model $(W, R), w, \sigma$ refuting the interpretation $\iota_\boxplus(\mathcal{G} \mid \Gamma_1, \Box\Sigma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n, \Box\Sigma_n \Rightarrow \Delta_n)$ of the conclusion of a rule induced by S. Suppose that $\mathcal{G} = \Theta_1 \Rightarrow \Omega_1 \mid \cdots \mid \Theta_m \Rightarrow \Omega_m$. Then w.l.o.g. there are $k \leq m$ and $\ell \leq n$ and worlds $v_1, \dots, v_\ell$ and $w_1, \dots, w_k$ with wRv_i and wRw_j for $i \leq \ell, j \leq k$ such that

- $(W, R), w, \sigma \not\Vdash \bigwedge \Theta_j \to \bigvee \Omega_j$ for $k < j \leq m$
- $(W, R), w, \sigma \not\Vdash \bigwedge \Gamma_i \wedge \bigwedge \Box\Sigma_i \to \bigvee \Delta_i$ for $\ell < i \leq n$
- $(W, R), w_j, \sigma \not\Vdash \bigwedge \Theta_j \to \bigvee \Omega_j$ for $1 \leq j \leq k$
- $(W, R), v_i, \sigma \not\Vdash \bigwedge \Gamma_i \wedge \bigwedge \Box\Sigma_i \to \bigvee \Delta_i$ for $1 \leq i \leq \ell$.

Since the frame (W, R) satisfies $\forall\vec{v}\exists u\varphi_S$, there is a $\langle S_R, S_=\rangle \in S$ and a world $u \in W$ such that v_iRu for every $i \in S_R, i \leq \ell$ and $v_i = u$ for every $i \in S_=, i \leq \ell$ and wRu (resp. $w = u$) if $S_R \cup \{\ell+1, \dots, n\} \neq \emptyset$ (resp. $S_= \cup \{\ell+1, \dots, n\} \neq \emptyset$). Hence we have

$$(W, R), u, \sigma \not\Vdash \bigwedge_{i\in S_=} \bigwedge \Gamma_i \wedge \bigwedge_{j\in S_R} \bigwedge \Sigma_j \to \bigvee_{i\in S_=} \Delta_i \ .$$

But by construction $\mathcal{H} := \mathcal{G} \mid \bigsqcup_{i\in S_=} \Gamma_i, \bigsqcup_{j\in S_R} \Sigma_j \Rightarrow \bigsqcup_{i\in S_=} \Delta_i$ is a premiss of the (application of the) rule induced by S for $\mathcal{R}_\mathsf{K}$. Now if $S_= \neq \emptyset$ for all $\langle S_R, S_=\rangle \in S$, then either $v_i = u$ for some $i \leq n$ and we have wRu and hence $(W, R), w, \sigma \not\Vdash \Box(\bigwedge_{i\in S_=} \bigwedge \Gamma_i \wedge \bigwedge_{j\in S_R} \bigwedge \Sigma_j \to \bigvee_{i\in S_=} \bigvee \Delta_i)$; or $w = u$ and hence

$(W,R), w, \sigma \not\Vdash \bigwedge_{i\in S_=} \bigwedge \Gamma_i \wedge \bigwedge_{j\in S_R} \bigwedge \Sigma_j \to \bigvee_{i\in S_=} \bigvee \Delta_i$. In both cases we have $(W,R), w, \sigma \not\Vdash \iota_{\boxplus}(\mathcal{H})$ and are done. If on the other hand $S_= = \emptyset$ for all $\langle S_R, S_=\rangle \in S$, then the class of frames defined by $\forall \vec{v} \exists u \varphi_S$ is extensible and for the new world x in $(W,R)^\circ$ we have $xR^\circ u$ and $xR^\circ w$ as well as $xR^\circ w_j$ for $j \leq m$. Hence for a valuation σ° with $\sigma^\circ\!\upharpoonright_W = \sigma$ we have $(W,R)^\circ, x, \sigma^\circ \not\Vdash \iota_{\boxplus}(\mathcal{H})$. Finally, in the transitive case the interpretation of the conclusion has the form $\iota_{\boxplus}(\mathcal{G} \mid \Gamma_1, \Box\Sigma_1, \Box\Pi_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n, \Box\Sigma_n, \Box\Pi_n \Rightarrow \Delta_n)$ and constructing u in the same way as above by transitivity we have

$$(W,R), u, \sigma \not\Vdash \bigwedge_{i\in S_=} \bigwedge \Gamma_i \wedge \bigwedge_{j\in S_R} (\bigwedge \Sigma_j \wedge \bigwedge \Box\Pi_j) \to \bigvee_{i\in S_=} \Delta_i \ .$$

But since by transitivity also wRu the model $(W,R), w, \sigma$ refutes the interpretation of the corresponding premiss. □

To obtain the simplest axioms we observe that given $\mathsf{H}\mathcal{R}_{\mathsf{K}}\mathsf{Cut}$ (resp. $\mathsf{H}\mathcal{R}_{\mathsf{K4}}\mathsf{Cut}$) by Lemma 4.14 the set of rules induced by a simple property is equivalent (in both cases!) to a *single* rule $\{(\bigsqcup_{i\in S_R} p_i \Rightarrow ; \vec{\mathcal{C}}_{\langle S_R, S_=\rangle}) : \langle S_R, S_=\rangle \in S\}/(\Box p_1)^\varepsilon \Rightarrow \mid \cdots \mid (\Box p_n)^\varepsilon \Rightarrow$ with $\mathcal{C}^i_{\langle S_R, S_=\rangle} = \mathcal{C}_{\mathsf{id}}$ for $i \in S_=$ and $\mathcal{C}_\emptyset$ otherwise, and where $(\Box p_i)^\varepsilon$ is $\Box p_i$ if there is a $\langle S_R, S_=\rangle \in S$ with $i \in S_R$ and empty otherwise. Translating this rule gives the corresponding axiom. This restricts the shape of the resulting axioms.

Definition 6.3 (Simple axioms). A *ι-simple axiom* for an interpretation $\iota = \{\iota_n : n \geq 1\}$ is an axiom $\iota_n(\varphi_1, \ldots, \varphi_n)$ where $\mathsf{mrk}\,(\varphi_i) \leq 1$ and $\Box$ occurs only negatively in the φ_i.

It is not too hard to see that the translations of rules of the above mentioned form indeed give ι-simple axioms.

Proposition 6.4 (Translation to simple axioms)**.** *Let $\mathcal{L}_S$ (resp. $\mathcal{L}^4_S$) be the logic of the class* F *of frames (resp. transitive frames) satisfying the simple frame property S. Then:*

1. *$\mathcal{L}^4_S$ is axiomatised over* $\mathsf{K4}$ *by one $\iota_{\boxplus}$-simple axiom*
2. *$\mathcal{L}^4_S$ is axiomatised over* $\mathsf{K4}$ *by one $\iota_{\Box}$-simple axiom if $\mathcal{L}^4_S$ is reflexive or* F *is p-extensible*
3. *$\mathcal{L}_S$ is axiomatised over* K *by one $\iota_{\Box}$-simple axiom if*
   (a) F *is extensible; or*
   (b) *$\mathcal{L}_S$ is reflexive and $S_= \neq \emptyset$ for all $\langle S_R, S_=\rangle \in S$ and $\mathcal{R}_{\mathsf{K}}$ is hssp for $(\mathcal{L}_S, \iota_{\Box})$.*

Proof. By collecting the conditions for regularity of the interpretation using Lemma 5.7, soundness of the rules $\mathcal{R}_{\mathsf{K}}$ from Lemma 5.12 and for the rules induced by S being hssp from Prop. 6.2. □

Thus in particular every extension of $\mathsf{S4}$ given by a simple frame property is axiomatised over $\mathsf{S4}$ by a single $\iota_{\Box}$-simple axiom. This extends to finite sets of simple frame properties (if using extensibility to show soundness we need the frame class obtained by adding *all* properties to be extensible). While seemingly restrictive, the conditions capture all transitive examples of [37], and most non-transitive ones, including the following frame conditions:

- Directedness $(\forall w_1 \forall w_2 \exists u (w_1 R u \wedge w_2 R u))$ with rule $\{(p_1, p_2 \Rightarrow ; \mathcal{C}_\emptyset, \mathcal{C}_\emptyset)\}/\Box p_1 \Rightarrow \mid \Box p_2 \Rightarrow$. The $\iota_{\Box}$-simple axiom is $\Box(\Box((p_1 \wedge p_2 \to \bot) \wedge p_1) \to \bot) \vee \Box(\Box((p_1 \wedge p_2 \to \bot) \wedge p_2) \to \bot)$ which as an axiom is equivalent to the standard axiom $\Box\neg\Box p \vee \Box\neg\Box\neg p$ (set $p_1 = p$ and $p_2 = \neg p$).

- Universality $(\forall w_1 \forall w_2 \exists u (w_1 R u \wedge w_2 = u))$; with rule $\{(p_1 \Rightarrow s; \mathcal{C}_\emptyset, \mathcal{C}_\emptyset)\}/\Box p_1 \Rightarrow \mid \Rightarrow \cdot s$ where $\cdot$ is the dummy modality replacing the restriction $\mathcal{C}_{\mathsf{id}}$. The axiom is $\Box(\Box((p_1 \to s) \wedge p_1) \to \bot) \vee \Box((p_1 \to s) \to s)$ which as an axiom is equivalent to the standard axiom $\Box\neg\Box p \vee \Box p$ (set $p_1 = p$ and $s = p$).

- Linearity $(\forall w_1 \forall w_2 \exists u ((w_1 R u \wedge w_2 = u) \vee (w_2 R u \wedge w_1 = u)))$ with rule $\{(p_1 \Rightarrow s_1; \mathcal{C}_\emptyset, \mathcal{C}_\emptyset)\}/\Box p_1 \Rightarrow \cdot s_1 \mid \Box p_2 \Rightarrow \cdot s_2$. The axiom is $\Box(\Box(\varphi \wedge p_1) \to (\varphi \to s_1)) \vee \Box(\Box(\varphi \wedge p_2) \to (\varphi \to s_2))$ where φ is $(p_1 \to s_2) \wedge (p_2 \to s_1)$. As an axiom this is equivalent to the standard axiom $\Box(\Box p \to q) \vee \Box(\Box q \to p)$ (set $p_1 = s_2 = p$ and $p_2 = s_1 = q$).

- Bounded top width ($\forall w_1 \dots \forall w_n \exists u \bigwedge_{1 \leq i < j \leq n}(w_i R u \wedge w_j R u)$) with rule $\{(p_i, p_j \Rightarrow ; \mathcal{C}_\emptyset, \dots, \mathcal{C}_\emptyset) : 1 \leq i < j \leq n\}/\Box p_1 \Rightarrow \mid \cdots \mid \Box p_n \Rightarrow$. The axiom is $\bigvee_{1 \leq k \leq n} \Box(\Box(\bigwedge_{1 \leq i < j \leq n}(p_i \wedge p_j \to \bot) \to p_k) \to \bot)$ or equivalently $\bigvee_{1 \leq k \leq n} \Box\neg\Box(\bigwedge_{1 \leq i < j \leq n} \neg(p_i \wedge p_j) \to p_k)$.

In the other direction, given a $\iota_\Box$-simple or $\iota_\boxplus$-simple axiom A, translating the axiom into a set of rules and applying the saturation procedure using Lemma 4.14 yields a set of rules which can be seen as a set of rules induced by a set of simple frame properties (provided the interpretation $\iota_\Box$ resp. $\iota_\boxplus$ is regular for all the successively constructed logics). Since the resulting rule sets are saturated and pspace-tractable, this automatically gives cut elimination and complexity results.

Theorem 6.5 (Simple axioms to rules). *Let $\mathcal{A}$ be a finite set of $\iota_\Box$-simple formulae and let $\mathcal{R}^4_{\mathcal{A}}$ be the result of the translation and saturation process with the rules in $\mathcal{R}_{\mathsf{K4}}$. Then $\mathsf{H}\mathcal{R}_{\mathsf{KT}}\mathcal{R}^4_{\mathcal{A}}$ is sound and complete for $\mathcal{L}_{\mathsf{KT4}\mathcal{A}}$, admits cut elimination, and yields an exponential space decision procedure for the respective logic. The analogous result holds also for $\iota_\boxplus$-simple axioms with the rules resp. axioms for T omitted and in the non-transitive case for the result $\mathcal{R}_{\mathcal{A}}$ of the translation and saturation process with $\mathcal{R}_{\mathsf{K}}$ provided $\mathcal{R}_{\mathsf{K}}$ is hssp for $(\mathcal{L}_{\mathsf{KT}\mathcal{A}}, \iota_\Box)$.*

Proof. In the case of $\iota_\Box$-simple axiom, reflexivity of the logic yields regularity of $\iota_\Box$. Translations of $\iota_\Box$-simple resp. $\iota_\boxplus$-simple axioms introduce only one box on the left hand side of the sequent arrow per component. Thus saturating under cuts with rules from $\mathcal{R}_{\mathsf{K}}$ resp. $\mathcal{R}_{\mathsf{K4}}$ yields rules in the format of rules induced by simple frame properties. Now cut elimination and complexity follow as above. $\Box$

This furthermore provides a method for *extracting a semantic characterisation* out of a $\iota_\Box$-simple or $\iota_\boxplus$-simple axiomatisation of a logic: Use Thm. 6.5 to turn the axioms into a set of rules, and then read of the corresponding simple frame properties from the rules according to [37].

6.2. Modal Logic S5

Let us briefly come back to perhaps the main example for the use of hypersequents in modal logic, the logic $\mathsf{S5}$. Starting from a characterisation of $\mathsf{S5}$ as $\mathsf{KTB5}$ (where (B) $p \to \Box\Diamond p$ is the well known axiom for symmetry) it is also possible to construct the calculus for $\mathsf{S5}$ from [54] for the interpretation $\iota_\boxplus$ in a direct way. To do this consider the axiom $\varphi := \boxplus\neg\Box p \vee \boxplus p$. Spelling out the definition of $\boxplus$ and using propositional equivalences (and the duality between $\Box$ and $\Diamond$) this is equivalent to the axiom

$$(\Diamond\Box p \to \Box p) \wedge (\Diamond\Box p \to p) \wedge (\neg\Box p \vee \Box p) \wedge (\Box p \to p)$$

and thus to the conjunction of the axioms $(\mathsf{5}), (\mathsf{B})$ and (T). Thus $\mathsf{S5} = \mathsf{K} \oplus \varphi$ and we may simply add the axiom φ to the hypersequent calculus for K under the interpretation $\iota_\boxplus$. Translating the axiom into a rule then gives the rule $\mathsf{5}$ $(p \Rightarrow ; \mathcal{C}_\emptyset, \mathcal{C}_{\mathsf{id}})/\Box p \Rightarrow \mid \Rightarrow$ from [54], but this time the interpretation is $\iota_\boxplus$ instead of $\iota_\Box$.

Since the rule K_n is derivable using n applications of the rule $\mathsf{5}$ followed by one application of K_0 and a number of applications of internal Weakening and external Contraction, it is clear that the calculus $\mathsf{H5K_0}$ from [54] is a cut-free complete calculus for $\mathsf{S5}$ which is hssp for $(\mathsf{S5}, \iota_\Box)$ as well as for $(\mathsf{S5}, \iota_\boxplus)$. The advantage of this calculus is that it can be used in a decision procedure of (optimal) coNP complexity (which seems not to have been considered in [54]):

In a first step, slightly modifying the result of using Kleene's Trick (Sec. 4.1) and omitting in the modified application of the rule $\mathsf{5}$ the copy of the second component in the premiss yields the rule applications $\mathsf{5}^*$ and K_0^* below left. Then, closing the rule $\mathsf{5}^*$ under external contraction of the two components of the principal part (and omitting the superfluous component in the premiss) results in the familiar rule T_1^* with applications as shown below right.

$$\frac{\mathcal{G} \mid \Gamma, \Box\varphi \Rightarrow \Delta \mid \varphi, \Sigma \Rightarrow \Pi}{\mathcal{G} \mid \Gamma, \Box\varphi \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}\ \mathsf{5}^* \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Box\varphi, \Delta \mid \Rightarrow \varphi}{\mathcal{G} \mid \Gamma \Rightarrow \Box\varphi, \Delta}\ \mathsf{K}_0^* \qquad \frac{\mathcal{G} \mid \Gamma, \Box\varphi, \varphi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Box\varphi \Rightarrow \Delta}\ \mathsf{T}_1^*$$

Note that the rules $\mathsf{5}^*$ and K_0^* are used in [54] for a completeness proof via model construction. Note also that apart from the $\Box p$ in the premiss of K_0^* this is the calculus given in [51, 52]. Let $\mathcal{R}^*_{\mathsf{S5}}$ be $\{\mathsf{5}^*, \mathsf{K}_0^*, \mathsf{T}_1^*\}$ and let $\mathsf{H}^*\mathcal{R}^*_{\mathsf{S5}}$ be the calculus given by these rules and the modified versions of the propositional rules, where similarly the copy of the principal component in the premiss is omitted. Admissibility of all the structural rules in the rule set without these rules as primitive is now shown as for modified rule applications, with the only difference that external contractions involving the component $\Sigma \Rightarrow \Pi$ in the rule $\mathsf{5}^*$ above are permuted into the premiss using depth-preserving admissibility of internal weakening (the obvious adaption of Lemma 3.8) and the rule T^*_1.

Theorem 6.6. *Backwards proof search for the calculus* $\mathsf{H}^*\mathcal{R}^*_{\mathsf{S5}}$ *can be implemented in* coNP.

Proof. Since internal contraction is admissible, we again work with set-set sequents as in the proof of Thm. 4.26. The main idea is first that we may fix the order of applications of rules and thus eliminate the existential guessing steps in the algorithm given in the proof of Thm. 4.26, and second that since the rule K^*_0 is the only rule introducing a new component, the number of new components is bounded by the number of subformulae of the input hypersequent. In detail the procedure with input hypersequent $\mathcal{G}$ is as follows: apply the rule K^*_0 backwards to the first formula which gives rise to a new component; apply the rule $\mathsf{5}^*$ backwards to all possible pairs of components and formulae such that the premiss of this rule application properly contains the conclusion; apply rule T^*_1 and all possible propositional rules such that each premiss properly contains the conclusion and universally guess one of the premisses; accept if you see an axiom; reject if no more rule applications are possible; otherwise go to the first step. If the size of $\mathcal{G}$ is n, the rule K^*_0 is applied at most n times, and so the whole loop is executed at most n times. Hence the number of components in every hypersequent occurring in the procedure is at most $2n$. Since there are at most n boxed formulae occurring on the left hand side of a component, in each iteration of the loop the rule $\mathsf{5}^*$ is applied at most $(2n)^2 \cdot n$ times. Finally, applying rule T^*_1 and all possible propositional rules to all components can be done in $2n \cdot n \cdot 2n$ steps. Thus in total we have a polynomial number of steps, and since all guesses were universal, the algorithm is in coNP. $\square$

While this result is nice in the sense that it shows that hypersequent calculi *can* be used in decision procedures of optimal complexity it is perhaps not so surprising: As pointed out by R. Kuznets, viewing hypersequents as flat nested sequents and using the correspondence between nested sequents and prefixed tableau from [25], this calculus corresponds to the prefixed tableau calculus for $\mathsf{S5}$ given e.g. in [26, p. 54], and it seems to be accepted in the prefixed tableau community that the latter can be used in a coNP decision procedure for $\mathsf{S5}$.

6.3. Convergent and connected normal modal logics

The methods developed in this paper allow us to construct in a (almost) purely syntactical way cut-free hypersequent calculi for extensions of the normal logics $\mathsf{K4}$, $\mathsf{KD4}$ and $\mathsf{S4}$ with additional axioms stating (weak) connectedness or convergence of the accessibility relation, as given in Table 4 (see e.g. [29]). In particular we will construct a simple and apparently new hypersequent calculus for the logic $\mathsf{K4.2}$. While the construction of the rule sets for all the logics is purely syntactical, in the non-reflexive case we appeal to the semantics to show that the interpretation $\iota_\Box$ is indeed regular for the logics under consideration. The result for the reflexive logics already follows from Thm. 6.5, but the explicit rule sets might be of independent interest.

Lemma 6.7. *The interpretation* $\iota_\Box$ *is regular for the logics* $\mathsf{KD4.2}, \mathsf{S4.2}, \mathsf{K4} \oplus (\mathsf{Lem}_0), \mathsf{KD4} \oplus (\mathsf{Lem}_0)$ *and* $\mathsf{S4.3}$.

Proof. The classes of frames characterising the logics $\mathsf{KD4.2}$, $\mathsf{K4} \oplus (\mathsf{Lem}_0)$, and $\mathsf{KD4} \oplus (\mathsf{Lem}_0)$ are p-extensible (Def. 5.5) via the extension

$$(W, R), w \quad \mapsto \quad (W \cup \{z\}, R \cup (\{z\} \times R[w]) \cup \{(z, w)\}) \ .$$

Thus Lemma 5.7 gives the result. Finally, since extensions of $\mathsf{S4}$ are reflexive, regularity of $\iota_\Box$ for $\mathsf{S4.2}$ and $\mathsf{S4.3}$ is immediate. $\square$

Table 4: Axioms for convergence and connectedness with the corresponding frame properties

(.2)	$\Box\neg\Box p \vee \Box\neg\Box\neg p$	convergence: $\forall x\forall y\forall z(xRy \wedge xRz \to \exists w(yRw \wedge zRw))$
(Lem_0)	$\Box(p \wedge \Box p \to q) \vee \Box(q \wedge \Box q \to p)$	weak connectedness: $\forall x\forall y\forall z(xRy \wedge xRz \wedge y \neq z \to yRz \vee zRy)$
(.3)	$\Box(\Box p \to q) \vee \Box(\Box q \to p)$	connectedness: $\forall x\forall y\forall z(xRy \wedge xRz \to yRz \vee zRy)$

Table 5: The hypersequent rules for logics of convergence and connectedness

$$\frac{(\vec{p},\vec{q} \Rightarrow\ ; \mathcal{C}_\Box, \mathcal{C}_\Box)}{\Box\vec{p} \Rightarrow \mid \Box\vec{q} \Rightarrow}\ 2^4_{n,m} \qquad \frac{(\ \Rightarrow\ \mathcal{C}_{\mathsf{id}}, \mathcal{C}_{\mathsf{id}}) \quad (\vec{p} \Rightarrow\ ; \mathcal{C}_\Box, \mathcal{C}_{\mathsf{id}}) \quad (\vec{q} \Rightarrow\ ; \mathcal{C}_{\mathsf{id}}, \mathcal{C}_\Box)}{\Box\vec{p} \Rightarrow \mid \Box\vec{q} \Rightarrow}\ \mathsf{Lem_{n,m}}$$

$$\frac{(\ \Rightarrow\ ; \mathcal{C}_{\mathsf{id}}, \mathcal{C}_{\mathsf{id}}) \quad (\vec{p},\vec{q} \Rightarrow\ ; \mathcal{C}_\Box, \mathcal{C}_\Box)}{\Box\vec{p} \Rightarrow \mid \Box\vec{q} \Rightarrow}\ 42_{n,m} \qquad \frac{(\vec{p} \Rightarrow\ ; \mathcal{C}_\Box, \mathcal{C}_{\mathsf{id}}) \quad (\vec{q} \Rightarrow\ ; \mathcal{C}_{\mathsf{id}}, \mathcal{C}_\Box)}{\Box\vec{p} \Rightarrow \mid \Box\vec{q} \Rightarrow}\ 3_{n,m}$$

$$\begin{array}{rlrl}
\mathcal{R}_{\mathsf{KD4.2}} &:= \mathcal{R}_{\mathsf{KD4}} \cup \{2^4_{n,m} : n,m \geq 1\} & \mathcal{R}_{\mathsf{K4}\oplus(\mathsf{Lem}_0)} &:= \mathcal{R}_{\mathsf{K4}} \cup \{\mathsf{Lem_{n,m}} : n,m \geq 1\} \\
\mathcal{R}_{\mathsf{S4.2}} &:= \mathcal{R}_{\mathsf{KT4}} \cup \{2^4_{n,m} : n,m \geq 1\} & \mathcal{R}_{\mathsf{KD4}\oplus(\mathsf{Lem}_0)} &:= \mathcal{R}_{\mathsf{KD4}} \cup \{\mathsf{Lem_{n,m}} : n,m \geq 1\} \\
\mathcal{R}_{\mathsf{K4.2}} &:= \mathcal{R}_{\mathsf{K4}} \cup \{42_{n,m} : n,m \geq 1\} & \mathcal{R}_{\mathsf{S4.3}} &:= \mathcal{R}_{\mathsf{K4}} \cup \{.3_{n,m} : n,m \geq 1\}
\end{array}$$

Moreover, since the logics are transitive, soundness of the rules $\mathcal{R}_{\mathsf{K4}}$ follows from Lemma 5.12 and we can apply the construction of rules corresponding to the axioms from Sec. 5.3. Converting the axioms $(.2), (\mathsf{Lem}_0)$ and $(.3)$ into hypersequent rules using regularity of $\iota_\Box$ and saturating under cuts yields the rule sets given in Table 5, with the exception of $\mathcal{R}_{\mathsf{K4.2}}$ which will be considered later. Note also that the rules T_n are derivable in $\mathsf{H}\mathcal{R}_{\mathsf{S4.3}}$.

Example 6.8. In the systems $\mathsf{H}\mathcal{R}_{\mathsf{KD.2}}$ and $\mathsf{H}\mathcal{R}_{\mathsf{S4.2}}$ the axiom (.2) $\Box\neg\Box p \vee \Box\neg\Box\neg p$ for convergence from Table 4 is derived as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \Rightarrow p}\ \mathcal{A}}{p, \neg p \Rightarrow}\ \neg_L}{\Box p \Rightarrow \mid \Box\neg p \Rightarrow}\ 2^4_{1,1}}{\Rightarrow \neg\Box p \mid \Rightarrow \neg\Box\neg p}\ \neg_R}{\Rightarrow \Box\neg\Box p \mid \Rightarrow \Box\neg\Box\neg p}\ \mathsf{K}_0}{\Rightarrow \Box\neg\Box p, \Box\neg\Box\neg p}\ \mathsf{IW, EC}}{\Rightarrow \Box\neg\Box p \vee \Box\neg\Box\neg p}\ \vee_R$$

It is not hard to check that these rule sets are indeed saturated and pspace-tractable, and so we uniformly obtain syntactic cut elimination and complexity results.

Corollary 6.9. *The hypersequent calculi for the logics* $\mathsf{KD4.2}$, $\mathsf{S4.2}$, $\mathsf{K4} \oplus (\mathsf{Lem}_0)$, $\mathsf{KD4} \oplus (\mathsf{Lem}_0)$ *and* $\mathsf{S4.3}$ *given in Table 5 are sound and complete for the respective logics, admit cut elimination and yield* EXPSPACE *decision procedures for the respective logics.* □

The rule sets themselves are not new: apart from $\mathcal{R}_{\mathsf{K4.2}}$ and modulo structural rules they appear in [37], the rule sets $\mathcal{R}_{\mathsf{K4}\oplus(\mathsf{Lem}_0)}, \mathcal{R}_{\mathsf{K4}\oplus(\mathsf{Lem}_0)}$ and $\mathcal{R}_{\mathsf{S4.3}}$ appear in [31], and the rule sets $\mathcal{R}'_{\mathsf{S4.2}} = \{\mathsf{T}_1, 4_0, 2^4_{0,0}\}$ and $\mathcal{R}'_{\mathsf{S4.3}} = \{\mathsf{T}_1, 4_0, 3_{0,0}\}$ obtained from $\mathcal{R}_{\mathsf{S4.2}}$ resp. $\mathcal{R}_{\mathsf{S4.3}}$ by omitting derivable rules are introduced in [36]. However, the method above gives a (almost) purely syntactic and uniform construction of these rules from the corresponding axioms. In contrast, the rules in [37] are constructed semantically, whereas in [31] and [36] no description of the construction is given, and only [36] mentions the interpretation of a hypersequent as a single formula. The fact that the semantic and syntactic constructions of calculi for these logics give the same results can be seen as an argument for the naturalness of the constructed calculi.

Remark 6.10 (Non-regularity). Seriality (axiomatically captured by $(\mathsf{D})\ \Diamond\top$) seems to be necessary for regularity of $\iota_\Box$ in convergent logics: The formula $\varphi_1 = \Diamond\top$ and the degenerate frame $(\{a\},\emptyset)$ witness that the rule $\Box\varphi/\varphi$ is not admissible in any logic $\mathcal{L}$ with $\mathsf{K.2} \subseteq \mathcal{L} \subseteq \mathsf{K4.2}$. The axiom $(\mathsf{G_0}) = \Diamond(p\wedge\Box q) \to \Box(p\vee\Diamond q)$ which captures *weak convergence* $(\forall x\forall y\forall z(xRy\wedge xRz\wedge y\neq z \to \exists w(yRw\wedge zRw)))$, see e.g. [29], also seems problematic: The formula $\varphi_2 = \Box\Diamond\top$ and the frame $(\{a,b\},\{(a,b)\})$ witness non-admissibility of $\Box\varphi/\varphi$. Similarly, the formula $\varphi_3 = \Box p \to p$ and the frame $(\{a\}\cup\mathbb{N}, \leq_\mathbb{N} \cup\{(a,n) : n\in\mathbb{N}\}$ with valuation σ such that $\sigma(p) = \mathbb{N}$ witness the same for every logic $\mathcal{L}$ with $\mathsf{K.3} \subseteq \mathcal{L} \subseteq \mathsf{KD4.3}$

While by the previous remark there is little hope of using the standard interpretation $\iota_\Box$ for the logic $\mathsf{K4.2}$, it is possible to reformulate the axiom (.2) in such a way that we can use the interpretation $\iota_\boxplus$:

Lemma 6.11. *Let* $\varphi := \boxplus(\neg p\vee\neg\Box q)\vee\boxplus(p\vee\neg\Box\neg q)$. *Then* $\mathsf{K4.2} = \mathsf{K4}\oplus\varphi$.

Proof. Spelling out the definition of $\boxplus$ and omitting the tautologous conjunct we have that adding the axiom φ to $\mathsf{K4}$ is equivalent to adding the axioms

$$(\mathsf{G_0}) := \Box(\neg p\vee\neg\Box q)\vee\Box(p\vee\neg\Box\neg q) \quad \text{and} \quad \psi := p\wedge\Box q \to \Box(p\vee\Diamond q)\ .$$

Both of these are (equivalent to) Sahlqvist formulae and using the standard methods (see e.g. [11]) can be seen to correspond to the frame properties $\forall x\forall y\forall z(xRy\wedge xRz\wedge y\neq z\to\exists w(yRw\wedge zRw))$ (also called *weak convergence*) and $\forall x\forall y(xRy\wedge x\neq y\to\exists w(xRw\wedge yRw))$. Thus writing Φ for the latter property, $\mathsf{K4}\oplus\varphi$ is the logic given by the class of transitive weakly convergent frames with Φ. Similarly, the axiom (.2) corresponds to the property of *convergence* given by $\forall x\forall y\forall z(xRy\wedge xRz\to\exists w(yRw\wedge zRw))$, and thus $\mathsf{K4.2}$ is the logic of transitive frames with this property. We show that these two classes coincide.

If $\mathfrak{F} = (W,R)$ is a transitive convergent frame, then trivially it is also weakly convergent. To see that Φ holds, assume that for $x,y\in W$ we have xRy and $x\neq y$. By convergence there is a $w\in W$ with yRw, and since R is transitive we also have xRw. Thus $\mathfrak{F}$ satisfies Φ. For the other direction, assume that $\mathfrak{F} = (W,R)$ is a transitive weakly convergent frame with Φ. To show that $\mathfrak{F}$ is convergent we only need to investigate the case that $x,y,z\in W$ with xRy and xRy and $y = z$. If $y\neq x$, then using Φ we have a $w\in W$ with xRw and in particular yRw, thus witnessing convergence. If $y = x$, then we know that xRx and clearly we have a w with yRw. Thus $\mathfrak{F}$ is convergent.

Thus we have that a formula is in $\mathsf{K4.2}$ iff it is valid in all transitive convergent frames iff it is valid in all transitive weakly convergent frames with Φ iff it is in $\mathsf{K4}\oplus\varphi$. $\Box$

Theorem 6.12. *The hypersequent calculus* $\mathsf{H}\mathcal{R}_{\mathsf{K4.2}}$ *is hssp and cut-free complete for* $(\mathsf{K4.2},\iota_\boxplus)$ *and yields an* EXPSPACE *decision procedure for* $\mathsf{K4.2}$.

Proof. Since $\mathsf{K4.2}$ is transitive, by Lemma 5.12 the rules $\mathcal{R}_\mathsf{K}$ are hssp for $(\mathsf{K4.2},\iota_\boxplus)$, and obviously the interpretation $\iota_\boxplus$ is regular. Using the previous Lemma we have that $\mathsf{K4.2} = \mathsf{K4}\oplus\varphi$ for $\varphi = \boxplus(\neg p\vee\neg\Box q)\vee\boxplus(p\vee\neg\Box\neg q)$, and converting the latter axiom into a hypersequent rule gives the rule $\{(\ \Rightarrow\ ; \mathcal{C}_{\mathsf{id}}, \mathcal{C}_{\mathsf{id}}), (p, q \Rightarrow\ ; \mathcal{C}_\emptyset, \mathcal{C}_\emptyset)\}/\Box p, \Box q \Rightarrow\ $. Finally, saturating under cuts with the rules $\mathcal{R}_{\mathsf{K4}}$ yields the rules $42_{n,m}$ given in Table 5. $\Box$

This result is particularly interesting because the logic $\mathsf{K4.2}$ is not given by a simple frame property and thus is not covered by the results in [37] (in particular since it is not serial it does not coincide with the logic of transitive *directed* frames considered there). Apart from the tableau system for this logic mentioned in [2] and apparently contained in [1] there seems not to be any sequent or hypersequent calculus for this yet.

6.4. Simply dependent bimodal logics

The method of cut elimination by saturation also allows for fairly straightforward combinations of different logics. As an example consider *simply dependent bimodal logics* [24]. Such logics are the combination of two normal modal logics $\mathcal{L}_1$ and $\mathcal{L}_2$ with modalities $\heartsuit$ and $\Box$ respectively into a bimodal logic given by the *fusion* of the two logics (i.e. the logic given by the union of the axioms) together with the additional inclusion axiom $\Box p \to \heartsuit p$. Following [24] we write $\mathcal{L}_1 \oplus_{\subseteq} \mathcal{L}_2$ for the resulting logic $\mathcal{L}_{\mathcal{L}_1 \cup \mathcal{L}_2 \cup \{\Box p \to \heartsuit p\}}$. Here we consider the logic $\mathsf{KT} \oplus_{\subseteq} \mathsf{S5}$ as an example. Since $\Box$ is an $\mathsf{S5}$-modality it is reflexive, and hence the interpretation $\iota_\Box$ is regular for $\mathcal{L}_1 \oplus_{\subseteq} \mathsf{S5}$. Thus we may add the rules $\mathcal{R}_{\mathsf{KT}}$ for $\heartsuit$ and the ground hypersequent $\Rightarrow \Box p \to \heartsuit p$ to the hypersequent calculus $\mathsf{H}\mathcal{R}_{\mathsf{KT4}\{\mathsf{5}_n : n \in \mathbb{N}\}}$ for $\Box$ to obtain a calculus for $\mathsf{KT} \oplus_{\subseteq} \mathsf{S5}$. Note that the inclusion axiom together with the fact that in presence of transitivity for $\Box$ the rules $\mathcal{R}_{\mathsf{K}}$ for $\Box$ are hssp for $(\mathsf{S5}, \iota_\Box)$ implies that the rules $\mathcal{R}_{\mathsf{KT}}$ for $\heartsuit$ are hssp for $(\mathsf{KT} \oplus_{\subseteq} \mathsf{S5}, \iota_\Box)$. Translating the inclusion axiom $\Rightarrow \Box p \to \heartsuit p$ then yields the hypersequent rule $(p \Rightarrow q; \mathcal{C}_\emptyset)/\Box p \Rightarrow \heartsuit q$ and saturating under cuts between rules yields the additional rules

$$\frac{(\vec{p}, \vec{q} \Rightarrow r; \mathcal{C}_\Box)}{\heartsuit\vec{p}, \Box\vec{q} \Rightarrow \heartsuit r}\ (||\vec{p}||, ||\vec{q}|| \geq 0) \qquad \text{and} \qquad \frac{(\vec{p}, \vec{q} \Rightarrow ; \mathcal{C}_{\mathsf{id}})}{\heartsuit\vec{p}, \Box\vec{q} \Rightarrow}\ (||\vec{p}|| + ||\vec{q}|| \geq 1) \tag{3}$$

with $||\vec{p}||, ||\vec{q}|| \geq 0$. Adding these to the rules for $\heartsuit$ and $\Box$ we obtain a saturated and hence cut-free hypersequent calculus for $\mathsf{KT} \oplus_{\subseteq} \mathsf{S5}$. While the logic is known to be $\mathsf{EXPTIME}$-complete [24], the decision procedure resulting from this calculus and Thm. 4.26 only yields an $\mathsf{EXPSPACE}$ bound and thus is of suboptimal complexity. But this example shows that the complexity bound of Thm. 4.26 cannot be lowered beneath $\mathsf{EXPTIME}$.

Remark 6.13. Similar to the construction of the rule set for the logic $\mathsf{T} \oplus_{\subseteq} \mathsf{S5}$ above it is possible to construct cut free hypersequent calculi for e.g. the family of simply dependent bimodal logics $\mathsf{S4} \oplus_{\subseteq} \mathcal{L}$ with $\mathcal{L} \in \{\mathsf{S4.2}, \mathsf{S4.3}, \mathsf{S5}\}$ which is considered in [36]. The only difference to the rules for $\mathsf{KT} \oplus_{\subseteq} \mathsf{S5}$ above is that the context restriction $\mathcal{C}_\Box$ in the left rule of (3) above is replaced with the restriction $\mathcal{C}_{\heartsuit\Box} = \langle \{\heartsuit p, \Box p\}, \emptyset \rangle$ and that the rules $\mathcal{R}_{\mathsf{KT4}\{\mathsf{5}_n : n \in \mathbb{N}\}}$ for $\Box$ are replaced with the rules $\mathcal{R}_{\mathsf{S4.2}}$ or $\mathcal{R}_{\mathsf{S4.3}}$ from Table 5, depending on the logic. The calculi from [36] are then easily obtained by omitting the derivable rules.

7. Conclusion

In this article we introduced the rule format of a hypersequent rule with context restrictions and established general theorems about fundamental properties such as cut elimination, decidability and complexity for calculi given by such rules. Furthermore, we constructed translations from Hilbert axioms into such rules and vice versa and used these results to establish some limits of this rule format. Finally, we applied the methods to a number of specific logics to obtain syntactical counterparts of the semantically driven results about modal logics given by simple frame conditions from [37], a closer analysis of hypersequent calculi for the logic $\mathsf{S5}$, a systematic construction of calculi for modal logics of convergent or connected frames including a novel calculus for the logic $\mathsf{K4.2}$ and calculi for simply dependent modal logics [24].

Future Work. The work presented here only forms a small piece of the research programme mentioned in the Introduction, and so there are many avenues for future research. One of the more obvious ones is the extension of the results in this article to logics with additional connectives based on intuitionistic instead of classical propositional logic. The corresponding results in the sequent framework [38, 40] suggest that this should be quite straightforward. Another such avenue is the adaption of the methods to successively more expressive frameworks such as nested sequents resp. tree-hypersequents [12, 52] or display calculi [7]. Such extensions will have to face the difficulty that unlike in the hypersequent or sequent framework where all the active components in a rule have the same "modal depth", in the nested formalism the active parts of the premisses might have different modal depth, and thus it will be difficult to use projective formulae in the translation from rules to axioms. While in the display formalism due to the display property this seems not to be a problem, in this case it is not clear whether *logical rules* of the form constructed by the method allow to capture more properties than purely *structural rules* in the spirit of [21], since the layer of logical connectives introduced in the conclusion might be converted into a layer of structural connectives instead. These questions are subject of ongoing research. A bit closer to the work presented in this article, it would be interesting to consider a more general format of hypersequent rules allowing to capture e.g. the rules for *symmetric* modal logic from [37]. While these calculi in general do not have full cut elimination, proving syntactically that analytic cuts suffice and identifying the class of axioms corresponding to such rules would fit in another piece of the puzzle concerning the expressivity of the different frameworks. On a more computational side, the huge disparity between the EXPSPACE complexity of the general decision algorithm from Thm. 4.26 and the coNP complexity of the algorithm for S5 prompts the question for general criteria as to when the complexity of the general algorithm can be lowered. It might also be possible to obtain better computational behaviour by adapting recent approaches applying the techniques of focusing [3] to nested sequent systems for modal logics [14, 42]. Finally, it is still an open question whether the exponential space bound on the general decision procedure of Thm. 4.26 is optimal or whether it can be sharpened to exponential time in general.

Acknowledgements. I would like to thank Agata Ciabattoni, Revantha Ramanayake and Roman Kuznets for their valuable input, and the anonymous reviewers for their many helpful comments. This work is funded by FWF project START Y544-N23 and by the European Union's Horizon 2020 programme under the Marie Skłodowska-Curie grant agreement No 660047.

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