On Semantic Games for Łukasiewicz Logic

Christian G. Fermüller* Ondrej Majer†

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Abstract

We explore different ways to generalize Hintikka's classic game theoretic semantic to a many-valued setting, where the unit interval is taken as the set of truth values. In this manner a plethora of characterizations of Łukasiewicz logic arise. Among the described semantic games is Giles's dialogue and betting game, presented in a manner that makes the relation to Hintikka's game more transparent. Moreover, we explain a so-called explicit evaluation game and a 'bargaining game' variant of it. We also describe a recently introduced backtracking game as well as a game with random choices for Łukasiewicz logic.

1 Introduction

Already in the 1960s Jaakko Hintikka [13, 14] introduced game theoretic semantic as an alternative characterization of the Tarskian notion of truth in a model. Two antagonistic players, where one is in the role of the verifier or proponent and the other one in the role of the falsifier or opponent of a given formula, stepwise reduce logically complex formulas until an atomic formula is reached, for that truth in the given model can be checked immediately. Roughly speaking, conjunction corresponds to a choice by the opponent, disjunction corresponds to a choice by the proponent, whereas negation corresponds to a switch of the players' roles; existential and universal quantification are analyzed via the choice of a witnessing domain element by the proponent or the opponent, respectively. Truth in a model in Tarski's sense turns out to be equivalent to the existence of a winning strategy for the initial proponent. At a first glance, this semantic game seems to be inherently classical. In particular there is no room for implication, other than defining it by $F \to G = \neg F \vee G$. Moreover, bivalence seems to be built into the game in an essential manner. However, from the very beginning Hintikka realized that the game triggers a generalization of classical logic by inviting us to consider what happens if the two players do not have perfect

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information about the choices made during a run of the game. These considerations lead to *Independence Friendly (IF-)logic*, as later worked out together with Sandu (see, e.g., [15, 18]).

Independently of Hintikka's game theoretic semantics, Robin Giles in the 1970s characterized a 'logic for reasoning about dispersive experiments', that coincides with infinitely valued *Łukasiewicz logic*, by another type of semantic game. Giles's game involves bets on the results of experiments that may show dispersion, i.e., repeated trials of the same experiments may show different results; but a fixed success probability is assumed for each experiment. At the end of the game the players pay a fixed amount of money to the other player for each their atomic claims, where the corresponding experiments fails. Regarding the rules for decomposing logically complex statements into simpler ones, Giles did not refer to Hintikka, but rather to the dialogue games suggested by Lorenzen [16, 17] as a foundation for constructive reasoning. While Giles initially proposed his game for logical reasoning within theories of physics, he later motivated the game as a semantic approach to fuzzy logic [11]. Nowadays, *Łukasiewicz logic* is indeed recognized as one of the most important, if not *the* most important example of a t-norm based logic over $[0,1]$ as set of truth values, i.e., a standard fuzzy logic in the sense of *mathematical fuzzy logic* [2].

In this paper we provide an overview of different types of semantic games for Łukasiewicz logic. We begin, in Section 2, with the observation that Hintikka's original game for classical logic in fact already characterizes the so-called weak fragment of Łukasiewicz logic by simply generalizing the players' payoffs from 0 or 1 (for 'win' or 'lose') to the unit interval $[0,1]$. In this fragment of Łukasiewicz logic we only have weak conjunction and weak disjunction, modeled by minimum and maximum, respectively, besides negation and the standard quantifiers. Providing a game based semantics for implication and for strong (t-norm based) conjunction and disjunction of full Łukasiewicz logic is a greater challenge. In Section 3 we present a so-called explicit evaluation game, or $\mathcal{E}$-game for short, where the players explicitly refer to a truth value associated with the current formula of the game. In Section 4 we present Giles's game ('$\mathcal{G}$-game') in a manner that supports the comparison with the other games of this paper. Since the $\mathcal{G}$-game deviates from Hintikka's game, but also from the $\mathcal{E}$-game, by considering more than one current formula at any given state, we ask whether the focus on a single formula (and a role distribution) can be restored in a game for full Łukasiewicz logic, where, unlike in the $\mathcal{E}$-game, we do not explicitly refer to truth values, but rather identify the payoff for the proponent of the initial formula with its truth value in the interpretation in question (like in the $\mathcal{G}$-game). A positive answer is provided in Section 5 by the so-called $\mathcal{B}$-game, where alternative game states are stored on a stack for backtracking. An alternative positive answer, where backtracking is avoided, is obtained in Section 6 by allowing for random choices in some rules of the game. This so-called $\mathcal{R}$-game is in fact rather close in its overall format to Hintikka's original game. The short Section 7 introduces a particular interpretation of the quantifier rules of semantic games, suggesting that the game can be seen as a kind of bargaining about the (partial) truth of the given formula. Finally, in the

Conclusion Section 8) we hint at an interesting relation between (propositional) IF-logic and the form of randomization in the $\mathcal{R}$-game.

While we will present the rules of the various games in some detail, we refer to the literature for the proofs the corresponding adequateness theorems.

2 Hintikka's game and (many) truth values

Hintikka's semantic game, called $\mathcal{H}$*-game* here, characterizes truth of a formula F in a model $\mathcal{J}$ for classical first order logic [13, 14, 15]. We will slightly simplify the original game by restricting attention to sentences (closed formulas) and by stipulating that there is a constant for each element c in the domain of $\mathcal{J}$. We will use the same name c for the constant.[1] We call the two players of the game *Myself* (or *I*) and *You*, respectively. The rules of the game do not refer directly to the players' identity, but rather to their current *roles* at any given state. With a nod to Lorenzen's related game based approach to logic [16], introduced already in the late 1950s, we will speak of the (current) *Proponent* **P** and *Opponent* **O**, respectively.[2] At every state of an $\mathcal{H}$-game the logical form of the *current formula* specifies who is to move at that state. The initial current formula is the one to be evaluated. Initially, *I* am in the role of **P** and *You* act as **O**. The succeeding states are determined according to the following rules:

($R^{\mathcal{H}}_{\wedge}$) If the current formula is $F \wedge G$ then **O** chooses whether the game continues with F or with G.

($R^{\mathcal{H}}_{\vee}$) If the current formula is $F \vee G$ then **P** chooses whether the game continues with F or with G.

($R^{\mathcal{H}}_{\neg}$) If the current formula is $\neg F$, the game continues with F, except that the roles of the players are switched: the player who is currently acting as **P**, acts as **O** at the the next state, and conversely for the current **O**.

($R^{\mathcal{H}}_{\forall}$) If the current formula is $\forall x F(x)$ then **O** chooses a domain element c and the game continues with $F(c)$.

($R^{\mathcal{H}}_{\exists}$) If the current formula is $\exists x F(x)$ then **P** chooses a domain element c and the game continues with $F(c)$.

Except for states with negated formulas, the roles of *Myself* and *You* remain unchanged. The game ends when an atomic formula A is hit. The player who is currently acting as **P** *wins* and the other player *loses* if A is true in the

[1] The game can straightforwardly be generalized to formulas with free variables and to languages, where there may not be a constant for every domain element, by explicitly augmenting formulas by assignments. However we find it more convenient to stick with sentences and to dispense with extra notation for assignments.

[2] Hintikka uses *Nature* and *Myself* as names for the players and *Verfier* and *Falisifer* for the two roles. To emphasize our interest in the connection to Giles's game (see Section 4) we use Giles's names for the players (*I*/*You*) and Lorenzen's corresponding role names (**P**/**O**) throughout the paper.

given model $\mathcal{J}$. The game starting with sentence F is called the *$\mathcal{H}$-game for F under $\mathcal{J}$*. Note that, like all other games described in this paper, it is a two-person constant-sum[3] extensive game of finite depth with perfect information. We may view each such game as a tree, where the branches correspond to the possible runs of the game. Each node of the tree corresponds to a game state and is labeled with the current formula of that state. A strategy for *Myself* may be identified with a subtree obtained by deleting all but one successor nodes (states) of every node where I can choose between different moves. If I win at all leaf nodes (final states), such a tree is called a winning strategy for *Myself*. Given these notions, one can straightforwardly show by backward induction that the $\mathcal{H}$-game characterizes classical logic in the following sense.

Theorem 1 (Hintikka)**.** *A sentence F is true in a (classical) interpretation $\mathcal{J}$ (in symbols: $\|F\|_{\mathcal{J}} = 1$) iff I have a winning strategy in the $\mathcal{H}$-game for F under $\mathcal{J}$.*

Regarding many-valued logics, our first important observation is that almost nothing has to be changed in Hintikka's game to obtain a characterization of so-called 'weak Łukasiewicz logic'. To make this precise, let us fix the following notions. (Full) Łukasiewicz logic Ł provides two forms of conjunction: *weak conjunction* ($\wedge$) and *strong conjunction* (&) and two forms of disjunction: *weak disjunction* ($\vee$) and *strong disjunction* ($\oplus$); moreover, we have negation ($\neg$), implication ($\rightarrow$), the constant "falsity" ($\bot$), and the standard quantifiers ($\forall$ and $\exists$). In weak Łukasiewicz logic $Ł^w$ only $\wedge$, $\vee$, $\neg$, $\forall$, and $\exists$ are considered.[4] The standard ('Tarskian') semantics of these connectives and quantifiers is given by:

$$
\begin{array}{ll}
\|F \wedge G\|_{\mathcal{J}} = \min(\|F\|_{\mathcal{J}}, \|G\|_{\mathcal{J}}) & \|F \,\&\, G\|_{\mathcal{J}} = \max(0, \|F\|_{\mathcal{J}} + \|G\|_{\mathcal{J}} - 1) \\
\|F \vee G\|_{\mathcal{J}} = \max(\|F\|_{\mathcal{J}}, \|G\|_{\mathcal{J}}) & \|F \oplus G\|_{\mathcal{J}} = \min(1, \|F\|_{\mathcal{J}} + \|G\|_{\mathcal{J}}) \\
\|\bot\|_{\mathcal{J}} = 0 & \|\neg F\|_{\mathcal{J}} = 1 - \|F\|_{\mathcal{J}} \\
\|F \rightarrow G\|_{\mathcal{J}} = \min(1, 1 - \|F\|_{\mathcal{J}} + \|G\|_{\mathcal{J}}) & \\
\|\forall x F(x)\|_{\mathcal{J}} = \inf_{c \in D}(\|F(c)\|_{\mathcal{J}}) & \|\exists x F(x)\|_{\mathcal{J}} = \sup_{c \in D}(\|F(c)\|_{\mathcal{J}})
\end{array}
$$

where D is the domain of $\mathcal{J}$, which we again identify with the set of constants. $\mathcal{J}$ now assigns a value $\|A\|_{\mathcal{J}} \in [0, 1]$ (and not just $\in \{0, 1\}$, as in classical logic) to each atomic formula A.

Note that the connectives of weak Łukasiewicz logic $Ł^w$ coincide with those considered in the $\mathcal{H}$-game. Therefore only the winning conditions have to be generalized in playing the $\mathcal{H}$-game with respect to the language of $Ł^w$. In fact, we may just identify the payoff for **P** with the truth value $\|A\|_{\mathcal{J}}$ and the payoff for **O** with $1 - \|A\|_{\mathcal{J}}$ if the game ends with the atomic formula A. We express this

[3] Hintikka's game, like the explicit evaluation game of Section 3, actually is a game where no payoff values are specified; rather it is sufficient to say that one player wins and the other player loses the game. This can be considered a special case of constant-sum by identifying winning with payoff 1 and losing with payoff 0.

[4] One can also find the name 'Kripke-Zadeh logic' for this fragment of Ł in the literature (see, e.g., [1]). The well-known textbook [19] even simply speaks of 'fuzzy logic'. We will focus on Łukasiewicz logic in this paper and thus prefer to use the name $Ł^w$.

by saying that the game is *under* $\mathcal{J}$. It is not hard to see that the backward induction argument that leads to Theorem 1 for classical logic, matches the above truth functions for $\wedge$, $\vee$, and $\neg$ (also) if we generalize payoffs from $\{0, 1\}$ to $[0, 1]$. However for the quantifiers a complication arises: there might be no domain element d such that $\|F(d)\|_{\mathcal{J}} = \inf_{c \in D}(\|F(c)\|_{\mathcal{J}})$ or e such that $\|F(e)\|_{\mathcal{J}} = \sup_{c \in D}(\|F(c)\|_{\mathcal{J}})$. A simple way to deal with this fact is to restrict attention to so-called witnessed models [12], where constants that witness all arising infima and suprema are assumed to exist. In other words: infima are minima and suprema are maxima in witnessed models. A more general solution refers to optimal payoffs up to some ϵ.

Definition 1. *Suppose that, for every $\epsilon > 0$, player $\boldsymbol{X}$ has a strategy that guarantees her a payoff of at least $r - \epsilon$, while her opponent has a strategy that ensures that $\boldsymbol{X}$'s payoff is at most $r + \epsilon$, then r is called the* value for **X** of the game.

This notion is closely related to the game theoretic concept of an epsilon-equilibrium, also known as near-Nash equilibrium. For the type of games considered in this paper (two-person constant-sum games of finite depth with perfect information) the existence of a unique value is guaranteed by general game theoretic results. The notion of a value allows us to concisely state the following generalization of Theorem 1 to a many valued setting. (A proof can be found in [8].)

Theorem 2. *An $Ł^w$-sentence F evaluates to $\|F\|_{\mathcal{J}} = r$ in an interpretation $\mathcal{J}$ iff the $\mathcal{H}$-game for F under $\mathcal{J}$ has value r for Myself.*

Characterizing not just $Ł^w$, but *full* Łukasiewicz logic Ł is a greater challenge. The following sections will present different ways to accomplish this task.

3 An explicit evaluation game

As we have seen in the previous section, the rules of $\mathcal{H}$-game do not change when we move from classical logic to weak Łukasiewicz logic. The difference between the two versions of the game can be interpreted as a difference of attitude regarding the truth of the initial claim. In the classical case my attitude is strict — either *I* fully succeed to defend truth or *I* completely fail. In the many-valued case my attitude to truth is rather loose, in the sense that *I* expect to be able to defend the truth of the claim only up to some (quantifiable) extent. Formally this is reflected by the fact that the first game is a win/lose game while the second one is a constant-sum game. The explicit evaluation game ($\mathcal{E}$-game) that we present in this section is closer to the classical case. *I* have to be strict about the initial claim again, but not about its full truth, but rather about its degree of truth. In particular *I* have to claim explicitly a minimal truth value of the claim *I* want to defend. This entails that the value of a formula is an explicit parameter of the initial state and consequently also of all the following states of the game. At the end of the game *I* either (fully) succeed to defend this value or *I* (completely) fail. So explicit evaluation games are win/lose games again. Loosely speaking, the difference between the $\mathcal{H}$-game and the $\mathcal{E}$-game in a many-valued setting can be paraphrased as follows: Instead of being partially satisfied about strict truth *I* have to be strict about the partial truth.

The formal set-up of an explicit evaluation game ($\mathcal{E}$-game) is similar as in the previous section. The game starts with my claim that the value $\|F\|_{\mathcal{J}}$ of a closed formula F of first-order Łukasiewicz logic Ł in a model $\mathcal{J}$ is at least r for some $r \in [0,1]$. To simplify the rules of the game we again assume that there is a constant for each element of the domain. *I* start in the role of Proponent **P**, while *You* are initially the Opponent **O**. At every state there is a unique rule to be applied, determined by the logical form of the current formula. While in the $\mathcal{H}$-game each rule refers to at most one action by one of the players, some $\mathcal{E}$-game rules consist of two actions — typically one of the players modifies the value of the formula and the other one chooses a subformula to continue with.

The rules for weak conjunction and disjunction remain the same as in $\mathcal{H}$-game. The only difference is, that the state of the game now also contains a reference to a value. The value however does not change in this kind of move.

($R^{\mathcal{E}}_{\wedge}$) If the current state is $(F \wedge G, r)$ then **O** chooses whether the game continues with (F, r) or with (G, r).

($R^{\mathcal{E}}_{\vee}$) If the current state is $(F \vee G, r)$ then **P** chooses whether the game continues with (F, r) or with (G, r).

The rule for strong disjunction consists of two actions. First (the current) **P** divides the value of the current formula between the disjuncts; then **O** chooses one of the disjuncts (with the corresponding value) for the next state of the game.

($R^{\mathcal{E}}_{\oplus}$) If the current state is $(F \oplus G, r)$, then **P** chooses r_F, r_G such that $r_F + r_G = r$ and **O** chooses whether the game continues with (F, r_F) or with (G, r_G).

Note that the rule $R^{\mathcal{E}}_{\vee}$ for weak disjunction can be seen as a restricted case of the rule $R^{\mathcal{E}}_{\oplus}$ for strong disjunction, where either $r_F = r$ and $r_G = 0$ or, conversely, $r_G = r$ and $r_F = 0$.

Negation corresponds to the role switch, as in $\mathcal{H}$-game. However switching formulas and the corresponding values is no longer enough, but now includes a change of the value as well. If **O** denies **P**'s claim that $\|\neg F\|_{\mathcal{J}} \geq r$ then she asserts that $\|\neg F\|_{\mathcal{J}} \leq r'$ for some r' strictly larger than r, which amounts to claiming $\|F\|_{\mathcal{J}} \geq 1 - r'$.

($R^{\mathcal{E}}_{\neg}$) If the current state is $(\neg F, r)$ then **O** chooses $r' > r$ and the game continues with $(F, 1 - r')$ with the roles of players switched.

The rule for the strong conjunction is dual to the one of strong disjunction. It again refers to two actions: modification of the value by **O** and a choice by **P**.

($R^{\mathcal{E}}_{\&}$) If the current state is $(F\&G, r)$ then **P** chooses r_F, r_G such that $r_F + r_G = r$ and **O** chooses whether the game continues with $(F, r + r_F)$ or with $(G, r + r_G)$.

The universal quantifier rule is analogous to the one for the $\mathcal{H}$-game. The state $(\forall x G(x), r)$ corresponds to **P**'s claim that $\inf\{\|G(c)\|_{\mathcal{J}} \mid c \in D\} \geq r$. **O** has to provide a counterexample, i.e., to find a d such that $\|G(d)\|_{\mathcal{J}} < r$. Clearly the choice of a counterexample is independent of the (non)existence of an witnessing element for the infimum.

($R^{\mathcal{E}}_{\forall}$) If the current state is $(\forall x F(x), r)$ then **O** chooses some $c \in D$ and the game continues with $(F(c), r)$.

The situation is different in the case of the existential quantifier. Now **P** has to provide a witness for the existential claim, i.e. for $\sup\{\|G(c)\|_{\mathcal{J}} \mid c \in D\} \geq r$. But as mentioned in Section 2, if the supremum is not a maximum, this poses a problem. It can happen, that **P**s claim is true, but that nevertheless there exists no witnessing element that would show this. The solution for the case of non-witnessed models is similar to the one from Section 2. We relax the winning condition for **P** and allow her to provide a witness for which the value of the formula might not be equal to r, but only arbitrarily close. To this aim we let **O** decrease the value of the formula (where, of course, it is in **O**'s interest to decrease it as little as possible) and only *then* require **P** to find a witness (for the decreased value). Note that this does not affect **O**'s winning condition. If in the state $(\exists x F(x), r)$ r is strictly greater than $\sup\{\|F(c)\|_{\mathcal{J}} \mid c \in D\}$ then **O** can always win by choosing an ϵ between the supremum and r. Formally the just discussed rule can simply be stated without explicit involvement of ϵ as follows.

($R^{\mathcal{E}}_{\exists}$) If the current state is $(\exists x F(x), r)$ then **O** chooses $r' < r$ and **P** chooses $c \in D$; the game continues with $(F(c), r')$.

Atomic formulas correspond to tests, like in the classical $\mathcal{H}$-game. If the current state is (F, r), where F is an atomic formula, then the game ends and (the current) **P** wins if $\|F\|_{\mathcal{J}} \geq r$, otherwise **O** wins.

The following adequateness theorem states that the game semantics given by the $\mathcal{E}$-game corresponds to the standard truth functional semantics for Łukasiewicz logic. Its proof can be found in [3].

Theorem 3. *An Ł-sentence F evaluates to $||F||_{\mathcal{J}} \geq r$ in interpretation $\mathcal{J}$ iff I have a winning strategy in the $\mathcal{E}$-game under $\mathcal{J}$ starting with (F, r).*

4 Giles's game

Already in the 1970's Giles [9, 10] presented a game based interpretation of Ł that in some aspects deviates more radically from Hintikka's game than the explicit evaluation game considered in Section 3. In fact Giles did not refer to

Hintikka, but rather to the logical dialogue game suggested by Lorenzen [16, 17] as a foundation for constructive reasoning. Initially Giles proposed his game as a model of logical reasoning within theories of physics; but later he motivated the game explicitly as an attempt to provide "tangible meaning" for fuzzy logic [11]. We briefly review the essential features of Giles's game, in a variant called $\mathcal{G}$-game, that facilitates comparison with the other semantic games mentioned in this paper.

Again the players are *Myself* (I) and *You*, and the roles are referred to as **P** and **O**. Unlike in the H- and in the $\mathcal{E}$-game, a game state now contains more that one current formula, in general. More precisely a state of a $\mathcal{G}$-game is given by

$$[F_1, \ldots, F_m \mid G_1, \ldots, G_n],$$

where $\{F_1, \ldots, F_m\}$ is the *multiset* of formulas currently asserted by *You*, called *your tenet*, and $\{G_1, \ldots, G_n\}$ is the multiset of formulas currently asserted by *Myself*, called *my tenet*. At any given state an occurrence of a non-atomic formula H is picked arbitrarily and distinguished as *current formula*.[5] If H is in my tenet then I am acting as **P** and *You* are acting as **O**. Otherwise, i.e. if H is in your tenet, *You* are **P** and I am **O**. States that only contain atomic formulas are called *final*. At non-final states the game proceeds according to the following rules:

($R^{\mathcal{G}}_{\wedge}$) If the current formula is $F \wedge G$ then the game continues in a state where the indicated occurrence of $F \wedge G$ in **P**'s tenet is replaced by either F or by G, according to **O**'s choice.

($R^{\mathcal{G}}_{\vee}$) If the current formula is $F \vee G$ then the game continues in a state where the indicated occurrence of $F \vee G$ in **P**'s tenet is replaced by either F or by G, according to **P**'s choice.

($R^{\mathcal{G}}_{\rightarrow}$) If the current formula is $F \rightarrow G$ then the indicated occurrence of $F \rightarrow G$ is removed from **P**'s tenet and **O** chooses whether to continue the game at the resulting state or whether to add F to **O**'s tenet and G to **P**'s tenet before continuing the game.

($R^{\mathcal{G}}_{\forall}$) If the current formula is $\forall x F(x)$ then **O** chooses an element c of the domain of $\mathcal{J}$ and the game continues in a state where the indicated occurrence of $\forall x F(x)$ in **P**'s tenet is replaced by $F(c)$.

($R^{\mathcal{G}}_{\exists}$) If the current formula is $\exists x F(x)$ then **P** chooses an element c of the domain of $\mathcal{J}$ and the game continues in a state where the indicated occurrence of $\exists x F(x)$ in **P**'s tenet is replaced by $F(c)$.

[5] It turns out that the powers of the players of a $\mathcal{G}$-game are not depended on the manner in which the current formula is picked at any state. Still, a more formal presentation of $\mathcal{G}$-games will employ the concepts of a regulation and of so-called internal states in formalizing state transitions. We refer to [7] for details.

No rule for negation is needed if $\neg F$ is defined as $F \to \bot$. Likewise, rules for strong conjunction & and $\oplus$ can either be dispensed with by treating these connectives as defined from the other connectives or by introducing corresponding rules. (See [5, 7] for presentations of a rule for strong conjunction.) If no non-atomic formula is left to pick as current formula, the game has reached a final state

$$[A_1, \ldots, A_m \mid B_1, \ldots, B_n],$$

where the A_i and B_i are atomic formulas. With respect to an interpretation $\mathcal{J}$ (i.e, an assignment of truth values $\in [0.1]$ to all atomic formulas) the payoff for *Myself* at this state is defined as

$$m - n + 1 + \sum_{1 \leq i \leq n} \|B_i\|_{\mathcal{J}} - \sum_{1 \leq i \leq m} \|A_i\|_{\mathcal{J}},$$

where empty sums are identified with 0. The $\mathcal{G}$-game is called *under* $\mathcal{J}$ if it has these payoff values.

Just like for the $\mathcal{H}$-game for $Ł^w$, we need to take into account that suprema and infima are in general not witnessed by domain elements. Note that Definition 1 (in Section 2) does not refer to any particular game. We may therefore apply the notion of the *value of a game* to $\mathcal{G}$-games as well. A $\mathcal{G}$-game where *my* tenet at the initial state consists of a single formula occurrence F, while *your* tenet is empty, is called a $\mathcal{G}$-game for F. This allows us to express the adequateness of $\mathcal{G}$-games for Łukasiewicz logic in direct analogy to Theorem 2.

Theorem 4 (Giles[6]). *An Ł-sentence F evaluates to $\|F\|_{\mathcal{J}} = r$ in an interpretation $\mathcal{J}$ iff the $\mathcal{G}$-game for F under $\mathcal{J}$ has value r for Myself.*

Readers familiar with the original presentation of the game in [9, 10] might be inclined to protest that we have skipped Giles's interesting story about betting money on the results of dispersive experiments associated with atomic assertions. Indeed, Giles proposes to assign an experiment E_A to each atomic formula A. While each trial of an experiment yields either "yes" or "no" as its result, successive trials of the same experiment may lead to different results. But for each experiment E_A there is a known probability $\langle A \rangle$ that the result of a trial of E_A is negative. Experiment $E_\bot$ always yields a negative result; therefore $\langle \bot \rangle = 1$. For each occurrence ('assertion') of an atomic formula in a player's final tenet, the corresponding experiment is performed and the player has to pay one unit of money (say 1€) to the other player if its result is negative. Therefore Giles calls $\langle A \rangle$ the *risk* associated with A. For the final state $[A_1, \ldots, A_m \mid B_1, \ldots, B_n]$ the expected total amount of money that *I* have to pay to *You* (my total risk) is readily calculated to be

$$\Big(\sum_{1 \leq i \leq m} \langle A_i \rangle - \sum_{1 \leq i \leq n} \langle B_i \rangle \Big) €.$$

[6] Giles [9, 10] in fact only sketched a proof for the language without strong conjunction. For a detailed proof of the propositional case, where the game includes a rule for strong conjunction, we refer to [7].

Note that the total risk at final states translates into the payoff specified above for $\mathcal{G}$-games via $\|A\|_{\mathcal{J}} = 1 - \langle A \rangle$. To sum up: Giles's interpretation of truth values as inverted risk values associated with bets on dispersive experiments is completely independent from the semantic game for the stepwise reduction of complex formulas to atomic sub-formulas. In principle, one can interpret the payoff values also for the $\mathcal{H}$-game as inverted risk values and speak of bets on dispersive experiments at final states also there. The only (technically inconsequential) difference to the original presentation is that one implicitly talks about *expected* payoff (inverted *expected* loss of money), rather than of certain payoff when the betting scenario is used to interpret truth values.

5 A backtracking game for Łukasiewicz logic

As we have seen above, characterizing full Łukasiewicz logic Ł (in contrast to weak Łukasiewicz logic $Ł^w$) by a semantic game seems to call for some non-trivial deviation from Hintikka's original game theoretic semantics. In the $\mathcal{E}$-game of Cintula and Majer (see Section 3) the players explicitly refer to some truth value at every state of the game, whereas in Giles's game (see Section 4) one has to take into account a whole multiset of formulas at a given state in general. In this and in the next section we indicate the possibility to define semantic games for Ł that focus on a single formula and a given role assignment at each state without explicitly referring to truth values. Moreover, unlike in the explicit evaluation game of 3, but like in the (many-valued) $\mathcal{H}$-game, there will be a direct match between payoffs and truth values.

We propose to 'sequentialize' the concurrent options for further moves that are considered at a given state of Giles's game. More precisely, we introduce a *(game) stack* on which information about an alternative state is stored at certain moves. Initially the stack is empty. Upon reaching an atomic formula the game only ends if the stack is empty. Otherwise, the game *backtracks* to the state indicated by the uppermost stack element. In addition to the stack, we need to keep track of the *preliminary payoff* $\sigma_{\mathbf{P}}$ for **P**. The preliminary payoff $\sigma_{\mathbf{O}}$ for **O** is $-\sigma_{\mathbf{P}}$ throughout the game. Initially, $\sigma_{\mathbf{P}} = 1$. When the game ends the preliminary payoff becomes final. We will call the resulting variant of Giles's game *backtracking game for* Ł or $\mathcal{B}$*-game* for short.

The rules $R^{\mathcal{H}}_{\wedge}$, $R^{\mathcal{H}}_{\vee}$, $R^{\mathcal{H}}_{\forall}$, and $R^{\mathcal{H}}_{\exists}$ (see Section 2) are taken over from the $\mathcal{H}$-game into the $\mathcal{B}$-game without change: no reference to the game stack or to $\sigma_{\mathbf{P}}$ and $\sigma_{\mathbf{O}}$ is needed in these cases. The rules for strong conjunction and implication are as follows:

($R^{Ł}_{\&}$) If the current formula is $F \& G$ then **P** can choose either (1) to continue the game with F and to put G together with the current role assignment on the stack, or (2) to continue the game with $\bot$.

($R^{Ł}_{\to}$) If the current formula is $F \to G$ then **O** can choose either (1) to continue the game with G and to put F on the stack together with the inverted role assignment, or (2) to continue the game with the top element of the stack. If the stack is empty, the game ends.

$\neg F$ is treated as $F \to \bot$ as therefore does not need a specific rule.

When the current formula is an atom A then $\|A\|_{\mathcal{J}} - 1$ is added to $\sigma_{\mathbf{P}}$ and the same value is subtracted from $\sigma_{\mathbf{O}}$.

We speak of the *$\mathcal{B}$-game for F under $\mathcal{J}$* if the game starts with the current formula F where initially *I* am **P** and *You* are **O**.

Theorem 5. *An Ł-sentence F evaluates to r in an interpretation $\mathcal{J}$, i.e., $\|F\|_{\mathcal{J}} = r$, iff the value of the $\mathcal{B}$-game for F under $\mathcal{J}$ for Myself is r.*

We refer to [4] for a proof of Theorem 5. An alternative proof is obtained by transforming Giles's game into a $\mathcal{B}$-game and vice versa.

6 A semantic game with randomized choice

Although the $\mathcal{B}$-game described in Section 5 focuses on a single formula at any given state, the backtracking mechanism, that entails the reference to a stack and to preliminary payoffs, renders the $\mathcal{B}$-game rather different from the $\mathcal{H}$-game. A game for full Łukasiewicz logic Ł that is much closer in spirit and structure to Hintikka's original game can be obtained by introducing a simple form of randomization. So far we have only considered rules where either *Myself* or *You* (as **P** or **O**) choose the sub-formula of the current formula with which the game continues. In game theory one often introduces *Nature* as a special kind of additional player, who does not care what the next state looks like when it is her turn to move and therefore is modeled by a uniformly random choice between all moves available to *Nature* at that state. As we will see below, introducing *Nature* leads to increased expressive power of semantic games. In fact, to keep the presentation of the games simple, we prefer to leave the role of *Nature* only implicit and just speak of random choices, without attributing them officially to a third player. The most basic rule of the indicated type refers to a new propositional connective π and can be formulated as follows.

($R_{\pi}^{\mathcal{R}}$) If the current formula is $F\pi G$ then a uniformly random choice determines whether the game continues with F or with G.

As shown in [6], adding rule $R_{\pi}^{\mathcal{R}}$ to the $\mathcal{H}$-game with payoffs in $[0, 1]$ yields a characterization of a logic that properly extends Ł^w, but is incomparable with Ł: the connective π is not definable from the connectives of Ł, nor can strong conjunction, strong disjunction, or Ł-implication be defined from π, $\wedge$, $\vee$, and $\neg$.

If we adapt Definition 1 (Section 2) by replacing 'payoff' with 'expected payoff', then the following truth function can be straightforwardly extracted from the game.

$$\|F\pi G\|_{\mathcal{J}} = (\|F\|_{\mathcal{J}} + \|G\|_{\mathcal{J}})/2.$$

Note that this is related to the truth function for strong disjunction $\oplus$ in Ł:

$$\|F \oplus G\|_{\mathcal{J}} = \min(1, \|F\|_{\mathcal{J}} + \|G\|_{\mathcal{J}}).$$

This observation suggests the following rule for strong disjunction.

($R^{\mathcal{R}}_{\oplus}$) If the current formula is $G \oplus F$ then a random choice determines whether to continue the game with F or with G. Moreover the payoff for **P** is doubled, but capped to 1.

For the resulting *$\mathcal{R}$-game* game, we retain the principle that the payoff for **O** is always inverse to that for **P**. In other words, like all other games considered in this paper, the $\mathcal{R}$-game is constant sum.

Since all other connectives can be defined from $\oplus$ and $\neg$ in Ł, it remains to specify a rule for negation. It turns out that Hintikka's original rule ($R^{\mathcal{H}}_{\neg}$), that simply consists in a role switch, suffices for this purpose.

Alternatively, one may synthesize explicit rules for the other connectives by combining role switch with the rule for $\oplus$. For example, the following rule for implication arises in this manner.

($R^{\mathcal{R}}_{\to}$) If the current formula is $F \to G$ then a random choice determines whether to continue the game with G or with F, where in the latter case the roles of **P** and **O** are switched. Moreover the payoff for **P** is doubled, but capped to 1.

Independently of whether the rules for connectives other than $\oplus$ and $\neg$ are made explicit or not, we speak of the *$\mathcal{R}$-game for F under $\mathcal{J}$* if the game starts with the current formula F where initially *I* am **P** and *You* are **O**.

Theorem 6. *An Ł-sentence F evaluates to r in an interpretation $\mathcal{J}$, i.e., $\|F\|_{\mathcal{J}} = r$, iff the value of the $\mathcal{R}$-game for F under $\mathcal{J}$ for Myself is r.*

A proof of Theorem 6 is implicit in the somewhat more general investigation of games with randomized choices presented in [4].

7 Semantic games as bargaining games

We finally briefly indicate an interpretation of the quantifier rules of semantic games that illustrates that game based evaluation can be interpreted as 'negotiating' the value of the given formula between the players. The aim of the Proponent **P** is to push the value of the formula up, while the aim of the Opponent **O** is to push it down.

Assume that we start with a formula in a prenex form. (In Łukasiewicz logic every formula can be transformed into an equivalent one that is in prenex form.) Like in the $\mathcal{H}$-game (as well as in the $\mathcal{G}$-, $\mathcal{B}$-, and $\mathcal{R}$-game) the quantifier moves consist of choices of witnesses in which players try to maximize (minimize) the value of the formula. In this sense we can see the 'quantifier part' of the game as a kind of negotiation or bargaining. When all variables have been replaced by constants, we stipulate that the value of the remaining (quantifier-free) formula is calculated in some manner and payoffs are distributed in the same way as in the $\mathcal{H}$-game for weak Łukasiewicz logic. Following [3], we call this version of a semantic game a *bargaining game*.[7] More formally, the bargaining game starts with a prenex Ł-formula F and refers to a given interpretation $\mathcal{J}$ with domain D. The only rules are the two quantifier rules of the $\mathcal{H}$-game.

($R_{\forall}^{\mathcal{H}}$) If the current formula is $\forall x F(x)$ then **O** chooses $c \in D$ and the game continues with $F(c)$.

($R_{\exists}^{\mathcal{H}}$) If the current formula is $\exists x F(x)$ then **P** chooses $c \in D$ and the game continues as $F(c)$.

If the current formula is a quantifier free formula F then the game ends with payoff $\|F\|_{\mathcal{J}}$ for **P** and payoff $1 - \|F\|_{\mathcal{J}}$ for **O**.

We remark that instead of taking the value under $\mathcal{J}$ of the quantifier free part of the formula as given, we may want to continue the game as in any of the other games for Łukasiewicz logic presented in the sections above.

Theorem 7. *A* Ł*-sentence F evaluates to $\|F\|_{\mathcal{J}} = r$ in interpretation $\mathcal{J}$ iff the bargaining game for F under $\mathcal{J}$ has value r for Myself.*

8 Conclusion

The results collected in this paper show that Hintikka's original semantic game for classical logic can be generalized in various ways to Łukasiewicz logic. In some sense, the very concept of game theoretic semantics invites the consideration of a many-valued setting, since game theory provides the tools for analyzing interactions, where at the final states certain payoffs are distributed to the participating players. Instead of just speaking of winning or losing a game, we may pay attention to particular payoff values. For semantic games, i.e., for games that proceed by reducing logically complex formulas to atomic ones, this suggests the identification of possible payoff values with truth values.

One may argue that the full power of game theoretic semantics only arises when the possibility of incomplete information about previous moves in a run of a game is taken into account. Indeed, the resulting IF-logic of Hintikka and Sandu [15, 18] is much more expressive than classical logic and shows features that cannot easily be captured by Tarskian semantics. A connection between many-valued logics and IF-logic, restricted to finite models, has recently been established by so-called equilibrium semantics (see, e.g., [18, 21]). In this approach weak Łukasiewicz logic arises by considering mixed strategies that induce intermediate expected payoff values in $[0, 1]$, even if each atomic formula is evaluated to either 0 or 1. Also the random choices in our $\mathcal{R}$-game can be seen as importing incomplete information into the game. The connection with (propositional) IF-logic can be made precise by the observation that the truth function

[7] The term 'bargaining game' has a different meaning in game theory (see, e.g., [20]). We do not want to allude to those types of (not logic-related) games here.

of $F\pi G$ coincides with that of $(F \vee_{/\{\wedge\}} G) \wedge (G \vee_{/\{\wedge\}} F)$ according to equilibrium semantics. (Here $\vee_{/\{\wedge\}}$ indicates that **P**, when choosing the right or left disjunct, does not know which conjunct has been chosen by **O**). We suggest that the relation between equilibrium semantics and semantic games with random choices should be explored more systematical in future work.

A further connection with IF-logic and topic for future research is suggested even more directly by the games considered in the paper. Which generalization(s) of Łukasiewicz logics arise(s) if we relax the assumption of complete information in those games? (See [3] for first hints in that direction.)

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