Randomized Game Semantics for Semi-Fuzzy Quantifiers

CHRISTIAN G. FERMÜLLER, *Theory and Logic Group, Institute of Computer Languages, Vienna University of Technology, Favoritenstrasse 9-11/E185-2, 1040 Vienna, Austria, E-mail: chrisf@logic.at*

CHRISTOPH ROSCHGER, *Theory and Logic Group, Institute of Computer Languages, Vienna University of Technology, Favoritenstrasse 9-11/E185-2, 1040 Vienna, Austria, E-mail: roschger@logic.at*

Abstract

We take up the challenge to extract particular truth functions for fuzzy quantifiers from a game semantic framework. To this aim, we start with a fresh look at Hintikka's evaluation game for classical first order logic and show that randomizing payoffs in that classical game results in a characterization of so-called weak Łukasiewicz logic. A further step of generalization, considering more than one formula as available for attack at a given state of the game, leads to Giles's game for full Łukasiewicz logic. Finally we extend this framework to random choices of witnesses for quantified statements. This allows us to characterize two families of extensions of Łukasiewicz logic with different semi-fuzzy proportionality quantifiers that include candidate models for vague natural language quantifiers like *about half*.

Keywords: Fuzzy quantifiers, semantic games, random choice

1 Introduction

Fuzzy quantification combines the theory of generalized quantifiers [39] with degree based reasoning. As we will explain in Section 2, largely following Glöckner's monograph [20], modeling vague natural language quantifiers like *many*, *few*, *about half*, etc, faces a number of challenges. These problems are alleviated if one focuses on so-called monadic semi-fuzzy proportionality quantifiers, as we will do here. 'Monadic' means that, like for the familiar universal and existential quantifier, we identify the range of a quantifier with the current universe of discourse (domain).[1] 'Semi-fuzzy' means that the scope of the quantifier is crisp; i.e., for any element of the domain it is assumed to be clear whether the predicate expressed by the scope of the quantifier applies or not; degrees of truth only emerge from the vagueness of the quantifier itself. 'Proportionality' refers to the fact that the truth value of the quantified sentence only depends on the proportion of elements of the domain that satisfy the scope of the quantifier.

[1] However, since binary quantifiers, where the range is expressed by a separate formula, are central from a linguistic point of view, we will make a few remarks on lifting our model from monadic to binary quantification in Section 5.

Even the limited realm of monadic semi-fuzzy proportionality quantifiers calls for further semantic principles—beyond insisting on the real unit interval as sets of truth values—to guide the search for linguistically adequate models that fit the framework of deductive fuzzy logics in the sense of Hájek [21, 22, 10]. We argue that game semantics is appropriate for this purpose. We start in Section 3 with the familiar (Henkin-)Hintikka evaluation game for classical first order logic and show that randomizing payoffs in that classical game results in a characterization of so-called weak Łukasiewicz logic. A further step leads to Giles's game for full Łukasiewicz logic in Section 4: at each state in such a game not just a single formula, but multisets of formulas asserted by the two players are available for attack. Giles's approach is then generalized in Section 5 by applying the concept of randomization also to the choice of witnesses for quantified formulas. This will allow us to characterize different types of semi-fuzzy proportionality quantifiers by variations of the basic game rules. We pick out two families of such quantifiers: blind choice quantifiers, which have linear truth functions, are investigated in Section 6; deliberate choice quantifiers, with more complex truth functions, are introduced in Section 7. We conclude in Section 8 by assessing our results with respect to the challenges outlined in Section 2. This will also provide a good occasion to hint at topics for further research triggered by the new approach to fuzzy quantification explored in this paper.

2 Fuzzy models of natural language quantifiers

Lotfi A. Zadeh, in a series of papers starting from the mid-1970s (see the collection [43] and further references there) developed a 'fuzzy logic' approach to natural language semantics. In particular the paper entitled 'A computational approach to fuzzy quantifiers in natural language' [44] has become an important point of reference. A wealth of ideas and methods for modeling reasoning with quantifiers like *few*, *most*, *about a half*, *about ten*, etc, with tools from fuzzy set theory are offered there. The essential feature of Zadeh's approach is the association of truth functions over the real unit interval $[0, 1]$ with such quantifiers.[2] For our purposes it suffices to remind the reader that pictures like the following (lifted from Figure 3 in [44]) nicely illustrate the gist of fuzzy quantifiers of the here relevant type.

In Figure 1, the value v on the x-axis refers to the proportion of domain elements satisfying the property expressed by the scope of the quantifier, whereas μ on the y-axis refers to the degree of truth of the resulting quantified statement. Such proportional quantifiers (including *about a half*, *at most roughly a third*, etc) are called 'fuzzy quantifiers of the second kind' in [44], to be distinguished from absolute quantifiers, like *about ten* or *at least (roughly) a thousand*, which Zadeh calls 'fuzzy quantifiers

[2] Interestingly Zadeh, right at the beginning of [44], refers to the work of Montague [32], Barwise and Cooper [3], Peterson [40] and others that had recently provided a new paradigm in linguistic research on semantics. We need not discuss the complex reasons for the well known fact that linguists largely choose to reject or ignore Zadeh's approach to natural language semantics. We think that sound methodological principles distract linguists from models that insist on a multitude of truth values and on a corresponding truth functional semantics. This in particular remains the case also when vagueness, ambiguity, and massive context dependence of certain quantifier expressions are taken into account. On the other hand, it is undeniable that the tools offered by fuzzy logic have been found valuable in many engineering applications involving or at least inspired by natural language processing. In this view, the lack of interaction between linguists and fuzzy logicians reflects different aims and a corresponding contrast of methods more than any ill-will on either side. We nevertheless think that an increased awareness of the various aims and tools employed in modeling vague language for various purposes might benefit fuzzy logicians as well as linguists. However this is a topic outside the scope of the current paper. ([14, 16] and the collection [6] document our own engagement in this debate.)

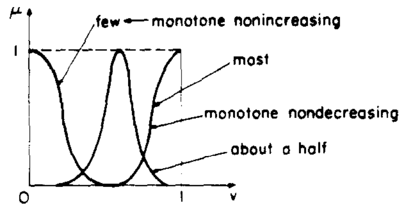


FIG. 1. reproduced from [44].

of the first kind'. Interestingly, *most* and *many* appear in both of Zadeh's lists of examples of quantifiers of these two kinds. The implied ambiguity in the reading of *many* has been independently discussed by linguists. In particular, Partee [38] reviews arguments for and against the ambiguity between a proportional and an absolute reading of *many* and argues in favor of such an ambiguity herself. Here, we will focus solely on proportional quantifiers, or rather on corresponding readings of quantifiers, leaving the discussion of models of (vague) absolute quantifiers to another occasion.

Zadeh is aware of the fact that linguistic adequateness requires to consider not only unary, but also binary quantifiers. In other words, one wants to model statements like *About half of the students failed* and *Most children are curious* rather than just statements like *Most* [*persons constituting the domain of discourse*] *are anxious*. We will refer to the first argument of a binary quantifier as its *range* and to the second argument as its *scope*. However for the class of quantifiers that we are interested in here, lifting models where the range coincides with the domain of discourse (unary quantifiers) to restricted ranges (binary quantifiers) is straightforward. We will comment on that feature briefly in Section 5, but otherwise focus on unary quantifiers for the sake of clearness.

To motivate our own models of a particular class of fuzzy quantifiers, we point out some worries about Zadeh's approach. We emphasize that the relevant features are not just present in [44], but rather are fairly typical of the fuzzy logic approach in general, as reviewed in more detail in [20].

Problems with fully fuzzy quantification. Following Liu and Kerre [28], one can distinguish four types of unary quantification with respect to their involvement of membership degrees and degrees of truth:

Type I: the quantifier is precise and its scope is crisp;

Type II: the quantifier is precise, but may have a fuzzy scope;

Type III: the quantifier is fuzzy, but its scope is crisp;

Type IV: the quantifier as well as its scope are fuzzy.

The extension of the classical universal and existential quantifiers ($\forall$ and $\exists$) from Type I to Type II by returning as truth value the infimum and supremum, respectively, of the membership degrees of the fuzzy set corresponding to the scope, is standard in first order fuzzy logics. However, *many*, *about half*, etc, require models of at least Type III, which we will call *semi-fuzzy* in this paper. Actually Zadeh aims at *fully fuzzy*, i.e. type IV quantification. But severe concerns about the linguistic adequateness of Zadeh's Type IV models have been raised in the literature on fuzzy quantifiers. To indicate at least one concrete problem, consider the statement *About half of the guests are hungry.* As already indicated, we will identify the set of guests with the universe of discourse and may thus consider *about half* as a unary fuzzy quantifier applied to the fuzzy predicate *hungry.* Following Zadeh, the truth value of the statement should only depend on some cardinality measure of the fuzzy set of hungry guests. If the later set is in fact crisp and exactly half of the guests are hungry (membership degree 1) and the other half are not hungry (membership degree 0) then clearly we should evaluate the statement as perfectly true (truth value 1). The problem arises when all guests are borderline hungry; i.e., when the predicate *hungry* uniformly applies with degree 0.5 to each guest. The approach favored by Zadeh, based on the so-called Σ-count for representing the cardinality of fuzzy sets evaluates *About half of the guests are hungry* as perfectly true also in this case, which clearly is in discrepancy with ordinary language use. Zadeh himself and a number of his followers suggest alternatives to the Σ-count measure. However, none of these models is in perfect correlation with pre-formal intuitions about the meaning of vague quantifiers in natural language. We do not want to enter the debate on the linguistic adequateness of various Type IV models, but rather refer to Glöckner's monograph [20] for an overview and detailed discussion of this literature. Glöckner suggests to focus on semi-fuzzy quantifiers first and let the generalization of Type III to Type IV be guided by an axiomatic framework. While we remain skeptical about the adequateness of (solely) fuzzy logic based models of vague natural language quantifiers with vague scope, we fully agree with the strategy to give priority to models of semi-fuzzy quantification over some finite, contextually fixed domains (universes of discourse).

Coherent interpretation of intermediate truth values. The challenge of finding a semantic frame that allows one to attach concrete meaning to truth values drawn from the real unit interval and in particular to justify the choice of corresponding truth functions is well known and certainly not specific to the study of quantifiers. Various proposals, like voting semantics [27], acceptability semantics [36], re-randomising semantics [26, 21], approximation semantics [4, 37], as well as Robin Giles's game based semantics [18, 19] address this challenge. To our best knowledge none of these semantics has so far been extended to cover also semi-fuzzy (let alone fully fuzzy) quantifiers. All mentioned approaches primarily refer to propositional logic and can only be (more or less) straightforwardly extended to Type II quantification ($\exists$ and $\forall$ as generalized disjunction and conjunction, respectively). Nevertheless we think that a uniform semantic framework that allows one to extract particular truth functions from more explicit models of reasoning with vague, uncertain, and/or underspecified propositions is called for, in particular also as a basis to tackle the problem outlined in the next paragraph.

An embarrassment of riches. Any function of type $\mathcal{P}([0,1]) \to [0,1]$ (where $\mathcal{P}(X)$ denotes the powerset of X) is a candidate for the truth function corresponding to a unary fuzzy quantifier. When we restrict attention to proportionality quantifiers over crisp domains, as done in this paper, this boils down to functions of type $[0,1] \to [0,1]$, as illustrated in (Zadeh's) Figure 1, above. Clearly, uncountably many candidates remain, even if we impose further general constraints, like continuity, monotonicity, symmetry, etc. Zadeh's figure is to be understood as a deliberately vague suggestion regarding the general form of plausible candidates of truth functions for *few*, *many*, and *about half*, rather than as fixing the meaning of these quantifiers by particular truth functions. Even so, the specific shape of the three graphs may be questioned. For example, why is *About half* [*elements of the domain*] *are X* evaluated as 1 only if *exactly* half of the elements satisfy property X? Should not the linguistic hedge '*about*' render the statement perfectly true also in cases where the proportion of elements satisfying X deviates from 0.5 very slightly? Independently of worries of this kind, the space of suitable candidates of truth functions that adequately model *about half* is not only uncountably large, but it is left unclear what parameters (if any) should or can be set in order to justify the choice of particular functions. No doubt further constraints, like efficient computability, will guide the choice of truth functions in practice. However even where, as in [20], concrete functions or parameterized families of functions are suggested, the choices remain *ad hoc*. To some extent this is probably unavoidable. Certainly one should not expect to be able to single out a particular truth function for, e.g., *about half* as clearly optimal in all respects. However, one may reasonably ask for a guiding semantic principle that leaves only a small number of parameters to be settled, where the space of possible parameter values is finite or at least discrete and tied to some general interpretation.

Compatibility with standard deductive fuzzy logics. Well after Zadeh's pioneering work, Petr Hájek and his collaborators developed a t-norm based approach to many-valued logics that connects fuzzy logic with a more traditional agenda of mathematical logic that includes axiomatic systems, soundness, completeness and complexity results, algebraic tools, proof theory, and the like. While Hájek's monograph [21] remains a classic reference, the Handbook of Mathematical Fuzzy Logic [7] provides a more recent overview of relevant work in this area. Although much of this research is centered on propositional logics and their algebraic counterparts, there is also a wealth of results on quantified logics. However the investigated first order systems focus on the (Type II) quantifiers $\forall$ and $\exists$. This holds in particular also for Łukasiewicz logic, which is often favored for applications, due to the fact that it can be characterized as the only t-norm based logic, where all connectives have continuous truth functions. On the other hand, research on fuzzy quantifiers seldom seeks to explicitly embed these quantifiers into deductive fuzzy logics.[3]

To sum up this discussion of some (potential) problems with traditional approaches to fuzzy quantifiers, we formulate a list of desiderata for an alternative approach. We aim at a framework—limited here to proportional quantifiers—that

- focuses on semi-fuzzy quantification,
- is based on a uniform semantic framework, able to justify specific truth functions with respect to first principles about reasoning with vague concepts and propositions,
- suggests candidates of truth functions for quantifiers like *about half* that arise from setting a small number of meaningful parameters,
- and straightforwardly extends standard Łukasiewicz logic.

As already indicated in the introduction, we think that a rather straightforward extension of Giles's game semantics for Łukasiewicz logic provides all needed ingredients. It

[3] The approach in [35], based on fuzzy type theory, is an exception. But the framework of [20]—the most comprehensive framework for fuzzy quantifiers, so far—is incompatible with full Łukasiewicz logic, due to an understanding of implication that is at variance with the truth function for implication in Łukasiewicz logic (the residuum of the Łukasiewicz t-norm).

thus remains to explain Giles's framework, before extending it to cover certain semi-fuzzy quantifiers. Since the relation between Giles's game and the standard semantic game for classical logic has to our best knowledge never been explained so far, we will first take the reader on a tour that starts with Hintikka's classic semantic game and leads to Giles's more elaborate setting in a few simple steps.

3 Randomizing Henkin-Hintikka games

As shown by Hintikka [24], building on an idea of Henkin, the Tarskian notion of truth can be characterized by a two person game played on a first order formula with respect to a given model. Such a characterization provides a semantic framework that goes beyond mere definitions of truth functions. It suggests an analysis of logical truth and validity in game theoretical terms and thus opens a formal pragmatic approach to logic that has proved to be very fruitful and led to the study of well motivated variants of classical logic, in particular IF-logic (independence friendly logic; see [25, 30]), that arises when the assumption of perfect information is dropped. We present the classical evaluation game in a slightly unusual terminology that will make the later transition to Giles's game more transparent.

The $\mathcal{H}$-game. There are two players, say *me* and *you*, who can both act in the roles of either the *attacker* or the *defender* of a formula. The game is played with respect to a given classical first order interpretation M, where all domain elements are witnessed by constants. M can thus be identified with an assignment of 0 (*false*) or 1 (*true*) to the variable free atoms of the language. By $v_M(F)$ we denote the truth value to which F evaluates in M.

At every state of the game either me or you act as the defender of some sentence (closed formula) F, the opponent player is the attacker. Accordingly, moves by the defender may be referred to as *defenses* and moves by the attacker as *attacks*. We will say that player $\mathbf{X}$ *asserts* F, if $\mathbf{X}$ is the defender of F at the given state. The game starts with my assertion of some formula and proceeds according to the following rules corresponding to the form of the currently considered formula.

($R_\wedge$) If I assert $F \wedge G$ then you attack by pointing either to the left or to the right subformula. As corresponding defense, I then have to assert either F or G, according to your choice.[4]

($R_\vee$) If I assert $F \vee G$ then I have to assert either F or G at my own choice.

($R_\neg$) If I assert $\neg F$ then you have to assert F. In other words, our roles are switched: the game continues with you as defender and me as attacker (of F).

($R_\forall$) If I assert $\forall x F(x)$ then you attack by picking c and I have to defend by asserting $F(c)$.

($R_\exists$) If I assert $\exists x F(x)$ then I have to pick a constant c and assert $F(c)$.

Note that ($R_\vee$) and ($R_\exists$) only involve a move by me. However, we may speak of an empty attack by you, followed by my defense, also in these cases. Also the role switch in ($R_\neg$) may be viewed as triggered by my defense to your attack of $\neg F$. In this manner we arrive at a uniform format of rules and corresponding rounds in a

[4] Note the duality of the rules for $\wedge$ and $\vee$. A version of conjunction where both conjuncts have to be asserted will be considered for $\mathcal{G}$-games, in Section 4.

run of the game: each round consists of an attack followed by a defense. We have only stated rules for states where I am the defender and you are the attacker of the currently considered formula. The rules for you defending a formula are completely dual.

More formally, each state of the $\mathcal{H}$-game determines the currently asserted sentence and a role assignment (either I am the defender and you the attacker, or vice versa). The role assignment remains unchanged in all state transitions, except for the one explicitly triggered by $(R_\neg)$. A run (terminal history) of the game is a sequence of states beginning with a sentence defended by me, where each successor state results from the previous one in accordance with the specified rules. Thus attack moves strictly alternate with corresponding defense moves. A round consists of two state transitions, where only the second one, the defense move, changes the currently asserted formula. Once we arrive at an atomic formula, the run of the game ends. In such a final state, where I assert an atomic formula A, we say that I win if $v_M(A) = 1$ and I lose if $v_M(A) = 0$. Analogously, in a final state, where you assert an atomic formula A, we say that I win if $v_M(A) = 0$ and I lose if $v_M(A) = 1$.

We call the game starting with my assertion of F the *$\mathcal{H}$-game for F under M*. Like all games that we will consider in this paper, it is a two-person zero-sum extensive game of finite depth with perfect information. We may view each such game as a tree where the branches correspond to the possible runs of the game. A strategy for me may be identified with a subtree obtained by deleting all but one successor nodes (states) of every node where I can choose between different moves. If I win at all final states, such a tree is called a winning strategy for me.

It might seem that we have deviated from Hintikka's classic semantic game. However, the differences between the above presentation of the game and the one found, e.g., in [25], are only superficial:

- We followed Giles [18, 17] by referring to the players as *me* and *you*, where Hintikka and Sandu use *Myself* and *Nature*, respectively.
- While the roles in the game are called *verifier* and *falsifier*, in [25], we prefer to speak of a *defender* and an *attacker* instead. Moreover we use "asserting F" as shorthand for "being the current defender of F". This is not only closer to Giles's terminology, but assists in disentangling the players' roles from the reference to the classical truth values.
- In stating the rules of the game, Hintikka and Sandu only refer to the roles, not to the identity of the players. While this allows for a more compact presentation, it hides the fact that, in any formal presentation of the game, one has to keep track of the current assignment of roles to the two players. In order to make the later transition to Giles's game more transparent, we prefer to include the reference to the players in stating the rules.

To sum up, while our *presentation* of the semantic game looks different from traditional descriptions, the game itself remains unchanged.

Theorem 3.1 (Hintikka)
A sentence F is true in an interpretation M (in symbols: $v_M(F) = 1$) iff I have a winning strategy in the $\mathcal{H}$-game for F under M.

Our aim is to provide a similarly elegant characterization of *graded truth* for first order fuzzy logics. While game semantics can be generalized to cover a wide range of different many-valued logics (see [11, 8, 15]) we will stick here to infinite valued Łukasiewicz logic, which is arguably the most important example of a deductive mathematical fuzzy logic in the sense of [7].

Łukasiewicz logic **Ł** provides two forms of conjunction: *weak conjunction* ($\wedge$) and *strong conjunction* (&); moreover, we have negation ($\neg$), implication ($\to$), (weak) disjunction ($\vee$), and the standard quantifiers ($\forall$ and $\exists$). The standard semantics of these connectives and quantifiers is given by:

$$
\begin{array}{l}
v_M(F \wedge G) = \min(v_M(F), v_M(G)) \\
v_M(F \vee G) = \max(v_M(F), v_M(G)) \\
v_M(F \,\&\, G) = \max(0, v_M(F) + v_M(G) - 1) \\
v_M(\bot) = 0 \\
v_M(\neg F) = 1 - v_M(F) \\
v_M(F \to G) = \min(1, 1 - v_M(F) + v_M(G)) \\
v_M(\forall x F(x)) = \inf_{c \in D}(v_M(G(c))) \\
v_M(\exists x F(x)) = \sup_{c \in D}(v_M(G(c)))
\end{array}
$$

where D is the domain of M (which we identify with the set of constants).

There are many good reasons to base **Ł** on the full syntax, as specified above.[5] In particular this nicely fits the general theory of t-norm based fuzzy logics as introduced by Hájek [21, 22] and developed into a prolific subfield of mathematical logic by many researchers since, as witnessed by the handbook [7]. However, in the vast literature on fuzzy logic and on many-valued logics in general one frequently considers only $\wedge$, $\vee$, and $\neg$ as propositional connectives. We will call this fragment of **Ł**, together with the standard quantifiers ($\forall$, $\exists$), *weak Łukasiewicz logic* $\mathbf{Ł}^w$ here.[6]

The restrictions of **Ł** and $\mathbf{Ł}^w$ to the propositional part will be denoted by $\mathbf{Ł}_p$ and $\mathbf{Ł}_p^w$, respectively.

In order to transfer $\mathcal{H}$-games into a many-valued setting we follow an idea of Giles [18, 17] and reformulate the winning condition in a way that will lead to an interesting interpretation of intermediate truth values in terms of expected risks of payments. We conceive of the evaluation of the atomic formula A at the final state of an $\mathcal{H}$-game as a *(binary) experiment* E_A that either *fails*, meaning $v_M(A) = 0$, or *succeeds*, meaning $v_M(A) = 1$. The experiment $\mathsf{E}_\bot$ always fails. Moreover, we stipulate that I have to pay 1€ to you if I lose the game. Hence winning strategies turn into strategies for avoiding payment.[7] So far this just amounts to an alternative way to present the original game. The main innovation of Giles is to let the experiments E_A be *dispersive*. This means that E_A may show different results upon repetition, where the individual trials of the experiment are understood as independent events. (Of course, $\mathsf{E}_\bot$ remains non-dispersive: it simply always fails.) The reader is invited to think about intended applications modeling vague language: while in concrete dialogues competent language users either (momentarily and provisionally) accept or don't accept grammatical utterances upon receiving them, vagueness results in a

[5] Actually one can define all connectives of **Ł** from just $\to$ and $\bot$ or alternatively from & and $\bot$. But neither $\to$ nor & can be defined from the remaining connectives.

[6] This logic is simply called "fuzzy logic" in [34]. It also coincides with "Zadeh-Kleene logic" [1], restricted to the unit interval.

[7] Note the asymmetry of the payoff scheme: even when the roles of attacker and defender are switched, it is *me*, not *you*, who has to pay upon losing the game. This is necessary to ensure that enforceable payments (inversely) correspond to truth values. Giles's extended game scenario allows one to restore perfect symmetry, as we will see in Section 4.

brittleness or dispersiveness of such highly context dependent decisions. (See, e.g, [41, 2].) In order to arrive at 'degrees of truth' for an atomic statement A in such a model, one assumes that the dialogue partners associate a fixed success probability $\pi(\mathsf{E}_A)$ to the experiment E_A. The result of E_A may be thought of as an answer to the question "Do you accept A (at this instance)?" By $\langle A \rangle = 1 - \pi(\mathsf{E}_A)$ we denote the *risk* associated with A, i.e., the *expected* (average) loss of money associated with my assertion of A. The function $\langle \cdot \rangle$ that maps each atomic sentence into a failure probability of the corresponding experiment is called *risk value assignment*. Note that risk value assignments are in 1-1-correspondence with (many-valued) interpretations via $\langle A \rangle_M = 1 - v_M(A)$.

The setting of randomized payoff for $\mathcal{H}$-games straightforwardly leads to a characterization of weak propositional Łukasiewicz logic $\mathbf{Ł}_p^w$, as shown in the following theorem.

THEOREM 3.2
A $\mathbf{Ł}_p^w$-sentence F is evaluated to $v_M(F) = x$ in interpretation M iff in the $\mathcal{H}$-game for F under the corresponding risk value assignment $\langle \cdot \rangle_M$ I have a strategy that limits my expected risk to $(1 - x)$€, while you have a strategy that ensures that my expected risk is not below this value.

PROOF. We use $\langle | \, G \rangle^*$ to denote my final risk when playing rationally in a game where I am defending and you are attacking G. If I am the attacker and you are the defender of G this value is denoted by $\langle G \, | \rangle^*$.[8]

If F is atomic then $\langle F \, | \rangle^* = 1 - \langle F \rangle_M$ and $\langle | \, F \rangle^* = \langle F \rangle_M$ and thus my risk is $v_M(F)$ in the former case and $1 - v_M(F)$ in the latter case, as required. Otherwise we argue by induction on the complexity of F that $\langle | \, F \rangle^* = 1 - v_M(F)$.

- If I assert $\neg G$, the game continues with your assertion of G and $\langle | \, \neg G \rangle^*$ reduces to $\langle G \, | \rangle^* = 1 - \langle | \, G \rangle^*$, just like in the truth function for $\neg$.
- If I assert $G \vee H$ then I will pick G or H according to where my associated expected risk is smaller. Therefore $\langle | \, G \vee H \rangle^* = \min(\langle | \, G \rangle^*, \langle | \, H \rangle^*)$, and thus $v_M(G \vee H) = \max(v_M(G), v_M(H)) = 1 - \min(1 - \langle | \, G \rangle^*, 1 - \langle | \, H \rangle^*)$.
- If I assert $G \wedge H$ then you will pick G or H according to where my associated risk, i.e., your expected gain, is higher. Therefore $\langle | \, G \wedge H \rangle^* = \max(\langle | \, G \rangle^*, \langle | \, H \rangle^*)$, corresponding to $v_M(G \wedge H) = \min(v_M(G), v_M(H))$.

The cases where you defend and I attack F are completely dual. ∎

Note that we are only interested in final risk as payoff values of the game, not in actual final payments due to particular results of experiments. Since individual trials of experiments are independent events, truth functionality is preserved. Consider a game for $A \vee \neg A$ for example. While I will finally have to pay either 1€ or nothing, depending on the result of E_A, my risk, i.e. my optimal expected loss under the risk value assignment corresponding to interpretation M is $\min(\langle A \rangle_M, 1 - \langle A \rangle_M)$€, which indeed amounts to $(1 - v_M(A \vee \neg A))$€.

There is a slight complication in lifting Theorem 3.2 to the first order level: in a $[0,1]$-valued interpretation M witnessing domain elements for quantified sentences may not exist. More precisely, we may have $v_M(\forall x F(x)) < v_M(F(c))$ and

[8] Note that probability (risk) is only involved in the definition of the payoff at final (atomic) states. The game itself does not contain any random moves. Since the game is of finite depth one can compute optimal *pure* strategies by backward induction as usual.

$v_M(\exists x F(x)) > v_M(F(c))$ for all constants c. For this reason we define the following general notion for games with randomized payoff (as in our new version of the $\mathcal{H}$-game, above, and in $\mathcal{G}$-games, introduced below).

DEFINITION 3.3
A game with randomized payoff is *r-valued for player $\boldsymbol{X}$* if, for every $\epsilon > 0$, $\mathbf{X}$ has a strategy that guarantees that her expected loss is at most $(r+\epsilon)$€, while her opponent has a strategy that ensures that the loss of $\mathbf{X}$ is at least $(r-\epsilon)$€. We call r the *risk for $\boldsymbol{X}$* in that game.

This notion allows us to state the generalization of Theorem 3.2 to $\mathbf{Ł}^w$ concisely:

THEOREM 3.4
A $\mathbf{Ł}^w$-sentence F is evaluated to $v_M(F) = x$ in interpretation M iff the $\mathcal{H}$-game for F under risk value assignment $\langle\cdot\rangle_M$ is $(1-x)$-valued for me.

PROOF. Building on the proof of Theorem 3.2, it only remains to consider the induction steps for quantified sentences:

- If I assert $\exists x F(x)$, then the game continues with my assertion of $F(c)$ for a constant c picked by me in a manner that minimizes my risk. In fact, since there might be no domain element witnessing the infimum $v_M(\exists x F(x)) = \inf_{c \in D}(v_M(F(c)))$, we can only ensure that, for any given $\delta > 0$, $\langle | \, \exists x F(x) \rangle^* = \langle | \, F(c) \rangle^* = 1 - v_M(\exists x F(x)) + \delta$.
- If I assert $\forall x F(x)$, the game continues with my assertion of $F(c)$, where c is chosen by you to maximize my risk. Therefore, analogously, we obtain $\langle | \, \forall x F(x) \rangle^* = \langle | \, F(c) \rangle^* = 1 - v_M(\forall x F(x)) - \delta$ for some $\delta > 0$.

The cases where you are the defender of a quantified formula are dual. ∎

Note that the value ϵ mentioned in Definition 3.3 does not directly correspond to δ as used in the above proof, but rather results from the accumulation of appropriate δs. In any case, since our intended applications assume *finite domains*, we may from now on safely ignore the fact that, in general, truth values of statements involving quantifiers are only approximated by expected risk in concrete instances of a game. We nevertheless retain the notion of the *value of a game*, but could actually simplify Definition 3.3 by dropping all references to ϵ.

4 From $\mathcal{H}$-games to $\mathcal{G}$-games

Already in the 1970s Robin Giles [18, 17] introduced an evaluation game that was intended to provide 'tangible meaning' to reasoning about statements with dispersive semantic tests as they appear in physics. For the logical rules of his game Giles referred not to Henkin or Hintikka, but to Lorenzen's dialogue game semantics for intuitionistic logic [29]. In particular, the following rule for implication was proposed:

$(R_\to)$ If I assert $F \to G$ then you may attack by asserting F, which obliges me to defend by asserting G. (Analogously if you assert $F \to G$.)

In contrast to $\mathcal{H}$-games, such a rule introduces game states, where more than one formula may be currently asserted by each of us. Since, in general, it matters whether we assert the same statement just once or more often, game states are now denoted as pairs of multisets of formulas. We call such games $\mathcal{G}$*-games*. A *final state* of a $\mathcal{G}$-game where $\{A_1, \ldots, A_n\}$ is the multiset of atomic assertions made by you and $\{B_1, \ldots, B_m\}$ is the multiset of atomic assertions made by me is denoted by

$$[A_1, \ldots, A_n \mid B_1, \ldots, B_m] .$$

Again we assume that a binary experiment E_A is associated with every atomic A with corresponding risk $\langle A \rangle = 1 - \pi(\mathsf{E}_A)$. We now make payments fully dual and stipulate that I have to pay 1€ to you whenever an instance of an experiment corresponding to one of my atomic assertion fails, while you have to pay me 1€ for each instance of a failing experiment corresponding to one of your atomic assertions. We obtain the following value for the expected total amount of money (in €) that I have to pay to you at the exhibited final state:

$$\langle A_1, \ldots, A_n \mid B_1, \ldots, B_m \rangle = \sum_{1 \leq i \leq m} \langle B_i \rangle - \sum_{1 \leq j \leq n} \langle A_j \rangle .$$

We call this value briefly my *risk* associated with that state. Note that the risk can be negative in $\mathcal{G}$-games, i.e., the risk values of the relevant propositions may be such that I expect a net payment by you to me.

Interestingly, the rules $(R_\wedge)$, $(R_\vee)$, $(R_\forall)$, and $(R_\exists)$ defined in Section 3 remain unchanged for $\mathcal{G}$-games. By adding the above implication rule $(R_\to)$ and defining $\neg F = (F \to \bot)$ we arrive at Giles's game for Łukasiewicz logic.

We like to point out that $(R_\to)$ contains a hidden *principle of limited liability*: the player opposing the defender of $F \to G$ may (instead of asserting F in return for the opponent's assertion of G) explicitly choose not to attack $F \to G$ at all. This option results in a branching of the game tree. The state $[\Gamma \mid \Delta, F \to G]$, where Γ and Δ are multisets of sentences asserted by you and me, respectively, and where the exhibited occurrence indicates that you currently refer to my assertion of $F \to G$, has the two possible successor states: $[F, \Gamma \mid \Delta, G]$ and $[\Gamma \mid \Delta]$. In the latter state you have chosen to limit your liability in the following sense. Attacking an opponent's assertion should never incur an expected (positive) loss, which were the case if the risk associated with asserting F is higher than that for asserting G. In such cases a rational player in the attacking role will explicitly renounce an attack on $F \to G$. For all other logical connectives the principle is ensured by the fact that—in all games considered here—each occurrence of a formula can be attacked at most once. (The attacked occurrence is removed from the state in the transition to a corresponding successor state.)

Another form of the principle of limited liability can be considered for defending moves. In defending any sentence F, the defending player has to be able to hedge her possible loss associated with the assertions made in defense of F to at most 1€. This is already the case for all logical rules considered so far. However, as shown in [11, 13], by making this principle explicit we arrive at a rule for strong conjunction, that is missing in Giles [18, 17]:

$(R_\&)$ If I assert $F \& G$, I have to assert either both F and G, or assert $\bot$ instead. (Analogously if you assert $F \& G$.)

The above description might yet be too informal to see in which sense every $\mathcal{G}$-game, just like an $\mathcal{H}$-game, constitutes an ordinary two-person zero-sum extensive game of finite depth with perfect information. For this purpose one has to be a bit more precise than Giles and introduce the notion of a *consistent regulation*, determining which player is to move next. Additionally, one has to model the selection of a non-atomic formula to be attacked as an explicit move in the game. For a detailed formal presentation, including examples, we refer to [13]. Here it suffices to point out that any given consistent regulation determines the tree representing a concrete $\mathcal{G}$-game for a formula F (meaning a game with initial state $[\mid F]$). By identifying the formula F selected for attack with the formula exhibited in the corresponding game rule we obtain for every such state the next two levels of successor states: the first level registers the possible choices (if any) for attacking F (including, in the case of an implication, the option to apply the principle of limited liability and thus just to remove F) and the second level registers the options for defending F according to the rule. (In the terminology introduced in Section 3, the two levels correspond to a *round* in the game. Like for randomized $\mathcal{H}$-games, every run of a $\mathcal{G}$-game is a sequence of states ending in a final state, the risk of which constitutes its payoff.)

We arrive at the following characterization of strong Łukasiewicz logic **Ł** by $\mathcal{G}$-games:

THEOREM 4.1 ([13], based on [18])
A **Ł**-sentence F is evaluated to $v_M(F) = x$ in interpretation M iff every $\mathcal{G}$-game for F under risk value assignment $\langle\cdot\rangle_M$ is $(1-x)$-valued for me.

5 Random witnesses for quantifiers

We argue that a simple generalization of the game for **Ł**, described in Section 4, allows one to address the challenges for fuzzy quantification outlined in Section 2 by singling out a class of semi-fuzzy quantifiers that fit Giles's idea to provide 'tangible meaning' to logical connectives in terms of bets on the results of dispersive experiments.

Remember that the only difference between the rules $(R_\forall)$ and (R_E) for defending assertions $\forall x F(x)$ and $\exists x F(x)$, respectively, is that either the defender or the attacker has to pick the constant c that determines the new sentence $F(c)$ that remains to be defended. Considering the randomized setting of $\mathcal{G}$-games, the following rule for a new type of (monadic) quantifier Π seems natural:

(R_Π) If I assert $\Pi x F(x)$ then I have to assert $F(c)$ for a randomly picked c.

The random choice refers to a uniform distribution of the finite domain. Note that, while all kinds of other forms of randomly picking domain elements might be considered in principle, we recall from the literature on generalized quantifiers (see, e.g., [39, 42]) that a necessary condition for a quantifier to be called *logical* is the *domain invariance* of its semantics.[9] As will get clear below, this is guaranteed for Π (and for the quantifiers considered in Sections 6 and 7 by insisting on random choices with respect to a uniform distribution.

[9] There is no agreement in the literature on when a generalized quantifier is to be called *logical*. However it is at least clear that invariance with respect to isomorphisms between domains is a necessary condition, because a quantifier should not be able to distinguish between elements of the universe. This principle has first been formulated by Mostowski [33]. See [39] for a discussion of this issue.

While (R_Π), in principle, can be applied to arbitrary **Ł**-formulas F in the scope of Π, we will view Π as a *semi-fuzzy quantifier* and hence insist on classical formulas in its scope for reasons explained in Section 2. More formally, we specify the language for logic $\mathbf{Ł}(Qs)$, where Qs is a list of (unary) quantifier symbols other than $\forall$ or $\exists$, as follows:

$$\begin{array}{lll} \gamma & ::= & \bot \mid \hat{P}(\vec{t}) \mid \neg\gamma \mid (\gamma \vee \gamma) \mid (\gamma \wedge \gamma) \mid \forall v\gamma \mid \exists v\gamma \\ \varphi & ::= & \gamma \mid \tilde{P}(\vec{t}) \mid \neg\varphi \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \to \varphi) \mid (\varphi \& \varphi) \mid \forall v\varphi \mid \exists v\varphi \mid \mathsf{Q}v\gamma \end{array}$$

where $\hat{P}$ and $\tilde{P}$ are meta-variables for classical and for general (i.e., possibly fuzzy) predicate symbols, respectively, $\mathsf{Q} \in Qs$; v is our meta-variable for object variables; $\vec{t}$ denotes a sequence of terms, i.e. either object variable or constant symbol, matching the arity of the preceding predicate symbol. Note the scope of the additional quantifiers from Qs is always a classical formula. Otherwise the syntax is as for **Ł** itself.

The following notion supports a crisp specification of truth functions for semi-fuzzy proportionality quantifiers over finite interpretations.

Definition 5.1
Let $\hat{G}(x)$ be a classical formula and $v_M(\cdot)$ a corresponding evaluation function over the finite domain D. Then

$$\mathrm{Prop}_x\, \hat{G}(x) = \frac{\sum_{c \in D} v_M(\hat{G}(c))}{|D|}$$

$\mathrm{Prop}_x\, \hat{G}(x)$ thus denotes the proportion of all elements in D satisfying the classical predicate $\hat{G}$. As we stipulated a uniform probability distribution over D, above, this matches the probability that a randomly chosen element satisfies $\hat{G}$.

The following theorem states that rule (R_Π) matches the extension of the valuation function for **Ł** to $\mathbf{Ł}(\Pi)$ by $v_M(\Pi x F(x)) = \mathrm{Prop}_x\, F(x)$.

Theorem 5.2
A $\mathbf{Ł}(\Pi)$-sentence F is evaluated to $v_M(F) = x$ in an interpretation M iff every $\mathcal{G}$-game for F augmented by rule (R_Π) is $(1-x)$-valued for me under risk value assignment $\langle\cdot\rangle_M$.

Theorem 5.2 will turn out to be an instance of a more general result to be proved in the next section.

As mentioned in Section 2, natural language quantifiers are usually binary, as in *About half of the students are present*, rather than unary as in *About half* [*of the elements in the domain of discourse*] *the students are present*. However, binary quantifiers like *about half*, *many*, *at least a third*, etc, are extensional. This means that, like in the above example, the first argument of the binary quantifier—its range—is only used to restrict the universe of discourse. More formally, let $\hat{A}$ denote the set of domain elements that satisfy the (crisp) predicate expressed by the classical formula $\hat{A}(x)$. If Q is a unary quantifier, then ${}^{\hat{A}}\mathsf{Q}xF(x)$ is a quantified statement defined by $v_M({}^{\hat{A}}\mathsf{Q}xF(x)) = v_{M'}(\mathsf{Q}xF(x))$, where M' denotes the interpretation that results from M by restricting the domain of M to $\hat{A}$, denoting the set of elements that satisfy the (crisp) predicate expressed by the classical formula $\hat{A}(x)$. This reduces extensional binary quantification to unary quantification in a manner that is readily modeled by a uniform modification of quantifier game rules, illustrated for Π as follows:

(R_{Π^2}) If I assert ${}^{\hat{A}}\Pi x \hat{B}(x)$ then I have to assert $\hat{B}(c)$ where c is a randomly picked element of $\hat{A}$.

If the classical formula $\hat{A}(x)$ is atomic then it is clear what it means to randomly pick an element of $\hat{A}$ (if one exists at all). If however $\hat{A}(x)$ is of arbitrary logical complexity, then we may remain within our game semantical framework by employing $\mathcal{H}$-games to find an appropriate random witness as follows:

1. Pick a random domain element c.
2. Initiate an $\mathcal{H}$-game where a Proponent **P** defends $\hat{A}(c)$ against an Opponent **O**.
3. If **P** wins the $\mathcal{H}$-game, then I and you continue the main $\mathcal{G}$-game with the constant c. Otherwise, continue at 1.

We deliberately changed the identity of the players (from I/You to **P**/**O**) in moving from the $\mathcal{G}$-game to the $\mathcal{H}$-game, since it is important to keep the objectives of the players **P** and **O** in the $\mathcal{H}$-game independent from our objectives in the $\mathcal{G}$-game. By Theorem 3.1 **P** wins the $\mathcal{H}$-game against the rational Opponent **O** if and only if $\hat{A}(c)$ is true, i.e., if $c \in \hat{A}$. Note that the indicated procedure and therefore the main $\mathcal{G}$-game will fail to terminate if the range $\hat{A}$ is empty. This is in accordance with the above definition that leaves $v_M({}^{\hat{A}}\mathsf{Q}xF(x))$ undefined if the range is empty. According to [3] this matches intuitions about natural language quantifiers applied to an empty range.

Remark. There is an interesting similarity between our notion of randomized witness selection and the "chance setups" used by Halpern and others in the context of reasoning about probability (see in particular [23]). Halpern introduces a first order logic that allows one to express statements[10] like "A randomly chosen element of the domain satisfies property $\hat{G}$ with probability at least p". Note that such statements are bivalent. Moreover they involve terms referring to probability values. Consequently, the underlying formal language differs substantially from fuzzy logics. Nevertheless probability logics and our generalization of semantic games share the basic idea of referring to random elements of the domain of discourse.

6 Blind choice quantifiers

Remember that in the context of our $\mathcal{G}$-games we have considered three types of challenges to the defender **X** of a quantified sentence $\mathsf{Q}xF(x)$. In each case **X** has to assert $F(c)$, but the constant (domain element) is either

(A) chosen by the attacker, or

(D) chosen by the defender, or

(R) chosen randomly.

We will speak of a challenge of type A, D, or R, respectively. The need to variate these three challenges arises when we allow the defender (and possibly also the attacker) of $\mathsf{Q}xF(x)$ to bet either *for* or *against* $F(c)$. Betting for $F(c)$ simply means to assert $F(c)$, betting against $F(c)$ is equivalent to betting for $\neg F(c)$ and thus amounts to

[10] In fact Halpern's logic is much more expressible and combines statistical reasoning over arbitrary distributions with reasoning about degrees of belief.

$$\left[\Gamma \mid \Delta, \mathsf{Q}x\hat{F}(x)\right]$$

$$\left[\Gamma, \hat{F}(c_i)^{r_1}, \bot^{s_1} \mid \Delta, \hat{F}(c_i')^{u_1}, \bot^{v_1}\right] \quad \cdots\cdots \quad \left[\Gamma, \hat{F}(c_i)^{r_n}, \bot^{s_n} \mid \Delta, \hat{F}(c_i')^{u_n}, \bot^{v_n}\right]$$

FIG. 2: Schematic blind choice quantifier rule — my possible defenses to a particular attack by you.

an assertion of $\bot$ in exchange for an assertion of $F(c)$ by the opposing player. We interpret this as follows: **X** pays 1€ for a betting ticket regarding $F(c)$ that entitles her to receive whatever payment by her opponent **Y** is due for **Y**'s assertion of $F(c)$ according to the results of associated dispersive experiments made at the end of the game.

As explained in Section 3, a round of a game consists of a player's *attack* of an assertion made by the other player, followed by a *defense* of that latter player, where the principle of limited liability states that asserting $\bot$ is always a valid defense. Moreover, by the other form of the principle of limited liability, the attacker, instead of attacking an assertion in some specific way, may grant the assertion which will consequently be deleted from the current state of the game. In general, when an assertion of $\mathsf{Q}x\hat{F}(x)$ is attacked, the round results in a state where both players are placing certain numbers of bets for or against various instances of $\hat{F}(x)$, where the constants replacing x can be of type A, D, or R. Thus we arrive at a rich set of possible quantifier rules. In this paper we are only interested in type R challenges. We will call $\hat{F}(c)$ a random instance of $\hat{F}(x)$ if c has been chosen randomly.

In this section we will investigate the family of *blind choice quantifiers* defined as follows.

DEFINITION 6.1
Q is a *(semi-fuzzy) blind choice quantifier* if it can be specified by a game rule satisfying the following two conditions:

(i) Only challenges of type (R) are allowed. An attack on $\mathsf{Q}x\hat{F}(x)$ followed by a defense move results in a state where both players have placed a certain number (possibly zero) of bets *for* and *against* random instances of $\hat{F}(x)$.

(ii) The identity of the random constants is revealed to the players only at the end of the round; i.e., after an attack has been chosen by the one player and a corresponding defense move has been chosen by the other player.

Figure 2 depicts possible state transitions involved in the application of a blind choice quantifier rule. Γ and Δ denote arbitrary multisets of formulas; $\hat{F}(x)$ is a classical formula that forms the scope of the sentence $\mathsf{Q}x\hat{F}(x)$ asserted by my and attacked by you; $\bot^k$ denotes k occurrences of $\bot$; and $\hat{F}(c_i)^k$ is used as an abbreviation for the k assertions of random instances $\hat{F}(c_1), \ldots, \hat{F}(c_k)$. Note that in general there is more than one way in which you may attack my assertion of $\mathsf{Q}x\hat{F}(x)$. Figure 2 only shows the scheme for one particular attack. A presentation of a full rule consists of a finite number of instances of this scheme. The root of all these trees is labeled by $\left[\Gamma \mid \Delta, \mathsf{Q}x\hat{F}(x)\right]$, which means that the effect of the different attacks is shown only at the end of a full round, i.e., only after also a corresponding defense has been

$$\begin{tikzcd}[column sep=small]
\left[\Gamma \mid \Delta, \mathsf{L}^k_m x \hat{F}(x)\right] \arrow[d, dash] & & \left[\Gamma \mid \Delta, \mathsf{L}^k_m x \hat{F}(x)\right] \arrow[dl, dash] \arrow[dr, dash] & \\
[\Gamma \mid \Delta] & [\Gamma \mid \Delta, \bot] & & \left[\Gamma, \hat{F}(c_i)^{k+m} \mid \Delta, \bot^m\right]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
\left[\Gamma \mid \Delta, \mathsf{G}^k_m x \hat{F}(x)\right] \arrow[d, dash] & & \left[\Gamma \mid \Delta, \mathsf{G}^k_m x \hat{F}(x)\right] \arrow[dl, dash] \arrow[dr, dash] & \\
[\Gamma \mid \Delta] & [\Gamma \mid \Delta, \bot] & & \left[\Gamma, \bot^m \mid \Delta, \hat{F}(c_i)^{k+m}\right]
\end{tikzcd}$$

FIG. 3. Games rules $R_{\mathsf{L}^k_m}$ and $R_{\mathsf{G}^k_m}$.

chosen. The principle of limited liability implies that you (the attacker) may choose to simply remove the exhibited occurrence of $\mathsf{Q}x\hat{F}(x)$ from the state. In other words, every rule includes an instance of Figure 2 that consists of only one branch ($n = 1$), where $r_1 = s_1 = u_1 = v_1 = 0$. For me as defender, the principle of limited liability implies that in any other instance of the schematic tree there is a branch i with $r_i = s_i = u_i = 0$ and $v_i = 1$, i.e., where I reply to your attack by asserting $\bot$.

Throughout the paper we assume that for every rule for my assertion of a formula, there is a corresponding rule for your assertion of the same formula, that arises by switching our roles. It therefore suffices to explicitly state and investigate rules for my assertions of quantified formulas only.

We define blind choice quantifiers L^k_m and G^k_m for as follows:

($R_{\mathsf{L}^k_m}$) If I assert $\mathsf{L}^k_m x\hat{F}(x)$ then you may attack by betting for k random instances of $\hat{F}(x)$, while I bet against m random instances of $\hat{F}(x)$.

($R_{\mathsf{G}^k_m}$) If I assert $\mathsf{G}^k_m x\hat{F}(x)$ then you may attack by betting against m random instances of $\hat{F}(x)$, while I bet for k random instances of $\hat{F}(x)$.

We insist on condition *(ii)* of Definition 6.1: the random constants used to obtain the mentioned instances of $F(x)$ are only revealed to the players after they have placed their bets. Moreover, although not explicitly mentioned, the principle of limited liability remains in force. Therefore, the defender may also respond to an attack by asserting $\bot$. However, if none of the players invokes the principle of limited liability the following successor game states are reached:

$$\begin{aligned}
\text{for } \mathsf{L}^k_m x\hat{F}(x): &\quad \left[\Gamma, \hat{F}(c_i)^{k+m} \mid \Delta, \bot^m\right] \\
\text{for } \mathsf{G}^k_m x\hat{F}(x): &\quad \left[\Gamma, \bot^m \mid \Delta, \hat{F}(c_i)^{k+m}\right]
\end{aligned}$$

Thus, for my assertion of $\mathsf{L}^k_m x\hat{F}(x)$ the rule can be depicted as shown in Figure 3 and analogously for your assertion of $\mathsf{L}^k_m x\hat{F}(x)$ and also for G^k_m.

We claim that these rules match the extension of Ł to $\text{Ł}(\mathsf{L}^k_m, \mathsf{G}^k_m)$ by

$$v_M(\mathsf{L}^k_m x\hat{F}(x)) = \min(1, \max(0, 1 + k - (m+k)\,\mathrm{Prop}_x\,\hat{F}(x))) \text{ and} \tag{6.1}$$

$$v_M(\mathsf{G}^k_m x\hat{F}(x)) = \min(1, \max(0, 1 - k + (m+k)\,\mathrm{Prop}_x\,\hat{F}(x))). \tag{6.2}$$

THEOREM 6.2
A $\text{Ł}(\mathsf{L}^k_m, \mathsf{G}^k_m)$-sentence F is evaluated to $v_M(F) = x$ in an interpretation M iff every $\mathcal{G}$-game for F augmented by the rules $(R_{\mathsf{L}^k_m})$ and $(R_{\mathsf{G}^k_m})$ is $(1-x)$-valued for me under risk value assignment $\langle\cdot\rangle_M$.

PROOF. Relative to the proof of Theorem 4.1 (see [18, 17, 13]) we only have to consider states of the form $\left[\Gamma \mid \Delta, \mathsf{L}^k_m x\hat{F}(x)\right]$ and $\left[\Gamma \mid \Delta, \mathsf{G}^k_m x\hat{F}(x)\right]$. (I.e., we only consider cases where the regulation of the game determines that my assertion of an L^k_m- or G^k_m-quantified sentences is to be considered next. The cases for your assertions of $\mathsf{L}^k_m x\hat{F}(x)$ or $\mathsf{G}^k_m x\hat{F}(x)$ are dual.) In fact, since G^k_m is treated analogously to L^k_m, we may focus on states of the form $\left[\Gamma \mid \Delta, \mathsf{L}^k_m x\hat{F}(x)\right]$ without loss of generality. Like for the other connectives, we obtain the total risk at such a state as the sum of the risk for the exhibited assertion and of the risk for the rest of the state:

$$\left\langle\Gamma \mid \Delta, \mathsf{L}^k_m x\hat{F}(x)\right\rangle = \langle\Gamma \mid \Delta\rangle + \left\langle \mid \mathsf{L}^k_m x\hat{F}(x)\right\rangle.$$

It remains to show that the reduction of the exhibited quantified formula to instances according to rule $(R_{\mathsf{L}^k_m})$ results in a risk that corresponds to the specified truth function if we play rationally. According to Figure 3 the three possible successor states are $\left[\hat{F}(c_i)^{k+m} \mid \bot^m\right]$, $[|]$, and $[| \bot]$. In the first case, revealing the constants to the players also reveals the amount of money I have to pay, since only classical formulas are involved: I have to pay m€ to you for my m assertions of $\bot$, while for each of your $k+m$ assertions you have to pay me either 0€ or 1€. In total I have to pay to you between $-k$€ and m€, depending on the random constants c_i. The risk value of the game state *before* the identities of the constants are revealed to the players is therefore calculated as the *expected* value for this amount. It is binomially distributed and readily computed as:

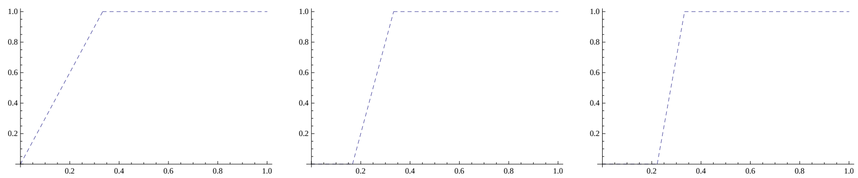
$$\begin{aligned}
&m - \sum_{i=0}^{k+m} i \cdot (\operatorname{Prop}_x \hat{F}(x))^{k+m-i}(1 - \operatorname{Prop}_x \hat{F}(x))^i \binom{k+m}{i} = \\
&m - (k+m)(1 - \operatorname{Prop}_x \hat{F}(x)) = -k + (k+m)\operatorname{Prop}_x \hat{F}(x)).
\end{aligned}$$

The second case (state $[|]$, carrying risk 0) arises if you choose to grant my assertion of the formula, which you will do if the above expression is below 0. The third case (state $[| \bot]$, carrying risk 1) arises if I invoke the principle of limited liability to hedge my expected loss. Thus we obtain

$$\left\langle \mid \mathsf{L}^k_m x\hat{F}(x)\right\rangle = \min(1, \max(0, -k + (k+m)\operatorname{Prop}_x \hat{F}(x))) = 1 - v_M(\mathsf{L}^k_m x\hat{F}(x))$$

which means that the claimed correspondence between the truth function and the risk resulting from playing rationally holds. ∎

At least about a third. As an example, let us take a closer look at quantifiers of the form G^s_{2s}. We argue that these quantifiers can be used to model the natural language expression *at least about a third*. Note that the attacker of $\mathsf{G}^s_{2s} x\hat{F}(x)$ is supposed to believe that $\hat{F}(x)$ holds for clearly less than a third of all domain elements (otherwise she would grant the assertion). Consequently she will agree to place $2s$



(a) $\mathsf{G}_2^1 x \hat{F}(x)$ (b) $\mathsf{G}_4^2 x \hat{F}(x)$ (c) $\mathsf{G}_6^3 x \hat{F}(x)$

FIG. 4. Truth functions for $\mathsf{G}_{2s}^s x \hat{F}(x)$

bets *against* random instances of $\hat{F}(x)$ if the defender places s bets *for* such random instances. Figure 4 shows the resulting truth functions for sample sizes $(2s+s)$ $3, 6$, and 9, where the horizontal axis corresponds to $\mathrm{Prop}_x\, \hat{F}(x)$ and the vertical axis to $v_M(\mathsf{G}_{2s}^s x \hat{F}(x))$. Functions like these are routinely suggested to represent natural language quantifiers like *at least about a third* in the fuzzy logic literature.[11] However no justification beyond intuitive plausibility is usually given. In contrast, our model allows one to extract such truth function from an underlying semantic principle: namely the willingness to bet on randomly chosen witnesses that support or refute the statement in question.

As noted above, the quantifiers L_m^k and G_m^k are only (very restricted) examples of blind choice quantifiers. Nevertheless, they turn out to be expressive enough to define *all* blind choice quantifiers in the context of weak Łukasiewicz logic $\mathbf{Ł}^w$:

THEOREM 6.3
All blind choice quantifiers can be expressed using quantifiers of the form L_m^k and G_m^k, conjunction $\wedge$, disjunction $\vee$, and $\bot$.

PROOF. As illustrated in Figure 2 above, the game state resulting from an attack and a corresponding defense of my assertion of a blind choice quantifiers is always of the form $\left[\Gamma, \hat{F}(c_i)^r, \bot^s \mid \Delta, \hat{F}(c_i')^u, \bot^v\right]$. Analogously to the proof of Theorem 6.2, the associated risk before the identities of the constants are revealed is computed as

$$\langle \Gamma \mid \Delta \rangle + v - s + (u - r)(1 - \mathrm{Prop}_x\, \hat{F}(x)).$$

Remember that $\hat{F}(c_i)^k$ is short hand notation for k (in general) different random instances of $F(x)$. As a first step towards a simplified uniform presentation of arbitrary blind choice quantifiers, note the following. Instead of picking $u+r$ random constants we can rather investigate the game state $\left[\Gamma, \hat{F}(c)^r, \bot^s \mid \Delta, \hat{F}(c)^u, \bot^v\right]$ where only one random constant c is picked, since this modification does not change the *expected* risk. As a further step, note that game states where assertions of $\hat{F}(c)$ are made by *both* players show redundancies in the sense that there are equivalent game states where $\hat{F}(c)$ occurs only in one of the two multisets of assertions that represent a state. Likewise for game states with assertions of $\bot$ made by both players. Depending on v, s, u,

[11] For example in [20] trapezoidal functions like the ones in Figure 4 are explicitly suggested for natural language quantifiers of this kind.

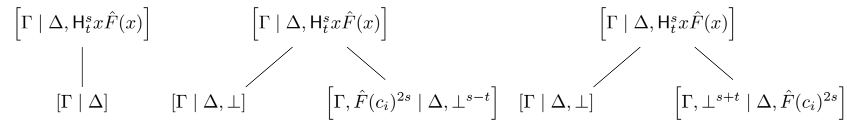


FIG. 5. The rule $R_{\mathsf{H}^s_t}$

and r, an equivalent game state is given by:

$$\begin{aligned}
&(1)\ \left[\Gamma, \hat{F}(c)^{r-u} \mid \Delta, \bot^{v-s}\right] \text{ if } v > s \text{ and } r > u,\\
&(2)\ \left[\Gamma, \hat{F}(c)^{r-u}, \bot^{s-v} \mid \Delta\right] \text{ if } v \leq s \text{ and } r > u\\
&(3)\ \left[\Gamma \mid \Delta, \hat{F}(c)^{u-r}, \bot^{v-s}\right] \text{ if } v > s \text{ and } r \leq u,\\
&(4)\ \left[\Gamma, \bot^{s-v} \mid \Delta, \hat{F}(c)^{u-r}\right] \text{ if } v \leq s \text{ and } r \leq u.
\end{aligned}$$

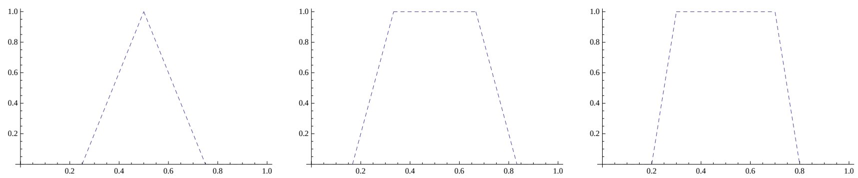
Note that states of type (2) are redundant, since you would rather invoke the principle of limited liability, resulting in $[\Gamma \mid \Delta]$, than to make an assertion without being compensated by any assertions made by me. On the other hand, states of type (3) reduce to state $[\Gamma \mid \Delta, \bot]$, since I may invoke the principle of liability. For states of type (1) I will invoke the principle of limited liability if $v - s > r - u$. Similarly, you will invoke the principle of limited liability to ensure that only those states of type (4) have to be considered where $s - v \leq u - r$. But, for appropriate choices of k and m, this leaves us with states that result from the rules for either $\mathsf{L}^k_m x \hat{F}(x)$ or for $\mathsf{G}^k_m x \hat{F}(x)$.

Finally observe that all of my defenses to your attack on $\mathsf{Q} x \hat{F}(x)$ lead to successor states which are reached also by suitable instances of $\mathsf{G}^k_m x \hat{F}(x)$, of $\mathsf{L}^k_m x \hat{F}(x)$, (or $\bot$). Hence my risk for that attack amounts to the minimum of the risk values for these successor states, which in turn equals the risk value for asserting the disjunction of these instances. Similarly, since you can choose between several attacks on $\mathsf{Q} x \hat{F}(x)$ in the first place, my risk for $\mathsf{Q} x \hat{F}(x)$ amounts to the maximum of the risks for these attacks. Hence it is equal to the risk of the conjunction of these disjunctions. ∎

About half. As an example consider the family of quantifiers H^s_t, defined by the game rule depicted in Figure 5.

We suggest that H^s_t induces plausible fuzzy models for the natural language quantifier *about half.* Figure 6 shows the truth functions for three different quantifiers of this family, where the horizontal axis corresponds to $\mathrm{Prop}_x \hat{F}(x)$ and the vertical axis to $v_M(\mathsf{H}^s_t x \hat{F}(x))$.

The two parameters of H^s_t can be interpreted as follows: s determines the sample size (i.e. the number of random instances involved in reducing the quantified formula), while t may be called the *tolerance*, since the smaller t gets, the closer $\mathrm{Prop}_x \hat{F}(x)$ has to be to $1/2$ if $\mathsf{H}^s_t x \hat{F}(x)$ is to be evaluated as perfectly true. If $t = 0$ (zero tolerance) then $v_M(\mathsf{H}^s_0 x \hat{F}(x)) = 1$ if only if $\mathrm{Prop}_x \hat{F}(x) = 1/2$ in M. By increasing t (while maintaining the same sample size s) the range of values for $\mathrm{Prop}_x \hat{F}(x)$ that guarantee $v_M(\mathsf{H}^s_0 x \hat{F}(x)) = 1$ grows symmetrically around $1/2$.



(a) $\mathsf{H}^2_1 x\hat{F}(x)$ (b) $\mathsf{H}^3_1 x\hat{F}(x)$ (c) $\mathsf{H}^4_2 x\hat{F}(x)$

FIG. 6. Truth functions for $\mathsf{H}^s_t x\hat{F}(x)$

As an instance of Theorem 6.3 we obtain that $\mathsf{H}^s_t x\hat{F}(x)$ is equivalent to $\mathsf{G}^{s+t}_{s-t} x\hat{F}(x) \wedge \mathsf{L}^{s+t}_{s-t} x\hat{F}(x)$. The tree at the center of Figure 5 corresponds to the rule for G^{s+t}_{s-t} and the one at the right hand side corresponds to the rule for L^{s+t}_{s-t}. The tree at left hand side corresponds to the fact that the attacker may choose to grant the formula.

Next we show how arbitrary blind choice quantifiers can be reduced to the quantifier Π introduced in Section 5 if the connectives of strong Łukasiewicz logic **Ł** are available.

THEOREM 6.4
The blind choice quantifiers G^k_m and L^k_m can be expressed in $\mathbf{Ł}(\Pi)$ via the following reductions:

$$v_M(\mathsf{G}^k_m x\hat{F}(x)) = v_M(\neg((\neg\Pi x\hat{F}(x))^{m+1})\&(\Pi x\hat{F}(x))^{k-1})$$
$$v_M(\mathsf{L}^k_m x\hat{F}(x)) = v_M(\neg((\Pi x\hat{F}(x))^{k+1})\&(\Pi x\neg\hat{F}(x))^{m-1})$$

for all natural numbers m and k and ϕ^n denoting $\phi\& \ldots \&\phi$, n times.

PROOF. Note that the truth functions of $\mathsf{G}^k_m x\hat{F}(x)$ and $\mathsf{L}^k_m x\hat{F}(x)$ depend only on $\mathrm{Prop}_x \hat{F}(x)$, while the random choice quantifier Π is directly represented by the truth function $\mathrm{Prop}_x \hat{F}(x)$. Hence the equivalences can easily be checked by computing the truth value of the respective right hand side formula and comparing it to the truth function for the corresponding quantifier. ∎

COROLLARY 6.5
All blind choice quantifiers can be expressed in $\mathbf{Ł}(\Pi)$.

The corollary follows directly from Theorems 6.3 and 6.4. By a less direct route, one could also employ (a constructive proof of) McNaughton's theorem [31] to obtain such reductions.

We finally point out a related fact: any linear function $f(x) = m_1 x + m_0$ with integer coefficients m_1 and m_0 capped to the unit interval $[0, 1]$ can be expressed via instances of G^k_m and L^k_m and $\perp$. We distinguish the two cases (1) $m_1 \geq 0$ and (2) $m_1 < 0$. Case (1): If $m_0 > 1$ we take the truth function of L^0_0, constantly yielding 1; if $m_0 + m_1 < 1$ we use $\perp$, constantly yielding 0; otherwise we use $\mathsf{G}^{1-m_0}_{m_0+m_1-1}$. Case (2): If $m_0 < 1$ we use $\perp$; if $m_0 + m_1 > 1$ we use L^0_0; otherwise we use $\mathsf{L}^{m_0-1}_{1-m_0-m_1}$. This can readily be checked by inserting the respective values for k and m into Formulas 6.1 and 6.2.

7 Deliberate choice quantifiers

In the previous section we surveyed the family of blind choice quantifiers and concluded that these quantifiers all amount to piecewise linear truth functions. A much more general class of quantifiers arises by dropping condition *(ii)* of Definition 6.1. As an example of this class we investigate the family of so-called *deliberate choice quantifiers*, specified by the following schematic game rule, where $\hat{F}$ is a classical formula:

($R_{\Pi^k_m}$) If I assert $\Pi^k_m x\hat{F}(x)$ then, if you attack, $k+m$ constants are chosen randomly and I have to pick k of those constants, say $c_1,\dots,c_k$, and bet for $\hat{F}(c_1),\dots,\hat{F}(c_k)$, while simultaneously betting against $\hat{F}(c'_1),\dots,\hat{F}(c'_m)$, where $c'_1,\dots,c'_m$ are the remaining m random constants. (Analogously for your assertion of $\Pi^k_m x\hat{F}(x)$.)

Although not mentioned explicitly, we emphasize that the principle of limited liability remains in place: after the constants are chosen, I may assert $\perp$ (i.e., agree to pay 1€) instead of betting as indicated above. Therefore I have $1+\binom{k+m}{k}$ possible defenses to your attack on my assertion of $\Pi^k_m x\hat{F}(x)$: either I choose to hedge my loss by asserting $\perp$ or I pick k out of the $k+m$ random constants to proceed as indicated.

We claim that this rule matches the extension of $\mathbf{Ł}$ to $\mathbf{Ł}(\Pi^k_m)$ by

$$v_M(\Pi^k_m\hat{F}(x)) = \binom{k+m}{k}(\mathrm{Prop}_x\,\hat{F}(x))^k(1-\mathrm{Prop}_x\,\hat{F}(x))^m.$$

Theorem 7.1
A $\mathbf{Ł}(\Pi^k_m)$-sentence F is evaluated to $v_M(F)=x$ in interpretation M iff every $\mathcal{G}$-game for F augmented by rule ($R_{\Pi^k_m}$) is $(1-x)$-valued for me under risk value assignment $\langle\cdot\rangle_M$.

Proof. Like in the proof of Theorem 6.2, we only have to consider states of the form $\left[\Gamma\mid\Delta,\Pi^k_m x\hat{F}(x)\right]$. Again, we can separate the risk for the exhibited assertion from the risk for the remaining assertions:

$$\left\langle\Gamma\mid\Delta,\Pi^k_m x\hat{F}(x)\right\rangle = \langle\Gamma\mid\Delta\rangle + \left\langle\mid\Pi^k_m x\hat{F}(x)\right\rangle.$$

It remains to show that my optimal way to reduce the exhibited quantified formula to instances as required by rule ($R_{\Pi^k_m}$) results in a risk that corresponds to the specified truth function. For the following argument remember that the principle of limited liability is in place. Moreover remember that $\hat{F}(x)$ is classical. This means that I either finally have to pay 1€ for my assertion of $\Pi^k_m x\hat{F}(x)$ or do not have to pay anything at all for it. The latter is only the case if all my bets for $\hat{F}(c_1),\dots,\hat{F}(c_k)$, as well as all my bets against $\hat{F}(c'_1),\dots,\hat{F}(c'_m)$, for $c_1,\dots,c_k$, $c'_1,\dots,c'_m$ as specified in rule ($R_{\Pi^k_m}$), succeed. Let the random variable K denote the number of chosen elements c on which my bet is successful; i.e., where $\left\langle\hat{F}(c)\right\rangle=0$. Then K is binomially distributed and the probability that this event obtains (the inverse of my associated risk) is readily calculated to be

$$\binom{k+m}{k}\mathrm{Prop}_x\,\hat{F}(x)^k(1-\mathrm{Prop}_x\,\hat{F}(x))^m.$$

This matches the relevant truth function. ∎

At a first glance, the deliberate choice quantifier Π^k_m might seem suitable for modeling the natural language quantifier *about* k *out of* $m+k$. However, a look at the corresponding graph for $\left\langle \Pi^1_1 x\hat{F}(x) \right\rangle$ reveals that the risk for asserting $\Pi^1_1 x\hat{F}(x)$ is always larger than 0.5. In other words the statement is always 'half-true' at best. This is clearly not in accordance with intuitions about the truth conditions for statements like *About half of the students passed*.

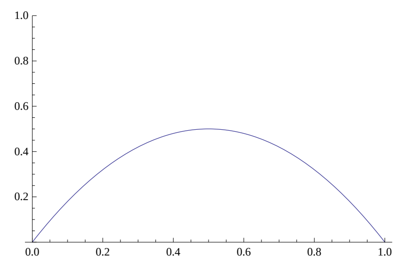


FIG. 7. Truth value for $\Pi^1_1 x\hat{F}(x)$ (depending on p)

An additional mechanism is needed to obtain more appropriate models of natural quantifier expressions like *about half*. While there are many ways to achieve the desired effect, we confine ourselves here to a particularly simple operator that nicely fits our semantic framework, since it arises by simply multiplying involved bets. Given a number $n \geq 2$ and a semi-fuzzy quantifier Q we specify the quantifier $\mathsf{W}_n(\mathsf{Q})$ by the following rule.

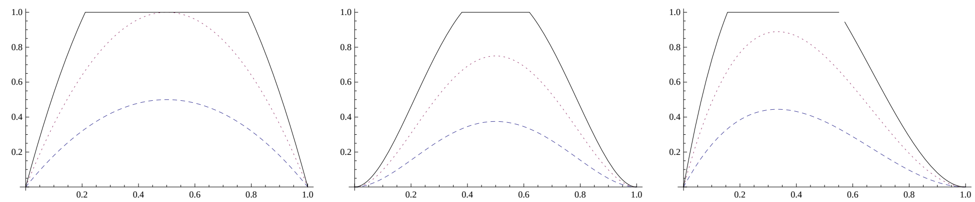
($\mathsf{W}_n(\mathsf{Q})x\hat{F}(x)$) If I assert $\mathsf{W}_n(\mathsf{Q})x\hat{F}(x)$ then you have to place n bets *against* $\mathsf{Q}x\hat{F}(x)$ while I have to bet *for* $\mathsf{Q}x\hat{F}(x)$ just once. (Analogously for your assertion of $\mathsf{W}_n(\mathsf{Q})x\hat{F}(x)$.)

Note that W_n is acting here as a *quantifier modifier*; for any semi-fuzzy quantifier Q, $\mathsf{W}_n(\mathsf{Q})$ still denotes a semi-fuzzy quantifier. The principle of limited liability remains in place, hence the game state $\left\langle \Gamma \mid \Delta, \mathsf{W}_n(\mathsf{Q})x\hat{F}(x) \right\rangle$ is reduced to $\left\langle \Gamma, \bot^n \mid \Delta, \mathsf{Q}x\hat{F}(x)^{n+1} \right\rangle$, or to $\langle \Gamma \mid \Delta \rangle$, depending on whether it is is preferable from the attacker's point of view to attack or to grant the assertion of $\mathsf{Q}x\hat{F}(x)^{n+1}$. (The defender never has to invoke the principal of limited liability in optimal strategies.) Moreover, similarly as in Theorem 6.4, W_n can be expressed using negation and strong conjunction by

$$v_M(\mathsf{W}_n(\mathsf{Q})x\hat{F}(x)) = v_M(\neg(\neg \mathsf{Q}x\hat{F}(x))^{n+1}).$$

The truth functions for some for quantifiers of type $\mathsf{W}_n(\Pi^k_m)$ are presented in the following figure:
The quantifier $\mathsf{W}_3(\Pi^2_2)$ may be considered as formal fuzzy counterpart of the informal expression *about half*. Likewise, $\mathsf{W}_3(\Pi^1_2)$ may be understood as model of *about a third*. Moreover, $\mathsf{W}_3(\Pi^1_1)$ might serve as a model of *very roughly half*, whereas $\mathsf{W}_2(\Pi^1_1)$ might be appropriate as fuzzy model of the (unhedged) determiner *half*.

We do not claim to have determined optimal fuzzy models of the natural language quantifiers in question. But our approach suggests a way to arrive at models that can



(a) $\mathsf{W}_i(\Pi_1^1)x\hat{F}(x)$ (b) $\mathsf{W}_i(\Pi_2^2)x\hat{F}(x)$ (c) $\mathsf{W}_i(\Pi_2^1)x\hat{F}(x)$

FIG. 8: W_i-modified proportional quantifiers — the graphs correspond to the cases $i = 1$, $i = 2$, and $i = 3$ from bottom to top in each diagram.

be interpreted in terms of bets on random instances of the quantified sentences. This obviously fits Giles's betting game based semantics of fuzzy logic [18, 17] and suggests corresponding extensions of Łukasiewicz logic in a principled manner. Admittedly, the involvement of the W_n-operator amounts to an ad hoc feature of the above models, that may one lead to prefer models based on blind choice quantifiers, studied in Section 6. But in any case, only a small number of discrete parameters, corresponding to numbers of (particular types of) bets, have to selected to obtain concrete game rules that in turn correspond to concrete truth functions.

In a similar manner, deliberate choice quantifiers can be used to generate plausible candidate models for the proportional reading of *many*. In particular, consider a model where asserting (the formal counterpart of) *Many* [*domain elements*] *are* $\hat{F}$ is expressed by a willingness to place a certain number of bets for random instances of $\hat{F}(x)$. This clearly amounts to considering the family of quantifiers Π_0^i. The corresponding truth functions are depicted in Figure 9.

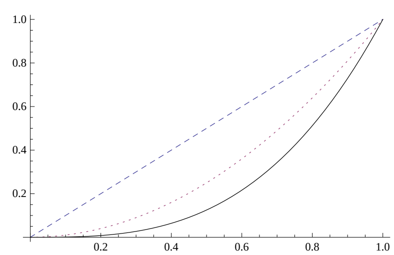


FIG. 9: Truth functions for $\Pi_0^i x\hat{F}(x)$ (depending on p) for $i = 1, 2, 3$ from top to bottom.

Like for *about half* etc, above, one may want to evaluate *Many* [*domain elements*] *are* $\hat{F}$ as perfectly true (truth value 1) even if $\mathrm{Prop}_x\, \hat{F}(x)$ is somewhat smaller than 1. Again, this can be achieved by employing the W_n-operator, which requires the attacker to place several bets against the contended assertion.

8 Conclusion

We began our investigation by pointing out some challenges for the traditional approach to fuzzy quantifiers as introduced by Zadeh in [44]. We do not claim to have satisfactorily solved all these problems. However we hope to have shown that the game semantic framework of Giles, if extended by the concept of randomized choices of witnessing constants, provides essential clues for at least partly meeting these challenges for the limited realm of semi-fuzzy proportionality quantifiers. We briefly revisit the four issues raised in Section 2 to assess and summarize our results, but also in order to hint at directions for future research.

Problems with fully fuzzy quantification. As Glöckner's discussion of these problems in [20] makes clear, the well justified criticism of Zadeh's models, that motivates the bulk of more recent literature on fuzzy quantifiers, did not result in generally accepted alternatives. Glöckner himself suggests to focus on semi-fuzzy quantifiers and to address the problem of lifting to fully fuzzy quantification within an axiomatic framework. We remark that, in principle, our game based models of semi-fuzzy quantifiers can be lifted to fully fuzzy quantification in a very straightforward manner. Indeed nothing at all has to be changed in the formulation of quantifier rules introduced in Sections 5, 6, and 7 if the restriction to crisp scopes is dropped. In fact, the syntax clearly even gets simplified by dropping the distinction between classical and fuzzy (sub)formulas. Since perfect information about the overall risk (payoff) associated with the final game states is maintained, one may extract particular truth functions also in this 'fully fuzzy' setting. However the linguistic adequateness of the resulting fuzzy quantifiers is questionable. In fact, we remain skeptical about the appropriateness of any attempt to capture the meaning of natural language quantifiers like *few*, *many*, and *about a half* applied to vague scopes by solely referring to truth functions. We rather think that it is more appropriate to treat the inherent vagueness of the quantifier expression separately from the possible vagueness of scope and range. To hint at a concrete direction for future related research, we suggest to study game based models, similar to the one for extending Łukasiewicz logic with a 'supertruth' modality in [12], where the evaluation of final game states refers to a context of possible precisifications endowed with a probability measure. Such models invite one to consider attack and defense moves for quantified statements that distinguish the choice of plausible precisifications of predicates from choices directly relating to the quantifier itself. In general, the resulting models will not be truth functional. However, truth functions appear as limiting cases. Moreover, particular truth functions can be extracted if the space of possible precisifications is completely homogeneous and does not reflect semantic dependencies among the relevant predicates. (We refer to [16] for an investigation of models that relate contexts of precisification to fuzzy logic.)

Coherent interpretation of intermediate truth values. Giles has motivated his game for Łukasiewicz logic (in particular in [19]) by the need to provide 'tangible meaning' to intermediate truth values and to corresponding truth functions as a precondition for judging the adequateness of fuzzy models in given contexts of applications. We have listed a number of alternative semantic frameworks in Section 2. We do not claim that game semantics is the only viable approach towards an interpretation of truth values and truth functions that covers at least certain kinds of fuzzy quantification. However, we point out that it remains open how, e.g., voting semantics [27], acceptability semantics [36], re-randomising semantics [26, 21], and approximation semantics [4, 37], fare in this respect. In any case we have demonstrated that Giles-style games amount to a semantic framework that leads rather straightforwardly to extensions of Łukasiewicz logic with different types of semi-fuzzy proportionality quantifiers.

An embarrassment of riches. By this term we mean to allude to the fact that the space of *prima facie* plausible candidates of truth functions modeling some vague natural language quantifier is vast and unstructured. Clearly, additional semantic principles are needed to support a systematic search for suitable truth functional models. So far, it remained unclear whether any of the semantic frameworks mentioned in last paragraph justifies the choice of particular truth functions for quantifiers other than $\forall$ and $\exists$. It is certainly unreasonable to hope that a single truth function for a vague quantifier like *about half* will pop out as clearly optimal candidate from any sufficiently general semantic approach. However, we contend that the rule guided reduction of a quantified assertion to assertions of random instances constitutes a simple semantic principle that suggests quite specific candidates for modeling, e.g., *about half.* In Sections 6 and 7 we have defined two families of quantifier game rules based on the mentioned principle and extracted corresponding truth functions from optimal strategies in the game. The difference between 'blind choice' and 'deliberate choice' of random instances illustrates a versatility of randomized game semantics that remains to be explored more fully in future work. In principle, all kinds of combinations of blind and deliberate choices as well as of instances resulting from witnesses picked by the players lead to new quantifier rules. We suggest to further investigate which families of truth functions receive 'tangible meaning' in this manner and whether this leads to models that plausibly match semantic intuitions about further natural language quantifiers.

Compatibility with standard deductive fuzzy logics. As pointed out in Section 2, research on fuzzy quantifiers, as initiated by Zadeh's [44], has largely been pursued independently from the t-norm based approach to deductive fuzzy logics developed by Hájek and his colleagues (see, e.g., [21, 7]). Conversely, results about the three fundamental t-norm based fuzzy logics—Łukasiewicz logic, Gödel logic, and Product logic—and about related logics, like Hajek's BL and MTL[12] so far focus either on the propositional level or on first order logics with universal and existential quantifiers only. By directly extending Giles's game, we have obtained characterizations of extensions of Łukasiewicz logic that include certain families of semi-fuzzy quantifiers. The question arises whether the game semantic approach can be adapted to cover a wider range of t-norm based logics. At the propositional level a variant of Giles's game that covers all three mentioned fundamental t-norm based fuzzy logics has been introduced in [5] and is studied in more detail in [11]. Another type of generalization of Giles's game, that allows one to characterize propositional logics close to Łukasiewicz logic, including cancellative hoop logic CHL [9], is investigated in [15]. We conjecture that all these game variants can be straightforwardly augmented by quantifier rules like those presented in this paper.

To summarize, the aim of this paper has been to explore a simple, but non-trivial extension of Giles's game semantics with respect to its capability to model semi-fuzzy

[12] We refer to the Handbook of Mathematical Fuzzy Logic [7] for information explaining the prominent roles of the five mentioned logics.

proportionality quantifiers. Refining these models in various directions, in particular in those suggested by potential applications, determining the full extent of randomized game semantics, and studying the properties of resulting quantifiers in more detail are obvious tasks for future research.

Funding.

This work has been supported by the ESF/Austrian Science Foundation (FWF), Grant [I143-G15] (LogICCC/LoMoReVI).

References

[1] S. Aguzzoli, B. Gerla, and V. Marra. Algebras of fuzzy sets in logics based on continuous triangular norms. In C. Sossai and G. Chemello, editors, *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*, volume 5590 of *Lecture Notes in Computer Science*, pages 875–886. Springer, 2009.

[2] C. Barker. The dynamics of vagueness. *Linguistics and Philosophy*, 25(1):1–36, 2002.

[3] J. Barwise and R. Cooper. Generalized quantifiers and natural language. *Linguistics and philosophy*, 4(2):159–219, 1981.

[4] A.D.C. Bennett, J.B. Paris, and A. Vencovska. A new criterion for comparing fuzzy logics for uncertain reasoning. *Journal of Logic, Language and Information*, 9(1):31–63, 2000.

[5] A. Ciabattoni, C.G. Fermüller, and G. Metcalfe. Uniform rules and dialogue games for fuzzy logics. In *Logic for Programming, Artificial Intelligence, and Reasoning*, volume 3452, pages 496–510. Springer, 2005.

[6] P. Cintula, C.G. Fermüller, L. Godo, and P. Hájek, editors. *Understanding Vagueness - Logical, Philosophical and Linguistic Perspectives.* College Publications, 2011.

[7] P. Cintula, P. Hájek, and C. Noguera, editors. *Handbook of Mathematical Fuzzy Logic.* College Publications, 2011.

[8] P. Cintula and O. Majer. Towards evaluation games for fuzzy logics. In O. Majer, A.-V. Pietarinen, and T. Tulenheimo, editors, *Games: Unifying Logic, Language, and Philosophy*, pages 117–138. Springer, 2009.

[9] F. Esteva, L. Godo, P. Hájek, and F. Montagna. Hoops and fuzzy logic. *Journal of Logic and Computation*, 13(4):532–555, 2003.

[10] F. Esteva, L. Godo, P. Hájek, and M. Navara. Residuated fuzzy logics with an involutive negation. *Archive for Mathematical Logic*, 39:103–124, 2000.

[11] C.G. Fermüller. Revisiting Giles's game. In O. Majer, A.-V. Pietarinen, and T. Tulenheimo, editors, *Games: Unifying Logic, Language, and Philosophy*, Logic, Epistemology, and the Unity of Science, pages 209–227. Springer, 2009.

[12] C.G. Fermüller and R. Kosik. Combining supervaluation and degree based reasoning under vagueness. In M. Hermann and A. Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning*, volume 4246 of *Lecture Notes in Computer Science*, pages 212–226. Springer, 2006.

[13] C.G. Fermüller and G. Metcalfe. Giles's game and the proof theory of Łukasiewicz logic. *Studia Logica*, 92(1):27–61, 2009.

[14] C.G. Fermüller and C. Roschger. Bridges between contextual linguistic models of vagueness and t-norm based fuzzy logic. In T. Kroupa and J. Vejnarova, editors, *Proceedings of the eight Workshop on Uncertainty Processing*, pages 69–79, 2009.

[15] C.G. Fermüller and C. Roschger. From games to truth functions: A generalization of Giles's game. *Studia Logica*, 2013. to appear.

[16] C.G. Fermüller and C. Roschger. Bridges between contextual linguistic models of vagueness and t-norm based fuzzy logic, in preparation.

[17] R. Giles. A non-classical logic for physics. *Studia Logica*, 33(4):397–415, 1974.

[18] R. Giles. A non-classical logic for physics. In R. Wojcicki and G. Malinkowski, editors, *Selected Papers on Łukasiewicz Sentential Calculi*, pages 13–51. Polish Academy of Sciences, 1977.

[19] R. Giles. Semantics for fuzzy reasoning. *International Journal of Man-Machine Studies*, 17(4):401–415, 1982.

[20] I. Glöckner. *Fuzzy quantifiers: A computational theory*, volume 193 of *Studies in Fuzziness and Soft Computing*. Springer Verlag, 2006.

[21] P. Hájek. *Metamathematics of Fuzzy Logic*. Kluwer Academic Publishers, 2001.

[22] P. Hájek. What is mathematical fuzzy logic. *Fuzzy Sets and Systems*, 157(157):597–603, 2006.

[23] J.Y. Halpern. An analysis of first-order logics of probability. *Artificial Intelligence*, 46(3):311–350, 1990.

[24] J. Hintikka. *Logic, language-games and information: Kantian themes in the philosophy of logic*. Clarendon Press Oxford, 1973.

[25] J. Hintikka and G. Sandu. Game-theoretical semantics. In A. ter Meulen and J. van Benthem, editors, *Handbook of Logic and Language*, pages 361–410. Elsevier, 2010.

[26] E. Hisdal. Are grades of membership probabilities? *Fuzzy Sets and Systems*, 25(3):325–348, 1988.

[27] J. Lawry. A voting mechanism for fuzzy logic. *International Journal of Approximate Reasoning*, 19(3-4):315–333, 1998.

[28] Y. Liu and E.E. Kerre. An overview of fuzzy quantifiers.(I). interpretations. *Fuzzy Sets and Systems*, 95(1):1–21, 1998.

[29] P. Lorenzen. Logik und Agon. In *Atti Congr. Internaz. di Filosofia*, pages 187–194. Sansoni, 1960.

[30] A.L. Mann, G. Sandu, and M. Sevenster. *Independence-friendly logic: A game-theoretic approach*. Cambridge University Press, 2011.

[31] R. McNaughton. A theorem about infinite-valued sentential logic. *Journal of Symbolic Logic*, 16(1):1–13, 1951.

[32] R. Montague. *Formal Philosophy; Selected Papers of Richard Montague*. New Haven, Yale University Press, 1974.

[33] A. Mostowski. On a generalization of quantifiers. *Fundamenta mathematicae*, 44(1):12–36, 1957.

[34] H.T. Nguyêñ and E. A. Walker. *A first course in fuzzy logic*. CRC Press, 2005.

[35] V. Novák. A formal theory of intermediate quantifiers. *Fuzzy Sets and Systems*, 159(10):1229–1246, 2008.

[36] J.B. Paris. A semantics for fuzzy logic. *Soft Computing*, 1(3):143–147, 1997.

[37] J.B. Paris. Semantics for fuzzy logic supporting truth functionality. In V. Novák and I. Perfilieva, editors, *Discovering the world with fuzzy logic*, pages 82–104. Physica-Verlag, 2000.

[38] B. Partee. Many quantifiers. In *Proceedings of ESCOL*, volume 5, pages 383–402, 1988.

[39] S. Peters and D. Westerståhl. *Quantifiers in language and logic*. Oxford University Press, USA, 2006.

[40] P.L. Peterson. *Intermediate quantifiers*. Aldershot, UK: Ashgate, 2000.

[41] S. Shapiro. *Vagueness in context*. Oxford University Press, USA, 2006.

[42] J. van Benthem. Questions about quantifiers. *Journal of Symbolic Logic*, 49(2):443–466, 1984.

[43] R.R. Yager, S. Ovchinnikov, R.M. Tong, and H.T. Nguyen, editors. *Fuzzy sets and applications—Selected papers by L.A.Zadeh*. John Wiley, New York, 1987.

[44] L.A. Zadeh. A computational approach to fuzzy quantifiers in natural languages. *Computers & Mathematics with Applications*, 9(1):149–184, 1983.

Received January 8, 2013