Note on a Dichotomy for the Classes $W[P](\mathcal{C})$ Defined via Symmetric Connectives

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Abstract. Adopting a generalised notion of connectives as ptime-computable symmetric boolean functions, for finite sets $\mathcal{C}$ of such connectives the classes $W[P](\mathcal{C})$ are defined via the parameterised weighted satisfiability problem for circuits with $\mathcal{C}$-gates. This note will prove the following dichotomy result: for all finite sets $\mathcal{C}$ of connectives $W[P](\mathcal{C}) = \text{FPT}$ or $W[P](\mathcal{C}) = W[P]$.

Keywords: Parameterised Complexity; Symmetric Gates; Majority

1 Introduction

One of the most important parameterised complexity classes is the class $W[P]$. It can be considered a natural parameterised analogue of the class NP and was originally defined in [3] via the parameterised weighted satisfiability problem for boolean circuits. What happens, if we change the computational power of the underlying circuits? One possibility is to allow other than the boolean gates, such as majority gates, which output 1 (true) if the majority of the inputs is 1 (true), and 0 (false) otherwise, or parity gates, which output 1 if the number of inputs set to 1 is odd, and 0 otherwise. Notice that these gates are symmetric in the sense that their output is invariant under permutations of the inputs. In the parameterised setting of [3] this amounts to the question of how difficult it is to solve the parameterised weighted satisfiability problem for circuits with symmetric gates of unbounded fan-in. In the following we will call such a family of symmetric gates which in addition is *ptime*-computable a connective. For a finite set $\mathcal{C}$ of such connectives the class of problems, which are *fpt*-reducible to the parameterised weighted satisfiability problem for $\mathcal{C}$-circuits, will be called $W[P](\mathcal{C})$. Of course now an interesting question is the relationship between these classes and $W[P]$. This note proves that a full dichotomy holds, i.e. that for every finite set $\mathcal{C}$ of connectives $W[P](\mathcal{C}) = \text{FPT}$ or $W[P](\mathcal{C}) = W[P]$. The proof will also give characterisations of the connectives for both alternatives.

Yet, the study of $W[P](\mathcal{C})$ is not just interesting in its own right. These classes were first defined in [5], where the authors also defined the $W(\mathcal{C})$-hierarchies via the notion of weft and showed that for sufficiently strong bounded $\mathcal{C}$ the levels of the $W(\mathcal{C})$-hierarchy and the W-hierarchy coincide. Unfortunately for unbounded $\mathcal{C}$ there are only a few exemplary (although very interesting) results. When trying

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to show such results for unbounded connectives it is vital to first have a look at the class $W[P](\mathcal{C})$, which contains the $W(\mathcal{C})$-hierarchy. Using the dichotomy, the connectives with $W[P](\mathcal{C}) = \mathrm{FPT}$ can be omitted from such an analysis straight away.

2 Preliminaries

In the following $\mathbb{N}$ denotes the set of natural numbers. As usual for a finite alphabet Σ the set Σ^* consists of the finite strings over Σ. If $\bar{x}$ is a string in Σ^* then $|\bar{x}|$ denotes the length of $\bar{x}$, and we write $\bar{x}^n$ for the concatenation of n copies of $\bar{x}$. The *weight* of a binary string $\bar{x} \in \{0,1\}^*$ is the number of 1's in $\bar{x}$.

A *parameterised problem* P is a subset of $\Sigma^* \times \mathbb{N}$ for a finite alphabet Σ. Here $\mathbb{N}$ is encoded in unary. Given an instance $(\bar{x}, k) \in \Sigma^* \times \mathbb{N}$ of the problem, $\bar{x}$ is the *input*, and k is the *parameter*. The problem P is *fixed-parameter-tractable*, if there are a computable function $h : \mathbb{N} \to \mathbb{N}$ and a polynomial p, such that for every instance $(\bar{x}, k)$ membership in P can be decided in time $h(k) \cdot p(|\bar{x}|)$. The class of fixed-parameter-tractable problems is denoted by FPT. Given two parameterised problems $P \subseteq \Sigma^* \times \mathbb{N}$ and $Q \subseteq (\Sigma')^* \times \mathbb{N}$ a mapping $f : \Sigma^* \times \mathbb{N} \to (\Sigma')^* \times \mathbb{N}$ is an *fpt-reduction* of P to Q, if the following holds:

- for all $(\bar{x}, k) \in \Sigma^* \times \mathbb{N}$: $(\bar{x}, k) \in P$ iff $f(\bar{x}, k) \in Q$
- there are a computable function $h : \mathbb{N} \to \mathbb{N}$ and a polynomial p such that f is computable in time $h(k) \cdot p(|\bar{x}|)$
- there is a computable function $g : \mathbb{N} \to \mathbb{N}$ such that for all $(\bar{x}, k)$ and $(\bar{x}', k')$ with $f(\bar{x}, k) = (\bar{x}', k')$ we have $k' \leq g(k)$.

If there is such an *fpt*-reduction, then P is *fpt-reducible* to Q and we write $P \leq^{fpt} Q$. For detailed information on parameterised complexity see [4, 7, 8].

3 The general framework

Arguably the most important property of boolean connectives is that they are symmetric, that is that the output depends only on the number of the inputs, which are set to '1' (true) respective '0' (false). Thus we generalise the notion of a boolean connective in the following way (following [5]).

Definition 1. *A* connective C *is a function* $C : \mathbb{N} \times \mathbb{N} \longrightarrow \{0,1\}$ *computable in time polynomial in the sum of its arguments.*

For $m, n \in \mathbb{N}$ we interpret the value $C(m, n)$ as the truth value of the connective C when its input consists of m ones and n zeros. For $n \in \mathbb{N}$ and $\bar{x} \in \{0,1\}^n$ with weight k we write $C[\bar{x}]$ for $C(k, n-k)$. For $\ell \in \mathbb{N}$ it will be convenient to write $C \restriction_\ell$ for the connective C restricted to tuples (m, n) with $m + n = \ell$.

Example 1. The standard connectives $\bigwedge, \bigvee, Maj, \oplus$ fit in this framework via:

$$\begin{aligned}
\bigwedge(m,n) = 1 &\iff n = 0 \\
\bigvee(m,n) = 1 &\iff m \neq 0 \\
Maj(m,n) = 1 &\iff m > n \\
\oplus(m,n) = 1 &\iff m \text{ is odd}
\end{aligned}$$

Substituting boolean gates in a boolean circuit by gates labelled with a connective in $\mathcal{C}$, we get the obvious notion of a $\mathcal{C}$-circuit as follows: Let $\mathcal{C}$ be a finite set of connectives. A *$\mathcal{C}$-circuit* is a finite directed acyclic graph with multiple edges and with labelled vertices, which we will call *gates*. There are two kinds of gates with in-degree 0: the *input-gates*, which are labelled with distinct consecutive natural numbers starting with 1, and the *constant-gates*, which are labelled with one of the constants 0 and 1, or a connective $C \in \mathcal{C}$. Gates with in-degree > 0 are labelled with a connective $C \in \mathcal{C}$ each. The gates with out-degree 0 are called the *output-gates* of the circuit. In the following all circuits are assumed to have exactly one output-gate. A $\mathcal{C}$-circuit D with ℓ input gates and one output gate computes a function $f_D : \{0,1\}^\ell \to \{0,1\}$ in the obvious way. As usual we say that the $\mathcal{C}$-circuit D is *satisfiable*, if there is a $\bar{x} \in \{0,1\}^\ell$, such that $f_D(\bar{x}) = 1$. The circuit is *k-satisfiable*, if there is a $\bar{x} \in \{0,1\}^\ell$ with weight k, such that $f_D(\bar{x}) = 1$.

Notice that boolean circuits are a special case of $\mathcal{C}$-circuits for $\mathcal{C} = \{\neg, \bigvee, \bigwedge\}$, where $\neg$ is the connective defined by $\neg(m,n) = 1 \Leftrightarrow m = 0$. Analogously to the parameterised weighted satisfiability problem for boolean circuits (see [4, 7]), the parameterised weighted satisfiability problem for $\mathcal{C}$-circuits is defined as:

p-WSat(Circ($\mathcal{C}$))
Input: a $\mathcal{C}$-circuit and $k \in \mathbb{N}$
Question: Is the circuit k-satisfiable?

Definition 2. *Let $\mathcal{C}$ be a set of connectives. A parameterised problem is in the class $W[P](\mathcal{C})$ iff it is fpt-reducible to the parameterised problem* p-WSat(Circ($\mathcal{C}$)).

If the set $\mathcal{C}$ contains only one connective C we write $W[P](C)$ instead of $W[P](\{C\})$. Since every ptime-computable boolean function with domain $\{0,1\}^*$ is computed by a family of boolean circuits computable in polynomial time in the length of the input [9, Theorem 11.5], we immediately get

Proposition 1. *$W[P](\mathcal{C}) \subseteq W[P]$ for all finite sets $\mathcal{C}$ of connectives.*

Proof. Replacing the $\mathcal{C}$-gates with the corresponding subcircuits yields an fpt-reduction to $W[P]$. □

4 The Dichotomy Theorem

Lemma 1. *Let C be one of the connectives $\bigvee, \bigwedge, \oplus$. Then $W[P](C) \subseteq \mathrm{FPT}$.*

Proof. Let C be one of the connectives $\oplus, \bigvee, \bigwedge$. Then for all $\bar{x}, \bar{y} \in \{0,1\}^*$ we have $C[C[\bar{x}]\,\bar{y}] = C[\bar{x}\,\bar{y}]$, and thus every C-circuit is equivalent to a C-circuit consisting of a single C-gate, which receives (possibly multiple) edges from the old input-gates and possibly from constant-gates. If $C \in \{\bigvee, \bigwedge\}$ we further simplify the circuit by substituting every multiple edge by a single edge. If $C = \oplus$, we substitute every multiple edge of odd cardinality by a single edge and delete every multiple edge of even cardinality. The resulting circuit is equivalent to the original circuit. As this clearly can be done in *fpt*-time, and as the k-satisfiability of the simplified circuit easily can be checked in polynomial time, we have p-WSat(Circ(C)) $\in \mathrm{FPT}$ and thus $W[P](C) \subseteq \mathrm{FPT}$. ☐

Theorem 1 (Dichotomy Theorem). *Let $\mathcal{C}$ be a finite set of connectives. Then $W[P](\mathcal{C}) = W[P]$ or $W[P](\mathcal{C}) = \mathrm{FPT}$.*

Proof. By Proposition 1, closure of $W[P](\mathcal{C})$ under *fpt*-reductions and availability of constant gates we have $\mathrm{FPT} \subseteq W[P](\mathcal{C}) \subseteq W[P]$. Call a connective C

- $\vee$-*closed* if there are $n, m \in \mathbb{N}$ with $C(m, n+2) = 0$ and $C(m+1, n+1) = C(m+2, n) = 1$
- $\wedge$-*closed* if there are $n, m \in \mathbb{N}$ with $C(m, n+2) = C(m+1, n+1) = 0$ and $C(m+2, n) = 1$
- *monotone*, if for all $m, n \in \mathbb{N}$ we have $C(m, n+1) \leq C(m+1, n)$.

If the connectives C_1, C_2, C_3 are $\vee$- and $\wedge$-closed and not monotone respectively, they simulate the boolean connectives via

$$x \vee y = C_1[1^{m_1}\, x\, y\, 0^{n_1}] \tag{1}$$
$$x \wedge y = C_2[1^{m_2}\, x\, y\, 0^{n_2}] \tag{2}$$
$$\neg x = C_3[\,1^{m_3}\, x\, 0^{n_3}] \tag{3}$$

A set $\mathcal{C}$ of connectives is monotone, if every connective in $\mathcal{C}$ is monotone, and $\vee$-closed (respective $\wedge$-closed), if there is a $\vee$-closed ($\wedge$-closed) connective in $\mathcal{C}$. For monotone $\mathcal{C}$ we have four cases.

Case 1: $\mathcal{C}$ is $\vee$-closed and $\wedge$-closed. As then $\mathcal{C}$ is able to simulate small disjunctions and conjunctions according to equivalences (1) and (2) above, the parameterised weighted satisfiability problem for boolean circuits without negation gates p-WSat(Circ$^+$) is *fpt*-reducible to p-WSat(Circ($\mathcal{C}$)). Since p-WSat(Circ$^+$) is $W[P]$-complete [1] we have $W[P] \subseteq W[P](\mathcal{C})$.

Case 2: $\mathcal{C}$ is $\vee$-closed, but not $\wedge$-closed. For all $C \in \mathcal{C}$ and for all $\ell \in \mathbb{N}$ we then have $C \restriction_\ell = \mathit{const}$ or $C \restriction_\ell = \bigvee \restriction_\ell$, because otherwise the connective C would be $\wedge$-closed as well. But then we have p-WSat(Circ($\mathcal{C}$)) $\leq^{fpt}$ p-WSat(Circ($\bigvee$)) via the following reduction: Given a $\mathcal{C}$-circuit we compute for every gate ν with ℓ_ν inputs and label $C \in \mathcal{C}$ the values $C[0^{\ell_\nu}]$ and $C[1\,0^{\ell_\nu - 1}]$. This can be done in polynomial time as a connective is *ptime*-computable. If both values are the same we replace the gate by a gate labelled with the respective constant 0 or 1 and delete the incoming edges. If the values differ, we know that $C \upharpoonright_{\ell_\nu} = \bigvee \upharpoonright_{\ell_\nu}$ and thus replace the gate by a gate labelled with $\bigvee$. Then we delete all gates from which there is no path to the output-gate. The resulting circuit is equivalent to the original circuit. By Lemma 1 we have $W[P](\bigvee) \subseteq \mathrm{FPT}$ and thus $W[P](\mathcal{C}) \subseteq \mathrm{FPT}$

Case 3: $\mathcal{C}$ is $\wedge$-closed, but not $\vee$-closed. Similar to Case 2 we get a reduction p-WSat(Circ($\mathcal{C}$)) $\leq^{fpt}$ p-WSat(Circ($\bigwedge$)) and therefore $W[P](\mathcal{C}) \subseteq \mathrm{FPT}$.

Case 4: $\mathcal{C}$ is neither $\vee$-closed nor $\wedge$-closed. Then every connective in $\mathcal{C}$ must be constant on inputs of length $\ell > 1$, and on inputs of length 1 either constant or identity. But then the problem p-WSat(Circ($\mathcal{C}$)) is in FPT: for every gate ν in the $\mathcal{C}$-circuit with $\ell_\nu > 1$ inputs and label $C \in \mathcal{C}$ we compute the value $C[1^{\ell_\nu}]$ and replace the gate by the according constant gate. For C-gates with one input we compute $C[1]$ and $C[0]$, and delete the gate if it is equivalent to identity, or replace it with an apropriate constant-gate otherwise. The parameterised weighted satisfiability problem for the resulting circuit clearly is in FPT.

If $\mathcal{C}$ on the other hand is not monotone, we know by equivalence (3), that $\mathcal{C}$ is able to simulate negation. Now if there is a $C \in \mathcal{C}$ and $m, n \in \mathbb{N}$, such that $C(m, n+2) \neq C(m+1, n+1) = C(m+2, n)$ or $C(m, n+2) = C(m+1, n+1) \neq C(m+2, n)$, then either $C(m, n+2) = 0$ and C is $\vee$- respectively $\wedge$-closed, or $C(m, n+2) = 1$ and $\neg C$ is $\wedge$- respectively $\vee$-closed. Thus substituting the gates in a boolean circuit by constant size $\mathcal{C}$-subcircuits, we get $W[P] \subseteq W[P](\mathcal{C})$.

If there are no such m, n as above, then for all $\ell \in \mathbb{N}$ and for all $C \in \mathcal{C}$ we have that $C \upharpoonright_\ell$ is constant or $C(\ell - t, t) \neq C(\ell - (t+1), t+1)$ for all $t < \ell$. In the non-constant case the values of $C \upharpoonright_\ell$ alternate between 0 and 1. Thus $C \upharpoonright_\ell = \oplus \upharpoonright_\ell$ or $C \upharpoonright_\ell = (\neg\oplus) \upharpoonright_\ell$. But then we have p-WSat(Circ($\mathcal{C}$)) $\leq^{fpt}$ p-WSat(Circ($\oplus$)) via the following reduction: For every gate ν with ℓ_ν inputs and label C compute the values $C(0, \ell_\nu)$ and $C(1, \ell_\nu - 1)$ and replace the gate by

- a gate labelled with c, if $C(0, \ell_\nu) = C(1, \ell_\nu - 1) = c$
- a gate labelled with $\oplus$, if $C(0, \ell_\nu) = 0$ and $C(1, \ell_\nu - 1) = 1$
- two gates computing $\oplus[1 \oplus [X_1 \ \dots \ X_{\ell_\nu}]]$, if $C(0, \ell_\nu) = 1$ and $C(1, \ell_\nu - 1) = 0$

Thus with Lemma 1 we have $W[P](\mathcal{C}) \subseteq \mathrm{FPT}$. □

Corollary 1 (Characterisation). *Let $\mathcal{C}$ be a finite set of connectives. If $\mathcal{C}$ has at least two of the properties of being $\vee$-closed, $\wedge$-closed, or not monotone, then $W[P](\mathcal{C}) = W[P]$. Otherwise $W[P](\mathcal{C}) = \mathrm{FPT}$.* □

5 Conclusion

We saw that for every finite set $\mathcal{C}$ of connectives the class $W[P](\mathcal{C})$ is equal to $W[P]$ or collapses to FPT. The proof also yielded characterisations of the connectives for both alternatives. A question left open is whether we get a similar dichotomy for the classes $W[\mathrm{Sat}](\mathcal{C})$, which are defined via the parameterised weighted satisfiability problems for $\mathcal{C}$-formulas ($\mathcal{C}$-circuits where all gates except for the input-gates have fan-out ≤ 1). Although there are results which settle the issue for connectives with bounded fan-in [2, Theorem 5.2.2], it is unclear how to handle connectives with unbounded fan-in. It is also not clear, whether for the dichotomy theorem to hold it is necessary to presuppose the constant gates 0 and 1.

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