Equilibrium Semantics for IF Logic and Many-Valued Connectives

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Abstract. We connect two different forms of game based semantics: Hintikka's game for Independence Friendly logic (IF logic) and Giles's game for Łukasiewicz logic. An interpretation of truth values in $[0, 1]$ as equilibrium values in semantic games of imperfect information emerges for a logic that extends both, Łukasiewicz logic and IF logic. We prove that already on the propositional level all rational truth values can be obtained as equilibrium values.

Keywords: Game semantics · IF logic · Fuzzy logic · Łukasiwicz logic · Giles's game

1 Introduction

Already in the 1960s Jaakko Hintikka [12] introduced a game based characterization of Tarski's central semantic notion of 'truth in a model'. The game features moves by two antagonistic players, one in the role of the verifier or proponent of a formula, the other one in the role of the falsifier or opponent. The game proceeds according to the outermost connective or quantifier of the formula currently at stake: disjunction and existential quantification trigger a move by the proponent, while conjunction and universal quantification elicit a move by the opponent; negation corresponds to a role switch. In this manner the formula currently in focus is replaced by one of its immediate sub-formulas in every round. At the atomic level a given model determines who won the actual run of the game. Other connectives, in particular implication, could be defined from the mentioned ones in classical logic, of course, but it is an important observation for our current endeavor that such a reduction of a richer set of connectives to just (this form of) conjunction, disjunction, and negation is no longer available in general, once we move on to a non-classical setting.

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Hintikka's game-theoretic semantics deploys its full capacity when we consider imperfect information: the players may not be fully informed about previous moves during a run of the game. In particular *Independence Friendly logic* (IF logic) results from attaching "slash sets" to the quantifiers, containing those variables for which the current player is ignorant of corresponding assignments of domain elements that result from previous moves. For example $\forall x \exists y/\{x\} y = x$ corresponds to a game, where first the opponent chooses an arbitrary domain element for the variable x and then the proponent has to choose an element for y without knowing which element has been picked for x. This entails that (in contrast to the game for $\forall x \exists y\, y = x$) the opponent has no winning strategy if there are two or more domain elements. The fact that games for IF formulas are not determined in general leads to *equilibrium semantics* [18,21], which arises if one considers mixed Nash equilibria for corresponding strategic games. Indeed, if we identify losing a game with payoff 0 and winning with payoff 1, we may associate a unique value $v \in [0, 1]$ to every IF formula φ[1] with respect to any given finite model, such that v is the expected payoff for the proponent of φ in the corresponding game when the players employ mixed strategies that are in equilibrium (i.e., neither player has an incentive to unilaterally change her strategy).

Motivated by the challenge to model reasoning in physics, Robin Giles developed another game based approach to logic in the 1970s [7,8]. Giles was seemingly unaware of Hintikka's game-theoretic semantics, but referred to Paul Lorenzen's attempts to justify intuitionistic logic in terms of an idealized dialogue between a proponent and an opponent of a given formula. Giles's game consists of two components: first, the players stepwise reduce logically complex formulas to their sub-formulas, similar as in Hintikka's game, but not bound by the restriction that at any state of the game only a single formula is asserted by the proponent and attacked by the opponent. Rather a whole multiset, called tenet, of formulas is asserted by each of the players at any given state. The second stage of the game commences when only atomic formulas are left in both players' tenets. For each atomic assertion a corresponding experiment is performed. If the experiment fails, then the asserting player has to pay 1€ (one Euro) to the other player. If for every given atomic formula the corresponding experiment either always fails or always succeeds then Giles's game leads to an alternative characterization of classical logic. However, Giles stipulates that any experiment may be dispersive: it may yield different results when repeated—only a specific failure probability (risk value) is known for each experiment. A player's payoff at the final state of Giles's game is identified with the *expected* amount of money that she has to pay minus the expected amount that she receives from her opponent. Giles proved that the payoffs enforceable by optimal strategies correspond to the (inverse of) truth

[1] In fact there are certain complications if one admits formulas corresponding to games where a player may not have access to her own previous moves. We will circumvent these problems by insisting on perfect recall. Moreover, we follow [18,21] in moving negations to the atomic level.

values resulting from evaluating the initial formula according to the truth functions for Łukasiewicz logic Ł.

At first sight, Hintikka's and Giles's games seem to serve different purposes and moreover are quite different in detail as well as in their overall structure. Nevertheless we propose a combination of the two games that corresponds to a rather expressive logic, which we shall call Ł(IF). The formulas of Ł(IF) are two-tiered: they can be thought of as formulas of Ł where the atomic formulas are replaced by (arbitrarily complex) IF formulas. Accordingly, the combined game proceeds in two stages: first, Giles's game is played until only IF formulas occur in the players' tenets and then an instance of Hintikka's game is employed as a dispersive experiment (in the sense of Giles) for each IF formula. The overall evaluation is like for Giles's game. In this setting intermediate truth values turn out to correspond to equilibrium values for IF formulas that in turn may be combined to yield truth values for formulas of an expressive many-valued logic. The adequateness of the combined game for Ł(IF) emerges as a corollary to the adequateness of Giles's and Hintikka's games for Ł and for IF logic, respectively. The achieved gain arises on a conceptual level in two different directions:

(1) Skeptics of many-valued logics rightfully challenge their defenders by asking for an explanation of intermediate truth values and of corresponding truth functions in terms of first principles about reasoning. Giles's game only provides a partial answer by replacing classical (bivalent) interpretations with assignments of risk values (probabilities) to atomic formulas. Our combined game can be understood as an explanation of risk values as equilibrium values, arising from evaluations with respect to classical interpretations under imperfect information.
(2) Equilibrium semantics for IF logic supports a many-valued interpretation of disjunction and conjunction as maximum and minimum, respectively. Somewhat indirectly also the truth function $1-x$ for negation is justified. However the more general format of Giles's game is needed to interpret the considerably richer set of connectives (including implication, strong conjunction, and strong disjunction) of Łukasiewicz logic. From this perspective IF logic provides only a limited way of modeling the effects of imperfect information. At least some of these limitations are lifted in Ł(IF). For example, simple schematic Ł(IF) formulas (but not IF formulas) express that instances of Hintikka's game are always constant-sum, but not determined in general.

In the light of item 1 it is important to note that indeed *all* rationals in $[0,1]$ can be obtained as equilibrium values [21]. (For the related framework of Dependence Logic a similar result is shown in [6].) The corresponding constructions involve first-order formulas and particular models. This triggers the question whether one can obtain all rational truth values already on the propositional level. We provide a positive answer by showing that for every rational $r \in [0,1]$ there is constant propositional IF formula φ_r with equilibrium value r. (Propositional IF formulas arise from classical propositional formulas if there is imperfect information about the choice of conjuncts and disjuncts in Hintikkas game. The formula is constant if it is built up from the atomic formulas $\bot$ and $\top$ only.)

In fact we will present two constructions: a simpler one for IF formulas with n-ary conjunction and disjunction for any $n \geq 2$ and a more involved one for ordinary binary connectives.

The rest of the paper is organized as follows. Section 2 reviews Hintikka's game for classical logic and moves on to explain equilibrium logic for (a particular type of) IF formulas. Section 3 is devoted to Giles's game for Łukasiewicz logic. The logic Ł(IF) and the corresponding combination the two types of semantic games is introduced in Sect. 4. The announced results regarding the realization of all rationals as equilibrium values are the topic of Sect. 5. We conclude in Sect. 6 with a short summary and some hints on directions for further research.

2 Hintikka's Game

Let us revisit Hintikka's game-theoretic semantics (cf. [12,13]). We will call the game that characterizes truth in a (classical) model the $\mathcal{H}$-*game*. There are two players, say *I* and *you*, who are either in the role of the *Proponent* **P** or the *Opponent* **O**[2]. Initially I am **P** and you are **O**. At each state of $\mathcal{H}$-game the player in role **P** seeks to defend the claim that a certain formula is true in a given model $\mathcal{I}$ under a given variable assignment ξ, while the player in role **O** aims at refuting this claim. We will use $D_{\mathcal{I}}$ to denote the domain of $\mathcal{I}$. Formulas are built up as usual from atomic formulas, including equalities, as well as $\top$ and $\bot$, using the propositional connectives $\wedge$, $\vee$, $\neg$, and the quantifiers $\forall$ and $\exists$. The game rules are symmetric in the sense that we only need to refer to the roles **P** and **O**, but not to the identity of the players. The formula φ together with the variable assignment ξ that is at stake at a given state is called the *current (augmented) formula*.[3] We will also say that **P** asserts the current formula $\varphi[\xi]$, while **O** attacks it.

($R^{\mathcal{H}}_{\wedge}$) If the current formula is $(\varphi \wedge \psi)[\xi]$, then **O** chooses whether the game continues with **P**'s assertion of $\varphi[\xi]$ or of $\psi[\xi]$.

($R^{\mathcal{H}}_{\vee}$) If the current formula is $(\varphi \vee \psi)[\xi]$, then **P** chooses whether the game continues with **P**'s assertion of $\varphi[\xi]$ or of $\psi[\xi]$.

($R^{\mathcal{H}}_{\neg}$) If the current formula is $\neg\varphi[\xi]$, then game continues with **P**'s assertion of $\varphi[\xi]$, except that the roles of the players are switched (i.e., **P** now is the player that attacked $\neg\varphi[\xi]$).

($R^{\mathcal{H}}_{\forall}$) If the current formula is $(\forall x \varphi)[\xi]$ then **O** chooses a $c \in D_{\mathcal{I}}$ and the game continues with **P**'s assertion of $\varphi[\xi[c/x]]$[4].

[2] Hintikka uses *Myself* and *Nature* as names for the players and *Verfier* and *Falisifer* for the two roles.

[3] It is more customary to attach the variable assignment to the interpretation instead of to the formula that is to evaluated. For the $\mathcal{H}$-game this does not make any difference. However we will later introduce games, where several formulas are to be evaluated over the same interpretation, but each with respect to a (possibly) different variable assignment.

[4] $\xi[c/x]$ denotes the variable assignment that is like ξ, except for assigning c to x.

($R^{\mathcal{H}}_{\exists}$) If the current formula is $\exists x\varphi[\xi]$ then **P** chooses a $c \in D_{\mathcal{I}}$ and the game continues with **P**'s assertion of $\varphi[\xi[c/x]]$.

($R^{\mathcal{H}}_{at}$) If the current formula is an atomic formula $A[\xi]$ then the game ends. **P** wins if A is true in $\mathcal{I}$ under assignment ξ, otherwise **O** wins.

We speak of the *$\mathcal{H}$-game for $\varphi[\xi]$ with respect to $\mathcal{I}$*, if the game starts with the augmented formula $\varphi[\xi]$. The adequateness of this game for classical logic is expressed as follows.

Theorem 1 (Hintikka). *I have a winning strategy in the $\mathcal{H}$-game for $\varphi[\xi]$ with respect to $\mathcal{I}$ iff φ is true in $\mathcal{I}$ under assignment ξ.*

Above, we have tacitly assumed that the players of the $\mathcal{H}$-game have perfect, complete and common knowledge. This means that they share knowledge not only about the rules, but also about all previous moves at each state of an instance of the game. A whole new branch of logic, called *I*ndependence *F*riendly logic (*IF* logic) arises by investigating the consequences of imperfect knowledge in the $\mathcal{H}$-game. Following [18], formulas of IF-logic are defined as follows.

Definition 1. *We fix a language with an infinite supply of constants and predicate symbols. Terms of the language are either constants or variables.*

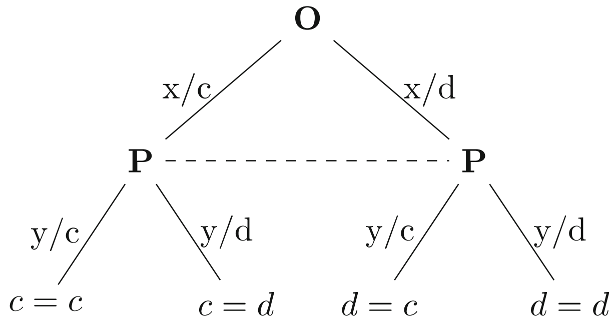
- $\top$ *and* $\bot$ *are IF formulas.*
- *If s and t are terms, then $s = t$ and $\neg(s = t)$, henceforth written as $s \neq t$, are IF formulas.*
- *If P is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms, then $P(t_1, \ldots, t_n)$ and $\neg P(t_1, \ldots, t_n)$ are IF formulas.*
- *If φ and ψ are IF formulas, then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are IF formulas.*
- *If φ is an IF formula, x a variable, and W a finite set of variables not containing x, then $(\exists x/W)\varphi$ and $(\forall x/W)\varphi$ are IF formulas, where φ is called the* scope *of the exhibited quantifier occurrence and W is called a* slash set. *We abbreviate $(\exists x/\emptyset)\varphi$ by $\exists x\varphi$ and $(\forall x/\emptyset)\varphi$ by $\forall x\varphi$.*

The intended (game-theoretic) semantics of IF formulas is specified with respect to a version of the $\mathcal{H}$-game, where the players have to choose the witness elements for bound variables without knowing the choices that may have been made for the variables in the corresponding slash sets at earlier stages of the game (see [18] for details). Moreover, we modify rule $R^{\mathcal{H}}_{at}$ and let the game end when the current formula is a *literal*, i.e. either an atomic formula or a negation of an atomic formula, augmented by a variable assignment. Consequently the negation rule ($R^{\mathcal{H}}_{\neg}$) is no longer needed and hence no role change takes place during the run of an $\mathcal{H}$-game for an IF formula. Therefore we may now identify the players with their initial roles: I am **P** and you are **O** throughout every run of the game. **P** wins and **O** loses the game for an IF formula φ with respect to a given interpretation $\mathcal{I}$ if the literal with which the game ends is true in $\mathcal{I}$; otherwise **P** loses and **O** wins. We identify winning with payoff 1 and losing with payoff 0. The effect of imperfect information is rather dramatic: in contrast to the $\mathcal{H}$-game for classical formulas, it may be the case that none of the players has a winning strategy in an $\mathcal{H}$-game for an IF formula, in general. In other words the game is not determined.

Remark 1. As just discussed, negation is pushed to the atomic level in IF formulas. Accordingly, we may define the dual $\varphi^\neg$ of a given IF formula φ by interchanging all occurrences of $\vee$ and $\wedge$ as well as $\forall$ and $\exists$, respectively, and replacing negated atomic formulas with unnegated ones and vice versa.

Example 1. Consider the IF formula $\forall x(\exists y/\{x\})\, x = y$. If the formula is evaluated with respect to an interpretation $\mathcal{I}$ with two domain elements, then it is called MP (for *Matching Pennies*). In the $\mathcal{H}$-game for MP, **O** starts by assigning one of the elements of $\mathcal{I}$ to the variable x. In the second stage of the game **P** has to choose a domain element for y, without knowing **O**'s choice for x.

For $D_\mathcal{I} = \{c, d\}$ the $\mathcal{H}$-game for MP is represented by the following tree:



The dashed line between the two **P**-nodes indicates that the two nodes are in the same information set. Consequently, **P** (just like **O**) has only two possible strategies. In contrast, we obtain the (perfect information) game for the classical formula $\forall x \exists y\, x = y$, in which **P** has four possible strategies, by simply deleting the dashed line.

Because of her imperfect knowledge, **P** has no winning strategy in the $\mathcal{H}$-game for MP. Clearly **O** does not have a winning strategy either. The dual $\mathsf{MP}^\neg = \forall x(\exists y/\{x\})\, x \neq y$ of MP is called IMP (for *Inverse Matching Pennies*); its $\mathcal{H}$-game is undetermined as well, of course. But also, e.g., the $\mathcal{H}$-game for $\forall x(\exists y/\{x\})x = y \vee \forall u(\exists v/\{u\})v \neq u)$ is undetermined whenever $D_\mathcal{I}$ consists of more than one element.

Throughout the paper, we will assume that each player has *perfect recall.* This means that a player is always aware of her own previous choices in any run of the game. Moreover, each bound variable should refer to a unique quantifier occurrence. This motivates the following definition.

Definition 2. *An IF formula φ is called* recall regular *if the following conditions are satisfied:*

- *For each variable x there is at most one occurrence of $(\mathsf{Q}x/W)$ in φ, where $\mathsf{Q} \in \{\forall, \exists\}$.*
- *If $(\forall x/W)$ occurs in φ then for each $v \in W$ this occurrence is in the scope of a quantifier occurrence of the form $(\exists v/V)$.*

– *If* $(\exists x/W)$ *occurs in* φ *then for each* $v \in W$ *this occurrence is in the scope of a quantifier occurrence of the form* $(\forall v/V)$.

Note that all formulas in Example 1 are recall regular. In the rest of the paper all IF formulas are assumed to be recall regular, even when not stated explicitly.

We will restrict attention to finite models. Consequently the $\mathcal{H}$-game is always finite. While the $\mathcal{H}$-game is presented as an extensive game, we may as well consider its *strategic form* and will simply speak of the *strategic* $\mathcal{H}$-game for a given formula and (finite) interpretation.

Example 2. Consider an interpretation $\mathcal{I}$, where $D_{\mathcal{I}} = \{c, d\}$. Then the strategic $\mathcal{H}$-game for $\mathsf{MP} = \forall x(\exists y/\{x\})\, x = y$ corresponding to the extensive $\mathcal{H}$-game depicted in Example 1 is given by the following payoff matrix:

$$\begin{array}{c c} & \begin{array}{cc} y/c & y/d \end{array} \\ \begin{array}{c} x/c \\ x/d \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

The matrix entries denote the payoff for **P**, where **O** chooses a row, while **P** chooses a column. Since the payoff for **O** is $1 - x$ whenever x is the payoff for **P**, we refrain from specifying the payoff for **O** explicitly from now on.

The payoff matrix for the strategic form of the (perfect information extensive form) $\mathcal{H}$-game for $\forall x \exists y\, x = y$ can be specified as follows:

$$\begin{array}{c c} & \begin{array}{cccc} y/cc & y/cd & y/dc & y/dd \end{array} \\ \begin{array}{c} x/c \\ x/d \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

where we have used $y/\xi\rho$, for $\xi, \rho \in \{c, d\}$, to denote the following strategy of **P**: "if **O** assigned c to x then assign ξ to y, otherwise assign ρ to y." Note that in contrast to the game for MP, **P** now has a strategy (namely y/cd) that guarantees her the payoff 1.

Mixed strategies for a player **X** in an extensive game come in two versions: (1) behavior strategies, where for each information set of **X**, a probability distribution over all possible moves is attached; (2) strategies that consist in a single probability distribution over all pure strategies that are available to **X** in the game. In general, only in the second case the strategies directly correspond to those of the strategic form of the game and consequently lead to unique a equilibrium value in finite, constant-sum games, like (instances of) the $\mathcal{H}$-game. However by Kuhn's Theorem [15] the two types of strategies are in one-one correspondence in games where all players have perfect recall. This justifies our focus on recall regular IF formulas.

Since the strategic $\mathcal{H}$-game is finite and constant-sum, von Neumann's Minimax Theorem can be applied to obtain the following result. (Cf. [18,21], where the term *equilibrium semantics* is introduced in this context.)

Theorem 2. *For every finite interpretation $\mathcal{I}$, every IF formula φ, and every corresponding variable assignment there is a unique value $v \in [0,1]$ such that v is the expected payoff for **P** and $1-v$ is the expected payoff for **O** under the (unique) mixed Nash equilibrium of the strategic $\mathcal{H}$-game for $\varphi[\xi]$ and $\mathcal{I}$.*

We will call the value v mentioned in Theorem 2 the *equilibrium value* of φ with respect to $\mathcal{I}$ and ξ and use $v_{\mathcal{I}}^{eq}(\varphi[\xi])$ to denote it. If φ is a closed formula then the reference to the (empty) assignment ξ is dropped.

Example 3. Let n be the cardinality of the domain of the interpretation $\mathcal{I}$. In the corresponding strategic $\mathcal{H}$-game for $\forall x(\exists y/\{x\})\, x = y$ (see Example 2) the only Nash equilibrium arises if **P** and **O** both randomize uniformly over their n strategies, which consist in picking an element of the domain of $\mathcal{I}$. The same holds for the strategic $\mathcal{H}$-game for the dual formula $\forall x(\exists y/\{x\})\, x \neq y$. Consequently $v_{\mathcal{I}}^{eq}(\forall x(\exists y/\{x\})\, x = y) = 1/n$ and $v_{\mathcal{I}}^{eq}(\forall x(\exists y/\{x\})\, x \neq y) = (n-1)/n$.

As shown in [18,21], equilibrium semantics provides a link to some standard truth functions for many-valued logics in the following sense.

Theorem 3. *Let $\mathcal{I}$ be a finite interpretation, φ and ψ two IF formulas, and ξ a variable assignment. Moreover remember that $\xi[c/x]$ denotes the variable assignment that is like ξ, except for assigning the element c to the variable x. Each of the following statements holds:*

- $v_{\mathcal{I}}^{eq}((\varphi \wedge \psi)[\xi]) = \min\{v_{\mathcal{I}}^{eq}(\varphi[\xi]), v_{\mathcal{I}}^{eq}(\psi[\xi])\}$,
- $v_{\mathcal{I}}^{eq}((\varphi \vee \psi)[\xi]) = \max\{v_{\mathcal{I}}^{eq}(\varphi[\xi]), v_{\mathcal{I}}^{eq}(\psi[\xi])\}$,
- $v_{\mathcal{I}}^{eq}(\forall x F) = \inf\{v_{\mathcal{I}}^{eq}(\varphi[\xi[c/x]]) \mid c \in D_{\mathcal{I}}\}$,
- $v_{\mathcal{I}}^{eq}(\exists x F) = \sup\{v_{\mathcal{I}}^{eq}(\varphi[\xi[c/x]]) \mid c \in D_{\mathcal{I}}\}$.

Theorem 3 can be read as a justification of minimum, maximum, infimum, and supremum as truth functions for conjunction, disjunction, existential and universal quantification in a many-valued logic, where the set of truth values is identified with the unit interval $[0,1]$. While negation only occurs in front of atomic formulas for IF formulas, it is clear that also $\lambda x(1-x)$ as truth function for negation fits the picture provided by equilibrium semantics.

3 Giles's Game for Łukasiewicz logic

In the last section we have seen that equilibrium semantics relates IF logic to a propositional many-valued logic, where an assignment $\mathcal{M}$ of truth values in the real unit interval $[0,1]$ to atomic formulas—in the following just called *many-valued interpretation*—is extended to logically complex formulas as follows.

$$\begin{aligned}
&v_{\mathcal{M}}(\varphi \wedge \psi) = \min(v_{\mathcal{M}}(\varphi), v_{\mathcal{M}}(\psi)),\\
&v_{\mathcal{M}}(\varphi \vee \psi) = \max(v_{\mathcal{M}}(\varphi), v_{\mathcal{M}}(\psi)),\\
&v_{\mathcal{M}}(\neg\varphi) = 1 - v_{\mathcal{M}}(\varphi),\\
&v_{\mathcal{M}}(\bot) = 0,\\
&v_{\mathcal{M}}(\top) = 1.
\end{aligned}$$

This logic is sometimes simply identified with 'fuzzy logic', (e.g. in [19]). Following [1], we call it *Kleene-Zadeh logic*, or KZ for short. Using the notation for variable assignments introduced in Sect. 2, KZ is extended to the first order level by

$$v_{\mathcal{M}}(\forall x\varphi) = \inf\{v_{\mathcal{M}}(\varphi[\xi[c/x]]) \mid c \in D_{\mathcal{I}}\},$$
$$v_{\mathcal{M}}(\exists x\varphi) = \sup\{v_{\mathcal{M}}(\varphi[\xi[c/x]]) \mid c \in D_{\mathcal{I}}\}.$$

If we restrict attention to interpretations over finite domains, these clauses correspond to equilibrium semantics as well (cf. Theorem 3).

From the point of view of mathematical fuzzy logic, the logic KZ is rather unsatisfying. Following a paradigm developed by Petr Hájek [10], the connectives of KZ should be augmented at least by an *implication* $\supset$ and a so-called *strong conjunction* $\&$, where $\&$ is interpreted by a continuous t-norm[5] and $\supset$ by its residuum. Arguably the most important logic of that kind is Łukasiewicz logic Ł, which is obtained from KZ by adding the following truth functions ($\oplus$ is called *strong disjunction*):

$$v_{\mathcal{M}}(\varphi \supset \psi) = \min(1, (1 - v_{\mathcal{M}}(\varphi)) + v_{\mathcal{M}}(\psi)),$$
$$v_{\mathcal{M}}(\varphi \,\&\, \psi) = \max(0, v_{\mathcal{M}}(\varphi) + v_{\mathcal{M}}(\psi) - 1),$$
$$v_{\mathcal{M}}(\varphi \oplus \psi) = \min(1, v_{\mathcal{M}}(\varphi) + v_{\mathcal{M}}(\psi)).$$

In fact all other propositional connectives could by defined in Ł, e.g., from $\supset$ and $\bot$, or from $\&$ and $\neg$, alone. But neither $\supset$ nor $\&$ nor $\oplus$ can be defined in KZ.[6] The increased expressiveness of Ł over KZ is particularly prominent at the first-order level: while in KZ there are no valid formulas at all, except those involving truth constants in some obvious manner, the set of valid first-order formulas in Ł (with or without truth constants) is not even recursively enumerable due to a classic result of Scarpellini [20].

Independently of Hintikka, Robin Giles devised a game-theoretic interpretation of Łukasiewicz logic in the 1970s [7,8]. Rather than referring to Hintikka's game (which he seemingly was not aware of) Giles refers to the logical dialogue game suggested by Lorenzen [16,17] as a foundation for constructive reasoning. Initially Giles was interested in modeling logical reasoning within theories of physics and only later motivated his game for Ł explicitly as an attempt to provide "tangible meaning" for fuzzy logic [9].

We briefly review the essential features of Giles's game, called $\mathcal{G}$-game here, but refer to [5,7,8] for more detailed presentations, including adequateness proofs. Like in the $\mathcal{H}$-game, I and you are the players and we can both act in the roles **P** or **O** with respect to given formulas augmented by variable assignments. In contrast to the $\mathcal{H}$-game, there may be more than one formula at stake at any state of the $\mathcal{G}$-game. We say that an augmented formula $\varphi[\xi]$ is currently *asserted* by you, if you act as **P** and I act as **O** with respect to it; and vice versa for a formula asserted by me. Since it will matter how often a formula is asserted at a

[5] A t-norm is a commutative and associative function $\circ : [0,1]^2 \to [0,1]$ such that $x \circ 1 = x$ and $x < y$ implies $x \circ z \leq y \circ z$.

[6] KZ is sometimes called the 'weak fragment of Łukasiewicz logic'.

given state, we collect the formulas currently asserted by you in a *multiset*, called *your tenet*. Likewise, *my tenet* consists of the multiset of augmented formulas currently asserted by me. We denote a state by

$$\left[\varphi_1[\xi_1], \dots, \varphi_m[\xi_m] \,\|\, \psi_1[\xi'_1], \dots, \psi_n[\xi'_n]\right],$$

where $\{\varphi_1[\xi_1], \dots, \varphi_m[\xi_m]\}$ is your tenet and $\{\psi_1[\xi'_1], \dots, \psi_n[\xi'_n]\}$ is my tenet. At any given state an occurrence of a non-atomic augmented formula is picked randomly either from my or from your tenet and distinguished as *current formula*.[7]

States that only contain atomic formulas are called *elementary*. At non-elementary states the game proceeds according to the following rules. Like for the $\mathcal{H}$-game, we do not have to refer to the players' identity directly, but only to their roles with respect to the current formula (which by definition is an occurrence of some non-atomic augmented formula in **P**'s tenet).

($R^{\mathcal{G}}_{\wedge}$) If the current formula is $(\varphi \wedge \psi)[\xi]$ then **O** chooses whether to replace it by $\varphi[\xi]$ or by $\psi[\xi]$ in **P**'s tenet.

($R^{\mathcal{G}}_{\vee}$) If the current formula is $(\varphi \wedge \psi)[\xi]$ then **P** chooses whether to replace it by $\varphi[\xi]$ or by $\psi[\xi]$ in **P**'s tenet.

($R^{\mathcal{G}}_{\neg}$) If the current formula is $\neg\varphi[\xi]$ then it is replaced by $\perp$ in **P**'s tenet and $\varphi[\xi]$ is added to **O**'s tenet.

($R^{\mathcal{G}}_{\supset}$) If the current formula is $(\varphi \supset \psi)[\xi]$ then **O** chooses whether to remove it or else to replace it by $\psi[\xi]$ in **P**'s tenet and add $\varphi[\xi]$ to **O**'s tenet.

($R^{\mathcal{G}}_{\&}$) If the current formula is $(\varphi \,\&\, \psi)[\xi]$ then **P** chooses whether to replace it by both, $\psi[\xi]$ and $\varphi[\xi]$, or by $\perp$ in **P**'s tenet.

($R^{\mathcal{G}}_{\oplus}$) If the current formula is $(\varphi \oplus \psi)[\xi]$ then **O** chooses whether to remove it or to replace it by $\psi[\xi]$ and $\varphi[\xi]$ in **P**'s tenet while adding $\perp$ to **O**'s tenet.

($R^{\mathcal{G}}_{\forall}$) If the current formula is $(\forall x \varphi(x))[\xi]$ then it is replaced in **P**'s tenet by $\varphi(x)[\xi[c/x]]$, where $c \in D_{\mathcal{I}}$ is chosen by **O**.

($R^{\mathcal{G}}_{\exists}$) If the current formula is $(\exists x \varphi(x))[\xi]$ then it is replaced in **P**'s tenet by $\varphi(x)[\xi[c/x]]$, where $c \in D_{\mathcal{I}}$ is chosen by **P**.

Note that rules $R^{\mathcal{G}}_{\wedge}$, $R^{\mathcal{G}}_{\vee}$, $R^{\mathcal{G}}_{\forall}$, and $R^{\mathcal{G}}_{\exists}$ directly correspond to $R^{\mathcal{H}}_{\wedge}$, $R^{\mathcal{H}}_{\vee}$, $R^{\mathcal{H}}_{\forall}$, and $R^{\mathcal{H}}_{\exists}$, respectively. However the rules for implication ($R^{\mathcal{G}}_{\supset}$), negation ($R^{\mathcal{G}}_{\neg}$), strong conjunction ($R^{\mathcal{G}}_{\&}$), and strong disjunction ($R^{\mathcal{G}}_{\oplus}$), involve more than just one formula at the succeeding state and therefore cannot be formulated in the format of the $\mathcal{H}$-game, where only one formula is asserted at any state.

If there is no non-atomic formula left to pick as current formula, the game has reached an *elementary state*

$$\left[A_1[\xi_1], \dots, A_m[\xi_m] \,\|\, B_1[\xi'_1], \dots, B_n[\xi'_n]\right],$$

where the $A_i[\xi_i]$ and $B_i[\xi'_i]$ are augmented atomic formulas. To define the players' payoffs at an elementary state Giles introduces the concept of *dispersive*

[7] The powers of the players of a $\mathcal{G}$-game do not depend on the manner in which the current formula is picked at any state. In more formal presentations of the $\mathcal{G}$-game one may introduce the concepts of a regulation and of so-called internal states in formalizing state transitions. We refer to [5] for details.

elementary experiments. For each (augmented) atomic formula $A[\xi]$ there is a corresponding experiment[8] $E_{A[\xi]}$ that yields either 'yes' or 'no' at each trial. Dispersiveness refers to the fact that the same experiment may give different answers when repeated. However a fixed probability (*risk*) $\langle A[\xi]\rangle$ of yielding a negative answer is associated with $E_{A[\xi]}$. Experiment $E_\bot$ always yields a negative result and thus $\langle\bot\rangle = 1$; similarly $\langle\top\rangle = 0$. It is stipulated that at the end of any run of the game, i.e. at an elementary state, the experiment $E_{A[\xi]}$ is performed for each occurrence of an augmented atomic formula $A[\xi]$ in my tenet and that I have to pay a fixed amount of money, say 1€, to you if $E_{A[\xi]}$ yields 'no'. Likewise you have to pay 1€ to me for each assertion in your tenet, where the corresponding experiment yields a negative answer. Therefore the expected (average) total amount of money (in €) that I have to pay to you is given by

$$\sum_{1\leq i\leq n} \langle B_i[\xi_i]\rangle - \sum_{1\leq i\leq m} \langle A_i[\xi_i']\rangle \,.$$

We call this value *my (total) risk* in a run of the $\mathcal{G}$-game that ends at the elementary state $\big[A_1[\xi_1],\ldots,A_m[\xi_m] \,\|\, B_1[\xi_1'],\ldots,B_n[\xi_n']\big]$. (This amount could also be negative, indicating that the total risk associated with my assertions is smaller than that associated with your assertions. Moreover, remember that empty sums evaluate to 0, reflecting the fact that empty tenets carry no positive risk.) Risk value assignments can be seen as inverted many-valued interpretations. More precisely, given a many-valued interpretation $\mathcal{M}$, we define a corresponding assignment of risk values $\langle\cdot\rangle_{\mathcal{M}}$ to augmented atomic formulas by $\langle A[\xi]\rangle_{\mathcal{M}} = 1 - v_{\mathcal{M}}((A[\xi]))$.

Definition 3. *Given a formula φ, a variable assignment ξ, and an assignment $\langle\cdot\rangle$ of risk values $\in [0,1]$ to all augmented atomic formulas, an instance of the $\mathcal{G}$-game starting in state $\big[\,\|\,\varphi[\xi]\big]$, where final (elementary) states are evaluated with respect to $\langle\cdot\rangle$, is called a $\mathcal{G}$-game* for *$\varphi[\xi]$ under $\langle\cdot\rangle$.*

The value *of such a game is $1-w$ if I have a strategy that guarantees that my risk at the final state is at most w, while you have a strategy that guarantees that my risk is at least $1-w$.*

Remember that we insist on finite domains. Under this assumption, the adequateness of the $\mathcal{G}$-game for Łukasiewicz logic (Giles's Theorem) can be formulated as follows.[9]

Theorem 4. *Let φ be an Ł formula, ξ be a variable assignment, and $\mathcal{M}$ a (many-valued) interpretation. Then any $\mathcal{G}$-game for $\varphi[\xi]$ has value w under the risk value assignment $\langle\cdot\rangle_{\mathcal{M}}$ iff $v_{\mathcal{M}}(\varphi[\xi]) = w$.*

[8] The idea is that for each atomic formula A there is schematic experimental setup that turns into a concrete experiment if elements of the domain of discourse are assigned to the free variables in A.

[9] Giles actually never considered strong conjunction and strong disjunction. For a detailed proof including strong conjunction we refer to [5]. That paper also features a link between the $\mathcal{G}$-game and an analytic proof system for Ł based on hypersequents.

4 Connecting the $\mathcal{G}$-game and the $\mathcal{H}$-game

In Sect. 2 we have seen that equilibrium semantics for IF logic provides an interpretation of the connectives of the many-valued logic KZ. However, as discussed in Sect. 3, game semantics for (full) Łukasiewicz logic Ł calls for Giles's more general concept of a game state consisting of multisets of formulas currently asserted by you and me, respectively.[10] In this section we want combine equilibrium semantics for IF logic with Giles's game for Ł.

Arguably the most straightforward way to connect IF logic with Ł allows for imperfect information about the choice of witness elements for the quantifiers in Ł-formulas in the same manner as for (classical) IF formulas. We would just have to attach slash sets to the quantifiers and treat these in a corresponding version of the $\mathcal{G}$-game exactly as in the $\mathcal{H}$-game: the players' choices of assignments for quantified variables have to remain independent of any assignment to variables in corresponding slash sets. While the resulting "Independence Friendly Łukasiewicz logic" might well be worth studying, we think that the following alternative way to connect equilibrium semantics and Giles's game for Ł is actually more interesting.

The $\mathcal{G}$-game allows one to derive the truth functions for all connectives and quantifiers of Ł from principles of reasoning about logically complex statements as encoded in the rules of the game. Notice that this derivation is completely independent of Giles's interpretation of truth values for atomic formulas in terms of the risk involved in claiming that certain dispersive experiments will yield positive results. Indeed, if one insists that atomic formulas are either simply true or false in any given interpretation, then this does not affect the rules of the $\mathcal{G}$-game, but leads to a characterization of classical logic ($\wedge$ and $\&$ both collapse to classical conjunction in this version of the $\mathcal{G}$-game; likewise $\vee$ and $\oplus$ both turn into classical disjunction). On the other hand, the $\mathcal{H}$-game for IF formulas provides an interpretation of intermediate truth values without departing from classical evaluation at the atomic level: the interpretation $\mathcal{I}$, with respect to which a given IF formula is to be evaluated, assigns either 1 (true) or 0 (false) to each (augmented) atomic formula. We propose to combine these two different semantic concepts by replacing the atomic formulas of Ł and corresponding dispersive experiments of the $\mathcal{G}$-game by IF formulas and corresponding instances of the $\mathcal{H}$-game.

To implement the idea sketched above, we define a two-tiered syntax for a new logic Ł(IF), where atomic subformulas of Ł formulas are replaced by IF formulas.[11]

[10] As shown in [2] and in [4] one may in fact formulate alternative semantic games for Ł that, like the $\mathcal{H}$-game, keep a single formula in focus at any given state, if either an explicit truth value or a stack of formulas is added. These and related variants of semantic games are discussed in [3], but they hardly are relevant in our context.

[11] This is somewhat reminiscent of [11], where an inner language for representing events and an outer, many-valued language for expressing assertions about the probability of such events is combined.

Definition 4. *With respect to any language as specified in Definition 1, the set of* Ł(IF) *formulas is defined as follows:*

– *Every recall regular IF formula (see Definition 2) is an* Ł(IF) *formula.*
– *If F and G are* Ł(IF) *formulas then also $\underline{\neg}F$, $F\underline{\wedge}G$, $F\underline{\vee}G$, $F\underline{\&}G$, $F\underline{\oplus}G$, $F\underline{\supset}G$, $\underline{\forall}xF$, and $\underline{\exists}xF$ are* Ł(IF) *formulas.*

Note that the logical connectives and quantifiers of Ł are underlined in order to clearly separate the outer (many-valued) level of Ł(IF) formulas from the inner level of (classical) IF formulas. Ł(IF) formulas are not evaluated with respect to a many-valued interpretation (like Ł formulas), but with respect to a (finite) classical interpretation $\mathcal{I}$, as for IF formulas. The intended semantics of Ł(IF) is given by the following combination of the $\mathcal{G}$-game and the $\mathcal{H}$-game, which we call $\mathcal{GH}$-game. Suppose we want to evaluate an (augmented) Ł(IF) formula $\chi[\xi]$, then the corresponding $\mathcal{GH}$-game proceeds as follows:

Phase 1: The $\mathcal{G}$-game with initial state $\left[\,\|\,\chi[\xi]\right]$ is played until the game reaches a state $S = \left[\varphi_1[\xi_1], \ldots, \varphi_m[\xi_m] \,\|\, \psi_1[\xi'_1], \ldots, \psi_n[\xi'_n]\right]$, in which all augmented formulas are (recall regular) IF formulas.

Phase 2: For each occurrence of an augmented IF formula $\varphi[\xi]$ in S a corresponding $\mathcal{H}$-game is played. If $\varphi[\xi]$ is in my tenet, then the $\mathcal{H}$-game starts with me as **P** and you as **O**, as usual. But if $\varphi[\xi]$ is in your tenet, then the initial roles are reversed: I act as **O** and you as **P**. No information about other instances of the $\mathcal{H}$-game initiated at state S is available to the players.

For the final evaluation we proceed like in the $\mathcal{G}$-game, where each instance $\varphi[\xi]$ of an augmented IF formula in S is treated like an atomic formula for which the corresponding dispersive experiment $E_{\varphi[\xi]}$ is the $\mathcal{H}$-game with initial formula $\varphi[\xi]$ as specified for Phase 2. If $\varphi[\xi]$ is in my tenet of S then I have to pay 1€ to you if you (initially acting as **O**) win the game. On the other hand, if $\varphi[\xi]$ is in your tenet of S then I initially act as **O** and you have to pay 1€ to me if I win the game. Assuming that we employ mixed strategies and play rationally, this setup guarantees that my risk associated with $\varphi[\xi]$ is equal to the inverse of my expected payoff at a Nash equilibrium of the $\mathcal{H}$-game for $\varphi[\xi]$. In other words, the risk value for $\varphi[\xi]$ is $1 - v^{eq}_{\mathcal{I}}(\varphi[\xi])$, where $v^{eq}_{\mathcal{I}}(\varphi[\xi])$ is the value of the $\mathcal{H}$-game for the IF formula $\varphi[\xi]$, as defined in Sect. 2.

Definition 5. *In analogy to Definition 3, we speak of a $\mathcal{GH}$-game for $\varphi[\xi]$ with respect to the (classical) interpretation $\mathcal{I}$ if the game starts in state $\left[\,\|\,\varphi[\xi]\right]$ and the evaluation is as indicated above: we compute the overall risk like for the $\mathcal{G}$-game that arises if each $\varphi'[\xi']$, where φ' is a largest sub-formula of φ that is an IF formula, is treated like an atomic formula for which the corresponding dispersive experiment consists in a run of the $\mathcal{H}$-game for $\varphi'[\xi']$ with respect to $\mathcal{I}$.*

The value of such a $\mathcal{GH}$-game is w if I have a strategy that guarantees that my overall risk evaluates to at most $1-w$, while you have a strategy that ensures that my risk is at least $1-w$.

Definition 6. *The truth value $v_{\mathcal{I}}(\varphi[\xi])$ of an augmented* Ł(IF) *formula $\varphi[\xi]$ under a classical interpretation $\mathcal{I}$ is defined as follows.*

– *If the outermost connective or quantifier of φ is an (underlined) connective or quantifier of* Ł *then $v_{\mathcal{I}}(\varphi[\xi])$ is obtained from the value(s) of the immediate sub-formula(s) just like specified for Ł at the beginning of Sect. 3.*
– *Otherwise, φ is a recall regular IF formula and we set $v_{\mathcal{I}}(\varphi[\xi]) = v^{eq}_{\mathcal{I}}(\varphi[\xi])$.*

The match between the game-theoretic semantics according to Definition 5 and the truth functional semantics specified in Definition 6 is obtained as a corollary to Giles's Theorem for Łukasiewicz logic (Theorem 4) and the adequateness of equilibrium semantics for IF formulas (Theorem 2).

Corollary 1. *Let φ be an* Ł(IF) *formula, $\mathcal{I}$ a classical interpretation, and ξ a variable assignment. Any $\mathcal{GH}$-game for $\varphi[\xi]$ with respect to $\mathcal{I}$ has value w iff $v_{\mathcal{M}}(\varphi[\xi]) = w$.*

Before analyzing some (schemes of) formulas in the light of Corollary 1, let us emphasize that neither implication nor strong disjunction can be expressed by IF formulas. Negation is represented, indirectly, by dualization. This latter fact can now be expressed in the object language by the (schematic) Ł(IF) formula $\underline{\neg}\varphi \underline{\leftrightarrow} \varphi^{\neg}$, where $\psi \underline{\leftrightarrow} \chi$ abbreviates $(\psi \underline{\supset} \chi) \wedge (\chi \underline{\supset} \psi)$. However, remember that $\psi \supset \chi$ is not equivalent to $\neg\psi \vee \chi$ in Ł (or in any other t-norm based logic for that matter). Therefore one should *not* define implication for IF formulas by $\psi^{\neg} \vee \chi$.

Example 4. Let φ be an arbitrary recall regular IF formula and consider the following Ł(IF) formulas:

(1) $\varphi \underline{\supset} \varphi$
(2) $\varphi \underline{\vee} \underline{\neg}\varphi$
(3) $\varphi \underline{\oplus} \underline{\neg}\varphi$

For (1), remember that $\psi \supset \psi$ is valid in Łukasiewicz logic for any Ł formula ψ. Consequently also (1) always evaluates to 1. In terms of the $\mathcal{GH}$-game starting in state $\left[\,\|\, \varphi \underline{\supset} \varphi\right]$ this can be seen as follows. According to the rule for implication, you (acting as **O**) can choose whether the next state of the game is the empty state $\left[\,\|\,\right]$ (resulting from removing $\varphi \underline{\supset} \varphi$ from my tenet) or else is $\left[\varphi \,\|\, \varphi\right]$. Clearly my risk is 0 in the first case. But it is also 0 in the second case, where we continue with two instances of the $\mathcal{H}$-game for φ: whatever amount of money I am expected to pay to you for the $\mathcal{H}$-game corresponding to the instance of φ in my tenet, it obviously equals the amount that you have to pay to me for the instance of φ in your tenet.

For (2), I can choose whether the $\mathcal{GH}$-game starting in state $\left[\,\|\, \varphi \underline{\vee} \underline{\neg}\varphi\right]$ will result in state $\left[\,\|\, \varphi\right]$ or in state $\left[\,\|\, \underline{\neg}\varphi\right]$, which further reduces to $\left[\varphi \,\|\, \bot\right]$ in Phase 1 of the game. In the first case my risk is $1 - v^{eq}_{\mathcal{I}}(\varphi)$, i.e., the inverse of the equilibrium value for the IF formula φ. In second case, I definitely have to pay 1€ to you, but expect to receive $(1 - v^{eq}_{\mathcal{I}}(\varphi))$€ from you, resulting from the $\mathcal{H}$-game for φ in your tenet where you are in role **P**. Consequently I have a strategy that limits my expected loss to $\min(1 - v^{eq}_{\mathcal{I}}(\varphi), v^{eq}_{\mathcal{I}}(\varphi))$. The value of the game is the inverse of that overall risk, i.e. $1 - \min(1 - v^{eq}_{\mathcal{I}}(\varphi), v^{eq}_{\mathcal{I}}(\varphi)) = \max(v^{eq}_{\mathcal{I}}(\varphi), 1 - v^{eq}_{\mathcal{I}}(\varphi))$, which matches the truth value calculated according to Definition 6.

In contrast to (2), formula (3) always evaluates to 1. This is obvious from Definition 6. To see it also for the corresponding $\mathcal{GH}$-game, recall that, by rule $R^{\mathcal{G}}_{\oplus}$, **O** can choose whether the initial state $[\,\|\, \varphi \underline{\oplus} \varphi]$ is succeeded by the empty state $[\,\|\,]$ or by the state $[\bot \,\|\, \varphi, \underline{\neg}\varphi]$, which reduces to $[\varphi, \bot \,\|\, \varphi\, \bot]$ in the next round. Clearly, I have no positive risk in either case. Therefore the value of the game is 1, as required.

Note that, since $\underline{\neg}\varphi$ is equivalent to $\varphi^{\neg}$ the validity of formula (3) corresponds to the fact that the $\mathcal{H}$-game is constant-sum. On the other hand, the fact that formula (2) is not valid corresponds to the indeterminateness of the $\mathcal{H}$-game, in general. Indeed, the value of $\varphi \underline{\vee} \underline{\neg}\varphi$ is below 1 iff I have neither a winning strategy in the $\mathcal{H}$-game for φ nor in the one for $\varphi^{\neg}$, where the players' roles are switched.

The above examples look at Ł(IF) as an extension of IF logic. The corresponding $\mathcal{GH}$-game widens the scope of equilibrium semantics by providing game based interpretations of a richer set of (truth functional) connectives for combining IF formulas.

On the other hand, one may understand the combined game as an extension of the original $\mathcal{G}$-game, where IF-formulas take the place of atomic Łukasiewicz formulas: results of previously unspecified dispersive experiments are now obtained as results of runs of Hintikka-styles games with imperfect information. This amounts to an interpretation of intermediate truth values as equilibria in games of imperfect information that involve only classical truth and falsity.

5 Propositional IF Logic and Realizable Truth Values

One of our motivations for introducing the $\mathcal{GH}$-game was to address philosophical worries about the nature of intermediate truth values by building a many-valued logic over classical models that evaluate all atomic formulas over $\{0,1\}$. An important question in this context is, whether IF logic is rich enough to cover a sufficient range of truth values. Equilibrium semantics for IF logic refers to constant-sum, two-player games with 0 and 1 as the only possible payoff values. It is a well known game-theoretic fact that the value of every such game is rational [24], hence the values of IF formulas under equilibrium semantics must be rationals from the interval $[0,1]$. As the functions of the connectives of Łukasiewicz logic are closed under rational numbers, we do not obtain the full real interval $[0,1]$, usually understood as the standard set of truth values for fuzzy logics. But can we get at least all rational values in the interval $[0,1]$? In particular, is there for any $q \in [0,1] \cap \mathbb{Q}$ an IF formula φ such that the value of φ is q?

Mann, Sandu and Sevenster in [18] deal with this question within the framework of predicate IF logic and give two solutions [p. 184]. The first one is based on a random quantifier expressed by an IF formula, which over an interpretation with a domain of size n and a unary predicate satisfied by exactly m elements of the domain has the value m/n. The second one is more general—it shows how to construct an IF formula which has the value m/n over every domain with more than two objects.

We present two solutions of the same problem within the framework of propositional IF logic: for any rational $q \in [0,1]$ we define a formula φ that evaluates to q according to equilibrium semantics under any (classical propositional) interpretation.

From a game-theoretic point of view there is no reason to limit imperfect information in semantic games to the quantifier moves: it is natural to consider independent choices already on the propositional level. This leads to propositional IF logic, discussed e.g. by Pietarinen [23] and Sandu and Pietarinen [22].

The minimal version of propositional IF logic introduces formulas expressing independence of disjunctions from immediately preceding conjunctions and, likewise, independence of conjunctions from immediately preceding disjunctions. The language of this logic is an extension of a standard propositional language by correspondingly slashed formulas.

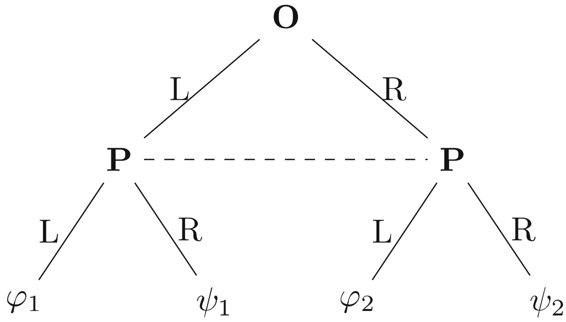
Definition 7. *The propositional IF formulas (IFP formulas) are built up over propositional variables and truth constants, $\top, \bot$ using $\wedge, \vee, \neg$ as usual. In addition we have two following clauses:*

- *If $\varphi_1, \varphi_2, \psi_1, \psi_2$ are IFP formulas, then* $(\varphi_1 \vee / \wedge \psi_1) \wedge (\varphi_2 \vee / \wedge \psi_2)$ *is an IFP formula*
- *If φ and ψ are IFP formulas, then* $(\varphi_1 \wedge / \vee \psi_1) \vee (\varphi_2 \wedge / \vee \psi_2)$ *is an IFP formula*

The interpretation of standard disjunction and conjunction remains the same: it consists of the choice by the Proponent **P** or the Opponent **O**, respectively. The slashed disjunction (conjunction) is, analogically to the first order case, interpreted as a game of imperfect information: one player chooses a disjunct (conjunct) without any information about the previous choice of the other player.

The moves for slashed disjunction (conjunction) cannot be labeled by the corresponding disjuncts (conjuncts), because perfect information would be recovered. If, in the semantic game for $(\varphi_1 \vee / \wedge \psi_1) \wedge (\varphi_2 \vee / \wedge \psi_2)$ **P** were asked to choose between φ_1 and ψ_1, she would know that in the previous move **O** must have chosen the left conjunct. Thus the players' choices are specified using labels ("Left disjunct", "Right disjunct", etc.).

The semantic game for the formula $(\varphi_1 \vee / \wedge \psi_1) \wedge (\varphi_2 \vee / \wedge \psi_2)$ has the following extensive form:



This game is the simplest (non-trivial) case of incomplete information as we can also see from its strategic form:

$$\begin{array}{ll} \mathbf{O}\backslash\mathbf{P} & \quad L \quad\; R \\ \begin{array}{l} L \\ R \end{array} & \begin{pmatrix} \varphi_1 & \psi_1 \\ \varphi_2 & \psi_2 \end{pmatrix} \end{array}$$

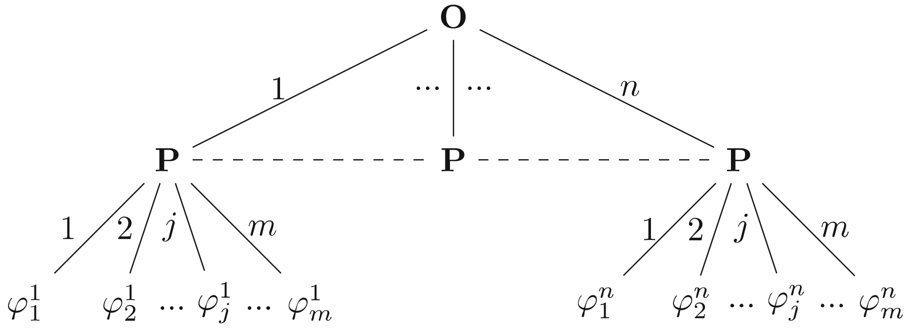
More general versions of propositional IF logic are discussed in the literature that, e.g., allow one to express that a disjunction is independent from any preceding conjunction. This requires a more substantial modification of syntax, that we will not introduce here, since the indicated minimal version is sufficient for our purposes. However, we will use a more concise notation suggested by Sandu and Pietarinen:

$$(\varphi_1 \vee / \wedge \psi_1) \wedge (\varphi_2 \vee / \wedge \psi_2) = W(\varphi_1, \psi_1, \varphi_2, \psi_2)$$

Our first solution for recovering arbitrary rationals in $[0,1]$ as equilibrium values requires an extension of the syntax from binary to n-ary conjunctions and disjunctions. We replace the clause for (binary) slashed disjunction from Definition 7 by the following one for n-ary disjunction:

– let $m, n \geq 2$ and let φ_i^j for $i = 1, \dots, m$, $j = 1, \dots, n$ be IFP formulas, then $(\varphi_1^1 \vee / \wedge \varphi_2^1 \vee / \wedge \cdots \vee / \wedge \varphi_m^1) \wedge \cdots \wedge (\varphi_1^n \vee / \wedge \varphi_2^n \vee / \wedge \cdots \vee / \wedge \varphi_m^n)$ is an IFP formula

and a similarly in the case of slashed conjunction. Like in the binary case, the independence of the slashed connective is with respect to the immediately preceding connective. Thus n-ary slashed disjunction corresponds to the following game tree:



The corresponding strategic form is represented by the following $m \times n$ matrix:

$$\begin{array}{cc} \mathbf{P}\backslash\mathbf{O} & \begin{array}{ccc} 1 & \dots & n \end{array} \\ \begin{array}{c} 1 \\ \vdots \\ m \end{array} & \begin{pmatrix} \varphi_1^1 & \dots & \varphi_1^n \\ \vdots & & \vdots \\ \varphi_m^1 & \dots & \varphi_m^n \end{pmatrix} \end{array}$$

We will use the matrix form of the game interpreting the n-ary conjunction/disjunction in our first proof of realizability of rationals.

Theorem 5. *For every non-negative rational number q there is a strategic two-person, zero-sum game with payoffs in $\{0,1\}$ such that:*

1. *the value of the game is q,*
2. *the equilibrium strategy for both players is the uniform distribution.*

Proof. For a given rational $q = m/n$, where $m, n \in \mathbb{N}, 0 \leq m < n$ we construct an $n \times n$ payoff matrix M with exactly m ones in each row and each column:

- $a_{i,j} = 1$ if $1 \leq i \leq j \leq i + m - 1$ and $j \leq n$
- $a_{i,j} = 1$ if $1 \leq j \leq (i + m - 1) \bmod n$ and $i + m - 1 > n$
- $a_{i,j} = 0$ otherwise

The pure strategies of both players consist in picking $i, j \in \{1, ..., n\}$. We denote their mixed strategies by $(p_1, ...p_n)$ and $(q_1, ..., q_n)$, respectively. The element $a_{i,j}$ is the payoff of the row player if she is playing i and the column player is playing j, the payoff of the column player for the same profile (couple) of pure strategies is $1 - a_{i,j}$.

It is clear that if both players play the mixed strategy corresponding to the uniform distribution ($p_i = q_j = 1/n$), the probability of each payoff $a_{i,j}$ is $1/n^2$. As there are m ones in each of the n rows, the payoff of the row player is $\frac{1}{n^2} m \cdot n \cdot 1 = m/n$.

It remains to check that the mixed strategy profile corresponding to uniform distributions for both players is an equilibrium pair. A standard characterization of equilibrium says that no player can improve her payoff by a unilateral deviation from her equilibrium strategy. As is well known, it is sufficient to check this condition with respect to *pure* strategies. If the first player deviates from uniform distribution playing i-th row against the uniform distribution played by the column player, her payoff is $1/n \cdot m$ (as there are exactly m ones in each row and each of them has probability $1/n$). This is the same as the equilibrium payoff, so no improvement is gained. The condition for the second player can be checked in a similar way.

The following theorem about realizability of rationals just translates the strategic game from Theorem 5 into the language of propositional IF logic.

Theorem 6. *For any $q \in [0, 1] \cap \mathbb{Q}$ there is an IFP formula ψ (using n-ary slashed disjunction) such that the value of ψ according to equilibrium semantics is q under any interpretation.*

Proof. Assume $q = m/n$, where $m, n \in \mathbb{N}, 0 \leq m < n$. From Theorem 5 we obtain an $n \times n$ matrix $(a_i^j), a_i^j \in \{0, 1\}$, representing a strategic game with the equilibrium value m/n. We can straightforwardly express this matrix in IF notation using n-ary slashed disjunction and the constants $\top$ and $\bot$ as follows: $\psi = (\varphi_1^1 \vee / \wedge \varphi_2^1 \vee / \wedge \cdots \vee / \wedge \varphi_n^1) \wedge ... \wedge (\varphi_1^n \vee / \wedge \varphi_2^n \vee / \wedge \cdots \vee / \wedge \varphi_n^n)$, where $\varphi_i^j = \top$ if $a_i^j = 1$ and $\varphi_i^j = \bot$ if $a_i^j = 0$.

We now present a second construction for realizing all rationals in $[0, 1]$ as equilibrium values, using IFP formulas logic with only binary connectives, as specified in Definition 7. It is based on iterating the connective W (encoding the simplest proper game of imperfect information). In analogy to the first-order case (in Sect. 2) we will denote by $v_I^{eq}(\varphi)$ the value of the formula φ according to equilibrium semantics. As the choice of a particular propositional interpretation I plays no role, we omit the index I. We also introduce the operator $\bar{W}$ of type $[0, 1]^4 \to [0, 1]$ corresponding to the connective W: $\bar{W}(x, y, z, u) = v^{eq}(W(\varphi_1, \varphi_2, \psi_1, \psi_2))$, where $v^{eq}(\varphi_1) = x, v^{eq}(\varphi_2) = y, v^{eq}(\psi_1) = z, v^{eq}(\psi_2) = u$.

Observe that W allows us to express random choice between two formulas φ, ψ. The equilibrium strategy of the game corresponding to $W(\varphi, \psi, \psi, \varphi)$ amounts to picking up with the same probability one of the elements φ, ψ. Consequently its equilibrium value is a mean of the values of φ and ψ: $v^{eq}(W(\varphi, \psi, \psi, \varphi)) = (v^{eq}(\varphi) + v^{eq}(\psi)/2)$. The resulting random choice connective is denoted by Π, where $\Pi(\varphi, \psi) = W(\varphi, \psi, \psi, \varphi)$, $\bar{\Pi}$ will be the corresponding function (operator), hence $v^{eq}(\Pi(\bot, \top)) = v^{eq}(W(\bot, \top, \top, \bot)) = \bar{W}(0, 1, 1, 0) = 1/2$. The corresponding game is the one of inverse Matching Pennies (IMP) with the following payoff matrix:

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} L \\ R \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

It is easy to see that iterating the Π-operator gives us powers of $1/2$: $\bar{\Pi}(0, \bar{\Pi}(0, 1)) = 1/4$, $\bar{\Pi}(0, \bar{\Pi}(0, \bar{\Pi}(0, 1))) = 1/8$ etc. We can represent this schematically as "pluggin-in" IMP into IMP :

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} L \\ R \end{array} & \begin{pmatrix} 0 & \mathsf{IMP} \\ \mathsf{IMP} & 0 \end{pmatrix} \end{array}$$

The choices (L, L) and (R, R) lead to the payoff for the first player, while for the choices (L, R) and (R, L) the game continues by playing IMP. This corresponds to symmetric iterations of the $\bar{W}$-operator: for example $(\bar{\Pi}(0, \bar{\Pi}(0, 1))) = \bar{W}(0, \bar{W}(0, 1, 1, 0), W(0, 1, 1, 0), 0)$. What happens in the case of "asymmetric" iterations? Consider the game which is the result of the simplest case of an asymmetric plug-in of IMP:

$$\begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} L \\ R \end{array} & \begin{pmatrix} 0 & \mathsf{IMP} \\ 1 & 0 \end{pmatrix} \end{array} \text{ with the payoff matrix } \begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} L \\ R \end{array} & \begin{pmatrix} 0 & 1/2 \\ 1 & 0 \end{pmatrix} \end{array}.$$

This corresponds to the substitution of the random choice operator at the second argument position of the $\bar{W}$-operator: $\bar{W}(0, \bar{\Pi}(0,1), 1, 0)$. We can easily check that the value of the game is 1/3 and the equilibrium strategy profile is $\langle(2/3, 1/3), (1/3, 2/3)\rangle$. We show that this simple kind of iteration in combination with negation is already sufficient to obtain all rationals. To simplify notation we introduce a unary connective O, defined by $O(\varphi) = W(\bot, \varphi, \top, \bot)$. Thus the formula corresponding to the above game can be written as $O(\varphi)$, where $v^{eq}(\varphi) = 1/2$ and $\bar{O}(1/2) = 1/3$. In fact we obtain $\bar{O}(1/n) = 1/(n+1)$ for every $n \in \mathbb{N}, n \geq 1$, as follows from the following Lemma.

Lemma 1. *The constant sum, two players strategic game represented by the payoff matrix*

$$\begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} L \\ R \end{array} & \begin{pmatrix} 0 & k/n \\ 1 & 0 \end{pmatrix} \end{array}$$

where $n, k \in \mathbb{N}, n \geq 1$ and $0 \leq k \leq n$, has the unique Nash equilibrium (equilibrium strategy profile) $\langle(n/(n+k), k/(n+k)), (k/(n+k), n/(n+k))\rangle$ and the corresponding equilibrium value for the row player is $k/(n+k)$.

Proof. The case for $k = 0$ is trivial: the column player has a pure winning strategy R and the payoff of the row player is $0 = 0/(n+0)$. Except for this trivial case no pure strategy is an equilibrium, so every mixed equilibrium strategy is proper—both pure strategies will be played with a non-zero probability (i.e., both of them belong to the support of mixed equilibrium strategies). It is a well known game-theoretic fact that in this case an equilibrium strategy $(p, 1-p)$ of the first player must yield the same payoff in response to both pure strategies of the second player, which gives us the equation: $p \cdot \frac{k}{n} = 1 - p$. This allows us to calculate the required probability values: $p = \frac{n}{n+k}, 1-p = \frac{k}{n+k}$. A similar line of reasoning leads to the values for the second player: $q = \frac{k}{n+k}, 1-q = \frac{n}{n+k}$. The equilibrium value of the game is $\frac{n}{n+k} \cdot \frac{k}{n+k} \cdot 0 + (\frac{n}{n+k})^2 \cdot \frac{k}{n} + (\frac{k}{n+k})^2 \cdot 1 + \frac{k}{n+k} \cdot \frac{n}{n+k} \cdot 0 = (\frac{n}{n+k})^2 \cdot \frac{k}{n} + (\frac{k}{n+k})^2 \cdot 1 = \frac{n \cdot k + k^2}{n+k^2} = \frac{k}{n+k}$.

Lemma 1 shows that $\bar{O}(k/n) = k/(k+n)$. To obtain all rationals in $[0,1]$ as equilibrium values of IFP formulas we have to use negation in addition to the connective O. The following theorem shows that this is sufficient.

Theorem 7. *Every $q \in [0,1] \cap \mathbb{Q}$ can be obtained as the result of iteratively applying the functions $\bar{O}(x)$ and $1-x$ to either 0 or 1.*

Proof. We show by induction on k that we can get all values $q = k/n$ for $0 \leq k \leq n$, where $n, k \in \mathbb{N}$ and $n \geq 1$. In fact it is sufficient to show we can get k/n for all $0 \leq k < n/2$ because the rest of the values is obtained by applying $1 - x$.

Base step: For the cases $k = 0, k = n$ we obtain from Lemma 1 that $\bar{O}(0) = 0$ and $\bar{O}(n/n) = \bar{O}(1) = 1/2$.

Induction step: Assume that we have all the values k'/n' where $1 \leq n' < n$ and $1 \leq k' \leq n'$. We show, that we can get k/n for all $k, 1 \leq k < n/2$. It follows from Lemma 1 that $\frac{k}{n} = \frac{k}{m+k} = \bar{O}(\frac{k}{m})$ for $m = n - k$. As we only need $k < \frac{n}{2}$, it holds that $2k < n$ and $k < n - k$. But then $k < n - k = m < n$ and the value $\frac{k}{m}$ is guaranteed by the induction hypothesis.

We obtain an expression of the form $\pm\bar{O}(\pm\bar{O}(\cdots \pm \bar{O}(x)))$, where $+\bar{O}(\_)$ is $\bar{O}(\_)$, $-\bar{O}(\_)$ is $(1 - \bar{O}(\_))$ and x equals 0 or 1. In fact we only need $x = 0$ in the case our $q = 0$, so we get either $\bar{O}(0)$ or $\pm\bar{O}(\pm\bar{O}(\cdots \pm \bar{O}(1)))$ The corresponding formula is $\pm O(\pm O(\ldots \pm O(\top)))$, or $O(\bot)$ where $+O(\_)$ is $O(\_)$, $-O(\_)$ is $(\neg O(\_))$.

The only remaining step is to translate the iterated $\bar{O}$-operator back to the language of propositional IF logic.

Theorem 8. *For every $q \in [0, 1] \cap \mathbb{Q}$ there is an IFP formula ψ built up from $\top$ and $\bot$ using only binary slashed disjunction and negation such that the value of ψ according to equilibrium semantics is q under any interpretation.*

Proof. Remember that $O(\varphi) = W(\bot, \varphi, \top, \bot) = (\bot \vee / \wedge \varphi) \wedge (\top \vee / \wedge \bot)$. Therefore the claim immediately follows from Theorem 7.

Example 5. We illustrate the previous results by constructing a propositional IF formula the value of which is 2/5. We start by expressing this value in the terms of the operator $\bar{O}$ using the formula $k/(n + k) = \bar{O}(k/n)$ from Lemma 1 iteratively. Our initial value can be expressed as $2/5 = \bar{O}(2/3)$. In the second step we need the value 2/3. As it is bigger than 1/2 we obtain it by complementation: $1 - \bar{O}(1/2) = 1 - 1/3$. We already know that $\bar{O}(1) = 1/2$. Putting together all these expressions we get $2/5 = \bar{O}(1 - \bar{O}(\bar{O}(1)))$. The translation to IF propositional logic is less compact, but it straightforwardly encodes the corresponding game tree. Using the connective O, corresponding to the $\bar{O}$-operator, we get $O(\neg O(O(\top)))$, which we can expand using $O(\varphi) = W(\bot, \varphi, \top, \bot) = (\bot \vee / \wedge \varphi) \wedge (\top \vee / \wedge \bot)$:

$$\begin{aligned} O(\neg O(O(\top))) &= W(\bot, \neg W(\bot, W(\bot, \top, \top, \bot), \top, \bot), \top, \bot) \\ &= (\bot \vee / \wedge \neg(\bot \vee / \wedge (\bot \vee / \wedge \top) \wedge (\top \vee / \wedge \bot)) \wedge (\top \vee / \wedge \bot)) \wedge (\top \vee / \wedge \bot) \end{aligned}$$

6 Conclusion

We have revisited two different types of semantic games: On the one hand, there is Hintikka's game-theoretic characterization of classical truth in a model, generalized by Hintikka and Sandu to IF logic that incorporates imperfect information, syntactically encoded by slashed quantifiers and connectives. Equilibrium semantics for IF logic provides an interpretation in which intermediate truth values arise from equilibrium strategies in the corresponding $\mathcal{H}$-game. On the other hand, there is Giles's game ($\mathcal{G}$-game) for Łukasiewicz logic, an expressive many-valued logic that, e.g., features two different forms of conjunction and disjunction. The $\mathcal{H}$-game and the $\mathcal{G}$-game are quite different, not only regarding their respective target logic, but also in their basic structure. Nevertheless they nicely fit together from a certain perspective. We introduced the $\mathcal{GH}$-game and the corresponding logic Ł(IF), which allows one to combine IF formulas with the connectives and quantifiers of Łukasiewicz logic. In this manner intermediary truth values retain their interpretation in terms of equilibria in imperfect information games, while featuring a set of propositional connectives and corresponding truth functions that reaches well beyond just min (weak conjunction), max (weak disjunction), and $1-x$ (negation). Thus Ł(IF) generalizes both, Łukasiewicz logic Ł as well as IF logic.

We have also addressed an interesting issue that already arises for IF logic: Can one represent all rational truth values already at the propositional level? We provided a positive answer in two different manners. If one allows for "slashed" conjunction or disjunction with arbitrary finite arity, formulas of minimal nesting depth built up from $\top$ and $\bot$ are sufficient to represent all rational truth values. If one insists on binary disjunction and conjunction a more elaborate construction, involving unbounded nesting of slashed connectives, is needed for this purpose.

We conclude by listing a number of possible directions for further investigations triggered by our considerations. As already indicated at the beginning of Sect. 4, it might be worthwhile to work out an independence friendly version of Łukasiewicz logic, which calls for a different generalization of the $\mathcal{H}$- and the $\mathcal{G}$-game. Yet another combination and generalization of the underlying games will arise if one considers arbitrary nestings of slashed (classical) connectives and Łukasiewicz connectives, instead of the strictly two-tiered syntax suggested here. One might also consider other many-valued logics for combining and/or generalizing IF formulas, e.g., Gödel or Product logic. Finally, we want to hint at subtle connections to Japaridze's Computability Logic (see, e.g., [14]). While Japaridze's game model of computation is quite different in several respects, there emerges some similarity in the options for representing various forms of combining (sub-)games by corresponding connectives, at least if one is willing to go beyond the truth functional setting induced by the two-tired syntax of Ł(IF).

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