Graphical Construction of Cut Free Sequent Systems Suitable for Backwards Proof Search *

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Abstract: We present a graphical representation of sequents and sequent rules, which aids in the discovery of cut-free sequent systems for non-iterative modal logics permitting backwards proof search in polynomial space. The technique is used to construct sequent systems for conditional logic V and KLM rational logic $\mathcal{R}$.

1 Introduction

Backwards proof search is one of the main techniques in theorem proving. The systems used for this usually are sequent systems which have cut elimination and admissibility of contraction. In this context the emergence of ever more specialised modal logics in computer science gives rise to the question of how to construct such systems.

One method of constructing cut-free sequent systems for modal logics is the method of *cut elimination by saturation*, previously used e.g. in [3]. As the name suggests, the method is based on saturating the rule set under the addition of rules which are necessary for the standard cut elimination proof to go through. Unfortunately, in the standard notation for sequent rules the construction of these new rules quickly becomes very tedious and prone to error. In this work we introduce a graphical representation of sequents and sequent rules, which makes the operations needed for saturating a rule set very simple and intuitive. Furthermore, we apply the method to Lewis' conditional logic V from [4] in the entrenchment language, yielding a sequent system suitable for backwards proof search in polynomial space. This sequent system moreover witnesses that deciding the flat fragment of V is in the class Π_3^P of the polynomial hierarchy. By translation this yields a purely syntactical Π_3^P-decision procedure for the KLM rational logic $\mathcal{R}$, which although of suboptimal complexity might still be of interest.

2 Cut Elimination by Saturation and Backwards Proof Search

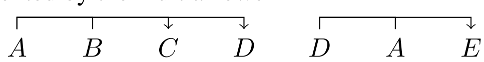
We consider formulae over the propositional connectives, the unary modality $\Box$ and the binary entrenchment modality $\preccurlyeq$. The results easily generalise to other signatures as well. Furthermore we assume the presence of the rules $\mathsf{G3}p$ of [5] for the underlying classical propositional logic. Let's recall some notions and facts from [3]. The general rule format considered is that of a *shallow rule*. Such a rule is given by

$$\frac{\{\Gamma, \Sigma_i \Rightarrow \Delta, \Pi_i \mid i \leq n\} \cup \{\Xi_j \Rightarrow \Omega_j \mid j \leq m\}}{\Gamma, \Phi \Rightarrow \Delta, \Upsilon},$$

where the $\Sigma_i \Rightarrow \Pi_i$ and $\Xi_j \Rightarrow \Omega_j$ are sequents (i.e. pairs of multisets) of propositional variables, the *contextual* and *noncontextual premisses* respectively, and $\Phi \Rightarrow \Upsilon$ is a sequent of modalised propositional variables, the *principal formulae* of the rule. The sequent $\Gamma \Rightarrow \Delta$ is the *context*. Since axioms without nested modalities can always be converted into equivalent sets of shallow axioms, this format suffices for all logics axiomatised by (finitely many) non-nested axioms. For two rules R_1, R_2 with principal formulae $\Phi_1 \Rightarrow \Upsilon_1, (A \preccurlyeq B)$ and $(A \preccurlyeq B), \Phi_2 \Rightarrow \Upsilon_2$ the rule $\mathsf{cut}(R_1, R_2, (A \preccurlyeq B))$ is the rule with principal formulae $\Phi_1, \Phi_2 \Rightarrow \Upsilon_1, \Upsilon_2$, whose premisses are built by cutting the combined premisses of R_1 and R_2 first on A and then on B. Here exactly those premisses are labelled contextual, whose construction involved at least one contextual premiss of R_1 or R_2. The definition for cuts on formulae of the form $\Box A$ is analogous. It can be shown that the so constructed rules are still sound. A set $\mathcal{R}$ of shallow rules is *cut closed* if for every two rules $R_1, R_2 \in \mathcal{R}$ with principal formulae $\Phi_1 \Rightarrow \Upsilon_1, C$ and $C, \Phi_2 \Rightarrow \Upsilon_2$ the rule $\mathsf{cut}(R_1, R_2, C)$ is derivable in $\mathsf{G3}p\mathcal{R}$ without using the cut rule. Similarly, the rule set is *contraction closed*, if for every rule R with principal formulae $\Phi \Rightarrow \Upsilon, C, C$ there is a rule $R' \in \mathcal{R}$ with principal formulae $\Phi \Rightarrow \Upsilon, C$, whose premisses are derivable from the premisses of R using only contraction and weakening (and similarly for contractions on the left of $\Rightarrow$). The method of cut elimination by saturation is based on the fact that in cut and contraction closed rule sets based on $G3p$ the cut rule can be eliminated, and proof search (in a slightly modified system) can be implemented in polynomial space. Thus the missing cuts between rules and contractions of rules are added until the rule set is saturated.

3 The Graphical Representation

In order to make the process of computing cuts between rules more intuitive we now introduce a graphical representation of sequents. The main idea is to represent a sequent $\Gamma \Rightarrow \Delta$ by a *multiarrow* with *tails* emerging from the formulae in Γ and *heads* pointing to the formulae in Δ. Thus for example the sequents $A, B \Rightarrow C, D$ and $D, A \Rightarrow E$ are represented by the multiarrows



An application of the cut rule to these two sequents with cut formula D now evidently is represented by connecting

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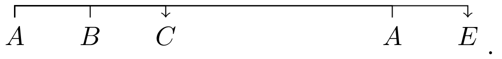
$$\frac{\{\ B_k \Rightarrow A_1, \dots, A_n, D_1, \dots, D_m \mid k \leq n\ \} \ \cup\ \{\ C_k \Rightarrow A_1, \dots, A_n, D_1, \dots, D_{k-1} \mid k \leq m\ \}}{\Gamma, (C_1 \preccurlyeq D_1), \dots, (C_m \preccurlyeq D_m) \Rightarrow \Delta, (A_1 \preccurlyeq B_1), \dots, (A_n \preccurlyeq B_n)}\ R_{n,m}$$

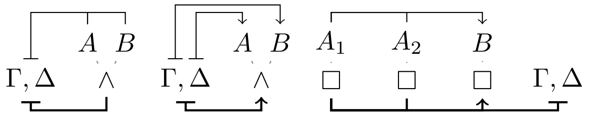
Figure 1: The general rule scheme for the set $\mathcal{R}_V := \{R_{n,m} \mid n \geq 1, m \geq 0\}$.

the head of the left multiarrow pointing do D to the tail of the right multiarrow emerging from D, "yanking the wire", and omitting the superfluous instances of the cut formula, resulting in the multiarrow

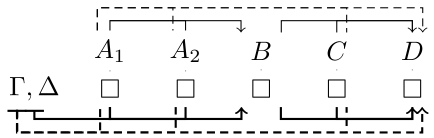


Similarly, the sequent resulting from now contracting the two instances of A is represented by the multiarrow

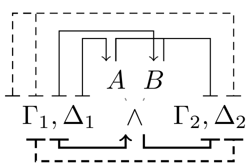
Rules are represented by writing the formulae as parse trees and drawing the multiarrows representing the premisses on top, the multiarrow representing the conclusion on the bottom. To mark contextual premisses we add an additional end to the arrows, marked as $\dashv$. This is to be read as an abbreviation for tails emerging from the formulae on the left side of the context and heads pointing to those on the right side of the context. Thus for example the left and right conjunction rules of G3p and an instance of the K rule of normal modal logic are represented by



Now the operation of cutting two instances of e.g. the K rule is visualised by performing cuts on the conclusion as well as on the corresponding elements of the premisses resulting in the dashed arrows in the diagram below



If we wanted to saturate the set of K rules, we would now have to add the rule represented by the dashed arrows above to our rule set. On the other hand, if we cut the left and right conjunction rules we get the rule represented by the dashed arrows on the right. Since this is simply an application of contraction, we do not need to add any rules to our rule set. In a similar way we get contractions of rules by contracting their conclusions as well as the corresponding formulae in their premisses.



4 Conditional Logic V and KLM Logic $\mathcal{R}$

This method can be applied to construct a cut- and contraction closed set of sequent rules for the conditional logic V in the entrenchment language. After turning the rules and axioms from [4, p.123,124] into shallow rules using the method mentioned in Section 2 and cutting and contracting rules, we arrive at the set $\mathcal{R}_V$ of sequent rules, whose traditional representation is given in Figure 1. By construction this set is guaranteed to be sound, and since it subsumes the original rules it is also complete. Cut and contraction closure can be seen by considering the graphical representation of the rules. Thus by the results of Section 2 the sequent system $G3p\mathcal{R}_V$ has cut elimination and admissibility of contraction and is suitable for backwards proof search in polynomial space. The translation $(A \mathrel{\Box\!\!\rightarrow} B) \equiv (\bot \preccurlyeq A) \vee \neg(A \wedge \neg B \preccurlyeq A \wedge B)$ of the more commonly used counterfactual $\Box\!\!\rightarrow$ into the entrenchment connective $\preccurlyeq$ from [4] unfortunately yields a blowup exponential in the nesting depth of $\Box\!\!\rightarrow$, but still can be used to show a PSPACE upper bound for the right nested fragment of V in the counterfactual language, thus syntactically reproducing the corresponding result from [2]. Also, since the translation is only linear for formulae without nested modalities, and since the alternation depth in the algorithm for backwards proof search is determined by the nesting depth of the modalities, we get a purely syntactical Π_3^P decision procedure for the flat fragment of V, and by the correspondence from [1] also for KLM rational logic $\mathcal{R}$. Whether this procedure can be pushed down to the optimal complexity coNP is subject of ongoing research.

5 Concluding Remarks

The described method of cut elimination by saturation using the graphical representation has also successfully been used to construct sequent systems for several extensions of V. Furthermore, by incorporating restrictions on the context it is also possible to treat certain nested axioms such as the axioms responsible for absoluteness in V and some examples of intuitionistic modal logics. A more thorough exploration of these issues will be subject of further research.

References

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