Semantic Games with Backtracking for Fuzzy Logics

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***Abstract*—Hintikka's game theoretic semantics for classical connectives and quantifiers has been generalized to many-valued logics in various ways. After providing a short overview, we introduce a new type of semantic games: backtracking games, where a stack of formulas is used to store information on how to continue the game even after reaching an atomic formula. We present backtracking games for the three fundamental t-norm based logics: Łukasiewicz-, Gödel and Product logic and provide corresponding adequateness theorems.**

I. Introduction

Already in the late 1960s Jaako Hintikka [15] suggested to interpret the classical logical connectives and quantifiers via a game where one player ('I') defends a formula against systematic attacks by another player ('Nature' or 'you'). With respect to a given interpretation $\mathcal{J}$ that decides who is winning for atomic formulas, I have a winning strategy for a formula F in the game iff F is true in $\mathcal{J}$. In the 1970s, Robin Giles [12], in an attempt to justify a particular form of approximate reasoning, introduced a quite different semantic game that characterizes infinite-valued Łukasiewicz logic. More recently, other forms of semantic games for various many-valued logic have been suggested [4], [10], [11], [7]. We will briefly review the most important of these games in Section IV. As we will see, these games deviate rather significantly from Hintikka's original game theoretic semantics in various aspects. In this paper, we introduce a new type of game that largely respects the structure of Hintikka's game, but allows for backtracking when reaching an atomic formula. We will show that different forms of backtracking and of cumulatively evaluating atomic formulas during the game leads to characterizations of all three fundamental t-norm based fuzzy logics: Łukasiewicz logic, Gödel logic, and Product logic.

II. Hintikka's game and Kleene-Zadeh logic

Hintikka's game for classical logic can be presented as follows. There are two players, called *Myself* (*I*) and *You*, here, who can both act either in the role of the *Proponent* **P** or of the *Opponent* **O**. Initially *I* act as **P** and *You* act as **O**. *My* aim — or, more generally, **P**'s aim at any state of the game — is to show that the initial formula is true in a given interpretation $\mathcal{J}$. Since we are mainly interested in *propositional* logics, we will only present propositional game rules, here. (However, in Section IX we will explain in which manner all our games can be extended to the first-order level.) More precisely, the following rules refer to the outermost outermost connective of the *current formula*, i.e. the formula that is at stake at the given state of the game. Together with a *role distribution* of the players, the current formula fully determines any state of the game.

$R^{\mathcal{H}}_{\wedge}$: If the current formula is $F \wedge G$ then **O** chooses whether the game continues with F or with G.

$R^{\mathcal{H}}_{\vee}$: If the current formula is $F \vee G$ then **P** chooses whether the game continues with F or with G.

$R^{\mathcal{H}}_{\neg}$: If the current formula is $\neg F$, the game continues with F, except that the roles of the players are switched: the player who is currently acting as **P**, acts as **O** at the the next state, and vice versa for the current **O**.

Except for rule $R^{\mathcal{H}}_{\neg}$, the players' roles remain unchanged. The game ends when an atomic formula A is hit. The player who is currently acting as **P** *wins* and the other player, acting as **O**, *loses* if A is true in the given model $\mathcal{J}$. We associate pay-off 1 with winning and pay-off 0 with losing. We also include the truth constants $\top$ and $\bot$, with their usual interpretation, among the atomic formulas. The game starting with formula F is called the *$\mathcal{H}$-game for F under $\mathcal{J}$*.

Theorem 1 (Hintikka): A formula F is true in a classical interpretation $\mathcal{J}$ (in symbols: $\|F\|_{\mathcal{J}} = 1$) iff *I* have a winning strategy in the $\mathcal{H}$-game for F under $\mathcal{J}$.

It has been observed, e.g., in [10], [11] the above game can be straightforwardly adapted to the so-called *Kleene-Zadeh logic* KZ. (In [10], [11] we call KZ the 'weak fragment of Łukasiewicz logic, but sometimes it is simply referred to as 'fuzzy logic' [19]. We prefer to follow the terminology of [1] and [7], here.) The connectives of KZ arise by extending an assignment $\mathcal{J}$ of either 0 or 1 to atomic formulas to an assignment of truth values in $[0, 1]$ and defining the following truth functions.

$$\|F \wedge G\|^{\mathsf{KZ}}_{\mathcal{J}} = \min(\|F\|^{\mathsf{KZ}}_{\mathcal{J}}, \|G\|^{\mathsf{KZ}}_{\mathcal{J}}),$$
$$\|F \vee G\|^{\mathsf{KZ}}_{\mathcal{J}} = \max(\|F\|^{\mathsf{KZ}}_{\mathcal{J}}, \|G\|^{\mathsf{KZ}}_{\mathcal{J}}),$$
$$\|\neg F\|^{\mathsf{KZ}}_{\mathcal{J}} = 1 - \|F\|^{\mathsf{KZ}}_{\mathcal{J}},$$
$$\|\bot\|^{\mathsf{KZ}}_{\mathcal{J}} = 0, \quad \text{and} \quad \|\top\|^{\mathsf{KZ}}_{\mathcal{J}} = 1.$$

The rules of $\mathcal{H}$-game are left unchanged, but pay-offs may now be any values in the unit interval $[0, 1]$.

Definition 1: If a player $\mathbf{X}$ of some game has a strategy that guarantees her a pay-off of at least w, while her opponent has a strategy that ensures that $\mathbf{X}$'s pay-off is at most w, then w is called the *value for **X** of the game*.

Theorem 2 (see, e.g., [11]*):* A formula F evaluates to w in a KZ-interpretation $\mathcal{J}$, i.e., $\|F\|^{\mathsf{KZ}}_{\mathcal{J}} = w$, iff the value of the $\mathcal{H}$-mv-game for F under $\mathcal{J}$ for $Myself$ is w.

III. T-norm based fuzzy logics

Contemporary mathematical fuzzy logic as documented, e.g., in [3] highlights the fact that the expressibility of logic KZ

is severely limited. In particular, in contrast to classical logic, one cannot define implication by $F \to G =_{df} \neg F \vee G$ if one wants to retain $F \to F$ as valid. Petr Hájek has prominently (e.g., in [13] formulated design principles for fuzzy logics that include the following:

- the truth function for conjunction is a continuous t-norm; i.e., a monotone, commutative and associative function $* : [0,1] \times [0,1] \to [0,1]$, where 1 acts as identity element,
- as truth function for implication one takes the (unique) *residuum* of the t-norm chosen for conjunction,
- negation is defined from implication by setting its right argument to 0.

While $\min$, the truth function for conjunction in KZ is indeed a continuous t-norm, its residuum is given by

$$x \Rightarrow_G y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}$$

The corresponding truth function for negation is given by:

$$x -_G y = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The logic with these truth functions, in addition to $\min$ for conjunction and $\max$ for disjunction, is called *Gödel logic* G.

Also multiplication over $[0,1]$ is a t-norm. Its residuum is

$$x \Rightarrow_P y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise;} \end{cases}$$

$-_G$ remains the corresponding truth function for negation, according to Hájek's principle. The resulting logic is called *Product logic* Π.

Łukasiewicz logic arises from the Ł-t-norm $x *_L y = \max(0, x + y - 1)$ and its residuum

$$x \Rightarrow_L y = \begin{cases} 1 & \text{if } x \leq y \\ 1 - y + x & \text{otherwise;} \end{cases}$$

The corresponding truth function for negation is $-_L(x) = 1 - x$. Since $\min$ can defined from the Ł-t-norm and its residuum, as well as from product and its residuum, the logics Ł and Π each have two conjunctions. Likewise, $\max$ can be expressed. We continue to denote the connectives interpreted as as $\min$ and $\max$ by $\wedge$ and $\vee$ respectively. The t-norm based or 'strong' conjunctions of Ł and Π will be denoted by $\&$. The signs for the implication and negation connectives remain $\to$ and $\neg$, respectively. For Ł and Π one can define an additional 'strong disjunction' interpreted as the corresponding co-t-norm. However we will not use that connective here.

IV. Semantic games for fuzzy logics A brief overview

The literature on semantic games for t-norm based fuzzy logic is mainly focused on Łukasiewicz logic Ł. Already in the 1970s Robin Giles [12], in an attempt to model reasoning about dispersive experiments in theory of physics, presented a game where two players stepwise reduce complex formulas to simpler ones, like in Hintikka's game. But unlike in the $\mathcal{H}$-game, Giles defined a rule for implication that entails that more than just one formula may have to be considered at a given state of the game. More precisely, a state of Giles's game is given by $[F_1, \ldots, F_m \mid G_1, \ldots, G_n]$, where $\{F_1, \ldots, F_m\}$ is the multiset of formulas currently asserted by *You* and $\{G_1, \ldots, G_n\}$ is the multiset of formulas currently asserted by *Myself*. The game ends if every F_i $(1 \leq i \leq m)$ and G_j $(1 \leq j \leq n)$ is atomic. For the definition of pay-offs Giles associates with each atomic formula a particular experiment that either 'succeeds' or 'fails'. These experiments are dispersive, which means that they may yield different results upon repetition. However, the players assign a fixed success probability to each experiment and therefore a concrete value in $[0,1]$ to each atomic formula. At the end of the game an associated experiment is performed for each occurrence of an atomic formula. It is stipulated that each player has to pay one unit of money to the other player whenever an experiment associated with an assertion by herself fails. The total payoff however is not directly identified with the amount of money gained (or lost) in that manner, but rather given by the *risk*, i.e. the expected (average) amount of money transferred to the other player in total. We will not need to refer to any details of Giles game here, but refer to [8], [6], [10], where also also generalizations of the game Łukasiewicz logic are considered.

A different type of semantic game for Łukasiewicz logic has been introduced in [4]. Like in Hintikka's game, but unlike in Giles's game, a state the game of Cintula and Majer is determined by a single formula and a distribution of roles to the two players. However, in contrast to all games mentioned so far, the rules of the game explicitly refer to some value from the unit value at each state of the game. It is a win-loose game without a direct connection between pay-offs and truth values as in Theorem 2. The game may be seen as modeling a negotiation about a given value that is to be assigned to a complex formula, given that we the know the values of atomic formulas.

Since the games of Giles and of Cintula and Majer deviate from Hintikka's original concept of game theoretic semantics by either considering more than one formula or by an explicit reference to a (truth) value at any given state, it has been investigated recently in [7], whether there is a way to stick to Hintikka's format of game states (avoiding explicit reference to values and multiple formulas) in semantic games that characterize logics that extend KZ, like in particular, Łukasiewicz logic. A positive answer is provided in [7] by considering a rule for a new binary connective π, where neither the Proponent **P** nor the Opponent **O** chooses whether to continue with F or with G when the current formula is $F \pi G$, but rather a random choice (made by *Nature*, so to speak) determines the successor state. By combining this rule with the possibility to double payoffs in the game (capped to 1) one obtains a game for Ł. Similar rules lead to other proper extensions of KZ and even of Ł,

For Gödel logic G yet another type of semantic game is presented in [9], where not single formulas but 'comparison claims' $F < G$ or $F \leq G$, for arbitrary formulas F and G, are stepwise reduced by the players to claims where only the values of atomic formulas in the given interpretation has to be compared. We will consider two new semantic games for G in Sections VI and VII, respectively, that arguably are closer to Hintikka's original game in spirit and structure.

We are not aware of any semantic game for Product logic **P**. (However, in [2], [6] a parameterized game is considered that can be used to characterize *validity*, rather than degrees of truth in a given interpretation, for all three fundamental t-norm based logics Ł, Π, and G.) In Section VIII we introduce a semantic game for **P** by variating a concept introduced in Section VII for Gödel logic G.

V. A BACKTRACKING GAME FOR Ł

As indicated in the last section, Giles's game for Łukasiewicz logic Ł deviates in a number of aspects from Hintikka's concept of game theoretic semantics. In particular there are two *multisets of formulas* to be considered for the continuation of the game at a given state. While the games for Ł presented in [4] and in [7] avoid this feature, those games also introduce features that are at variance with Hintikka's original concept: *explicit references to a truth value* at every state of the game or *randomized choices*, respectively.

In this section we show that Hintikka's principle of focusing on a single formula and a role distribution can be maintained in a game for Ł without introducing randomized choices or explicit references to truth values in rules associated with logical connectives. For this purpose we propose to 'sequentialize' the multiple concurrent options for further moves that are available at any state of Giles's game. More precisely, we introduce a *stack* on which information about an alternative state is stored (in a first-in first-out manner) when making particular moves. Initially the stack is empty. Upon reaching an atomic formula the game only ends if the stack is empty. Otherwise, the game *backtracks* to the state (formula and role distribution) indicated by the top element of the stack. That stack element is thereby pushed from the stack.

In addition to the stack, we need to keep track of the *preliminary pay-off* $\sigma_{\mathbf{P}}$ for **P**. The preliminary pay-off $\sigma_{\mathbf{O}}$ for **O** is $-\sigma_{\mathbf{P}}$ throughout the game. When the game ends the preliminary pay-off becomes final. Initially, $\sigma_{\mathbf{P}} = 1$. We will call the resulting variant of Giles's game *backtrack game for Ł* or *$\mathcal{B}$Ł-game* for short.

The rules for $R^{\mathcal{H}}_{\wedge}$, $R^{\mathcal{H}}_{\vee}$, and remain unchanged for the $\mathcal{B}$Ł-game; no reference to the game stack or to $\sigma_{\mathbf{P}}$ and $\sigma_{\mathbf{O}}$ is needed. This implies that the $\mathcal{B}$Ł-game (for Ł) actually is an extension of the $\mathcal{H}$-mv-game for KZ. The rules for strong conjunction and implication are as follows:

- $R^{Ł}_{\&}$: If the current formula is $F \& G$ then **P** can choose either (1) to continue the game with F and to put G together with the current role distribution on the stack, or (2) to continue the game with $\bot$.
- $R^{Ł}_{\to}$: If the current formula is $F \to G$ then **O** can choose either (1) to put G on the stack with the current role distribution and continue the game with F and inverted roles, or (2) to continue the game with the top element of the stack. If the stack is empty, the game ends.
- $R^{Ł}_{\text{at}}$: If the current formula is an atom A then $\|A\|^{Ł}_{\mathcal{J}} - 1$ is added to $\sigma_{\mathbf{P}}$ and the same value is subtracted from $\sigma_{\mathbf{O}}$. The game ends if stack is empty and is continued with the top element of the stack otherwise.

$\neg F$ is treated as $F \to \bot$.

Again, we speak of the *$\mathcal{B}$Ł-game for F under $\mathcal{J}$* if the game starts with the current formula F where initially *I* am **P** and *You* are **O**.

Theorem 3: A formula F evaluates to w in an Ł-interpretation $\mathcal{J}$, i.e., $\|F\|^{Ł}_{\mathcal{J}} = w$, iff the value of the $\mathcal{B}$Ł-game for F under $\mathcal{J}$ for *Myself* is w.

Proof: We generalize to $\mathcal{B}$Ł-games that may start with any formula, role distribution, preliminary pay-offs $\sigma_{\mathbf{P}} = -\sigma_{\mathbf{O}}$ and any stack content. We use $\mathcal{S}^I$ to denote the multiset of $|\mathcal{S}^I|$ formulas on the stack where *I* am assigned the role of **P**, and $\mathcal{S}^Y$ to denote the multiset of $|\mathcal{S}^Y|$ formulas on the stack where *You* are assigned the role of **P**. (Note that we ignore the order of stack elements, but not the number of occurrences of the same formula on the stack.) We define $s(F) = 1$ if F is atomic, $s(\neg F) = s(F) + 1$, and $s(F \circ F') = s(F) + s(F') + 1$ for $\circ \in \{\vee, \wedge, \&, \to\}$. We prove the following by induction on $n = s(F) + \sum_{H \in \mathcal{S}^I \cup \mathcal{S}^Y} s(H)$: u is the value for *Myself* of the $\mathcal{B}$Ł-game under interpretation $\mathcal{J}$ that starts with formula F and with *Myself* as **P** iff

$$
\begin{aligned}
u &= \sigma_{\mathbf{P}} + \|F\|^{Ł}_{\mathcal{J}} + \sum_{H \in \mathcal{S}^I} (\|H\|^{Ł}_{\mathcal{J}} - 1) - \sum_{H \in \mathcal{S}^Y} (\|H\|^{Ł}_{\mathcal{J}} - 1) \\
&= \sigma_{\mathbf{P}} + \|F\|^{Ł}_{\mathcal{J}} - |\mathcal{S}^I| + |\mathcal{S}^Y| + \sum_{H \in \mathcal{S}^I} \|H\|^{Ł}_{\mathcal{J}} - \sum_{H \in \mathcal{S}^Y} \|H\|^{Ł}_{\mathcal{J}}.
\end{aligned}
$$

The theorem follows for $\sigma_{\mathbf{P}} = 1$ and $|\mathcal{S}^I| = |\mathcal{S}^Y| = 0$. The case where *You* are initially in the role of **P** is analogous: one just needs to invert the (preliminary) pay-offs.

At the base case, $n = 1$, the stack is empty and F is atomic. Therefore $\|F\|^{Ł}_{\mathcal{J}} - 1$ is added to $\sigma_{\mathbf{P}}$. The game ends at that state and $\sigma_{\mathbf{P}} + \|F\|^{Ł}_{\mathcal{J}} - 1$ is the pay-off for *Myself* as well as the value of the game, as required. (Remember that in the game for F that we are interested in, we have $\sigma_{\mathbf{P}} = 1$.)

For the induction step we distinguish the following cases:

F is atomic, but $n > 1$: $\|F\|^{Ł}_{\mathcal{J}} - 1$ is added to $\sigma_{\mathbf{P}}$ and the game continues with the formula and role distribution that forms the top element of the stack. Clearly, the induction hypothesis is preserved.

$F = \neg F'$: The roles are inverted and the game continues with H and the inverted variant of the induction hypothesis applies.

$F = F' \wedge F''$: We continue either with the game where F' or the game where F'' is the initial formula, according to **O**'s choice. Therefore we have to replace $\|F\|^{Ł}_{\mathcal{J}}$ by $\min(\|F'\|^{Ł}_{\mathcal{J}}, \|F''\|^{Ł}_{\mathcal{J}})$ to obtain the value for **P** of the original game from the values for **P** of the two possible succeeding games. This clearly matches the truth function for $\wedge$.

$F = F' \vee F''$: like the case for $F = F' \wedge F''$, except for replacing $\|F\|^{Ł}_{\mathcal{J}}$ by $\max(\|F'\|^{Ł}_{\mathcal{J}}, \|F''\|^{Ł}_{\mathcal{J}})$ in the value for **P**, since now **P** herself can choose the successor game.

$F = F' \& F''$: By the induction hypothesis, if $\|F'\|^{Ł}_{\mathcal{J}} + \|F''\|^{Ł}_{\mathcal{J}} - 1$ is ≥ 0 then the value is maximized for **P** by choosing option (1): continue with F', while putting F'' on the stack. However, if $\|F'\|^{Ł}_{\mathcal{J}} + \|F''\|^{Ł}_{\mathcal{J}} - 1$ is below 0 then **P** is better off by continuing the game with $\bot$ as new initial formula; i.e., choosing option (2) of rule $R^{Ł}_{\&}$. Therefore, putting

the two options together, the value for **P** of the original game results from the values of the two possible succeeding games when we replace by $\|F\|^{Ł}_{\mathcal{J}}$ by $\max(0, \|F'\|^{Ł}_{\mathcal{J}} + \|F''\|^{Ł}_{\mathcal{J}} - 1)$. This matches the truth function for $\&$.

$F = F' \to F''$*:* If $\|F'\|^{Ł}_{\mathcal{J}} > \|F''\|^{Ł}_{\mathcal{J}}$ then the (negative) contribution of F' to the value of the game for **O** is higher than the (positive) contribution of F'' for **O** and therefore **O** will choose option (1) of rule $R^{Ł}_{\to}$ and let the game continue with F' and inverted roles, while F'' is put on the stack. If, on the other hand, $\|F'\|^{Ł}_{\mathcal{J}} \leq \|F''\|^{Ł}_{\mathcal{J}}$ then **O** will choose option (2) and discard F altogether. In latter case the game continues with the next formula/role distribution pair on the stack, unless the stack is empty and the game ends. Combing the two options we obtain the value for **P** from her values of the possible succeeding games as given by the induction hypothesis: we replace by $\|F\|^{Ł}_{\mathcal{J}}$ by $\max(1, 1 - \|F''\|^{Ł}_{\mathcal{J}} + \|F'\|^{Ł}_{\mathcal{J}})$. This matches the truth function for $\to$. ■

Remark. An alternative way of proving Theorem 3 consists in transforming Giles's game into a $\mathcal{B}$Ł-game and vice versa. However we prefer to present an independent proof here.

VI. A BACKTRACKING GAME FOR G

Like in KZ, but unlike in Ł, we only have to consider $\min$ and $\max$ as truth functions for conjunction and disjunction, respectively, Gödel logic G. In addition the semantics of implication is specified by $\|F \to G\|^{G}_{\mathcal{J}} = \|G\|^{G}_{\mathcal{J}}$ if $\|F\|^{G}_{\mathcal{J}} > \|G\|^{G}_{\mathcal{J}}$ and $\|F \to G\|^{G}_{\mathcal{J}} = 1$ otherwise. Negation is defined by $\neg F =_{df} F \to \bot$ and therefore does not need separate consideration. To characterize implication in our backtracking game format we define the following rule:

$R^{G}_{\to}$: If the current formula is $F \to G$ then the game is continued with G in the current role distribution and F is put on the stack together with the inverse role distribution.

Note that no choice of the players is involved in this rule. We will present an alternative implication rule with choice in Section VII. Here, choices remain restricted to conjunctive and disjunctive formulas, for which the rules $R^{\mathcal{H}}_{\wedge}$ and $R^{\mathcal{H}}_{\vee}$ remain in place.

R^{G}_{at}: If the current formula is atomic the game ends if the stack is empty and is continued with the top element of the stack otherwise.

Keeping track of pay-off values is more involved than in the $\mathcal{B}$Ł-game. An (ordered) tree τ of all formula occurrences visited during the game is built up for that purpose. At a state where the current formula F is a conjunction or a disjunction the subformula of F chosen by **O** or **P**, respectively, is attached to τ as successor node to F. If the current formula F is an implication $F' \to F''$ then F' and F'' are attached to τ as the right and left successor node to F, respectively. When an atomic formula A is reached then the corresponding leaf node A is labeled by $\|A\|^{G}_{\mathcal{J}}$. To compute the pay-off the values (labels) at the leaf nodes are finally propagated upwards in τ as follows. Let F be the non-atomic formula at an internal node of τ, where each successor node has already been labeled by a value:

- If $F = F' \to F''$ than F is labeled by 1 if $f' \leq f''$ and by f'' if $f' > f''$, where f' and f'' are the values that label F' and F'', respectively.
- If $F = F' \vee F'$ or $F = F' \vee F'$ than the same value that labels the successor node of F also labels F itself.

The $\mathcal{B}$G*-game for* F *under* $\mathcal{J}$ starts with empty stack, the current formula F (that is also the initial tree τ) and the role distribution where *I* am **P** and *You* are **O**. The pay-off for *Myself* in that game is given by the label f of F in τ (computed as explained above, once the game has ended). The pay-off for *You* is $-f$. (In other words: the $\mathcal{B}$G-game is a zero sum game.)

Theorem 4: A formula F evaluates to w in a G-interpretation $\mathcal{J}$, i.e., $\|F\|^{G}_{\mathcal{J}} = w$ iff the value of the $\mathcal{B}$G-game for F under $\mathcal{J}$ for *Myself* is w.

Proof: If F does not contain any occurrences of $\wedge$ or $\vee$ then the tree τ of the game is just the tree of *all* subformulas G of F, with G is labeled by $\|G\|^{G}_{\mathcal{J}}$. In particular, the pay-off for *Myself*, and therefore the value of the game, coincides with $\|F\|^{G}_{\mathcal{J}}$.

It remains to check that the values labeling formulas of the form $F' \wedge F''$ and $F' \vee F''$ correspond to $\min(\|F'\|^{G}_{\mathcal{J}}, \|F''\|^{G}_{\mathcal{J}})$ and $\max(\|F'\|^{G}_{\mathcal{J}}, \|F''\|^{G}_{\mathcal{J}})$, respectively. To this aim, we refer to the *polarity* $\pi_F(G) \in \{+, -\}$ of a subformula G in F, defined recursively as follows:

- $\pi_F(F) = +$,
- $\pi_F(G \circ G') = \pi_F(G) = \pi_F(G')$ for $\circ \in \{\wedge, \vee\}$,
- $\pi_F(G \to G') = \pi_F(G')$, but $\pi_F(G) = -$ if $\pi_F(G \to G') = +$ and $\pi_F(G) = +$ if $\pi_F(G \to G') = -$.

It is straightforwardly checked by induction that I am **P** and you are **O** in a state with current formula G iff $\pi_F(G) = +$. For $G = F' \vee F''$ this implies that *I* (as **P**) will choose the subformula labeled by the bigger (truth) value. On the other hand, for $G = F' \wedge F''$ *You* (as **O**) will choose the subformula labeled by the smaller value. The case where $\pi_F(G) = -$ is dual: *I* (as **O**) will choose the subformula of $G = F' \wedge F''$ that minimizes *You*r (i.e., **P**'s) pay-off and therefore maximizes *My* (**O**'s) own pay-off. Likewise, for $G = F' \vee F''$ *You* (as **P**) will choose the subformula with maximal value. ■

Remark. We may retain rule $R^{\mathcal{H}}_{\neg}$ in addition to negation as defined by $\neg A =_{df} A \to \bot$. This amount to a game for the Gödel logic $G^{\sim}$ augmented by involutative negation. $G^{\sim}$ is considered at various places in the literature (see, e.g., [5]), but has not yet been considered from a game semantic point of view.

VII. AN IMPLICIT BACKTRACKING IN A GAME FOR G

The BG $-game$ presented in Section VI is unsatisfying in a few aspects. As we have already mentioned, no choice of either player is involved in rule $R^{G}_{\to}$. In fact, if we focus on formulas where implication is the only binary connective, the $\mathcal{B}$G-game can be viewed as just a particular implementation of the evaluation algorithm for G-formulas. Thus a lot of the appeal of game semantics is lost. Another drawback is the comparatively complex way of computing the pay-off. In this section we seek to address these worries by defining an alternative semantic game for G where backtracking and thus the use of a stack is left *implicit* in the very same way as a stack for backtracking is implicit in recursive programs: the stack only gets explicit when the recursion is unraveled.

We use $\mathcal{I}\mathsf{G}(F, \rho)$ to denote the *implicit backtracking game* for logic G ($\mathcal{I}\mathsf{G}$-game) for formula F starting with role distribution ρ and use $\langle \mathcal{I}\mathsf{G}(F, \rho)\rangle_{\mathbf{P}}$ to denote the value for **P** of that game. Of course, $\mathcal{I}\mathsf{G}(F, \rho)$ also refers to given interpretation $\mathcal{J}$. However we prefer to keep that reference implicit in order to simplify notation. Like all other games described in this paper, the $\mathcal{I}\mathsf{G}$-game is zero-sum. given this clarification, it is sufficient to mention only the pay-off for **P** in the following: the pay-off for **O** is always inverse to that for **P**.

The rule for implication in the $\mathcal{I}\mathsf{G}$-game is as follows.

$\bar{R}^{\mathsf{G}}_{\to}$: In $\mathcal{I}\mathsf{G}(F \to G, \rho)$ **P** chooses whether (1) to continue the game as $\mathcal{I}\mathsf{G}(G, \rho)$ or (2) to play, in addition to $\mathcal{I}\mathsf{G}(G, \rho)$, also $\mathcal{I}\mathsf{G}(F, \widehat{\rho})$, where $\widehat{\rho}$ denotes the role distribution that is inverse to ρ. In the latter case the pay-off for **P** is 1 if $\langle \mathcal{I}\mathsf{G}(G, \rho)\rangle_{\mathbf{P}} \geq \langle \mathcal{I}\mathsf{G}(F, \widehat{\rho})\rangle_{\mathbf{P}}$ and -1 otherwise.

Remark. While the formulation of $\bar{R}^{\mathsf{G}}_{\to}$ looks quite different from that of the rules for the $\mathcal{B}$Ł- or the $\mathcal{B}\mathsf{G}$-game, the difference lies only in the fact that in $\bar{R}^{\mathsf{G}}_{\to}$ we hide details of implementation. If in choice (2) we insist in playing $\mathcal{I}\mathsf{G}(F, \widehat{\rho})$ first and thus putting G with $\widehat{\rho}$ on a stack we obtain a version of the rule that is analogous to those of the earlier games.

$\bar{R}^{\mathsf{G}}_{\mathrm{at}}$: The pay-off for **P** at $\mathcal{I}\mathsf{G}(A, \rho)$ is $\|A\|^{\mathsf{G}}_{\mathcal{J}}$.

Note that we do not insists that the game ends upon reaching an atomic formula. Indeed, the pay-off may be preliminary since it may only refer to a sub-game of the overall game, as indicated in rule $\bar{R}^{\mathsf{G}}_{\to}$.

The rules for conjunction and disjunction in the $\mathcal{I}\mathsf{G}$-game are virtually identical to $R^{\mathcal{H}}_{\wedge}$ and $R^{\mathcal{H}}_{\vee}$ and can be formulated as follows:

$\bar{R}^{\mathsf{G}}_{\wedge}$: In $\mathcal{I}\mathsf{G}(F \wedge G, \rho)$ **O** chooses whether to continue the game as $\mathcal{I}\mathsf{G}(F, \rho)$ or as $\mathcal{I}\mathsf{G}(F, \rho)$.

$\bar{R}^{\mathsf{G}}_{\vee}$: In $\mathcal{I}\mathsf{G}(F \vee G, \rho)$ **P** chooses whether to continue the game as $\mathcal{I}\mathsf{G}(F, \rho)$ or as $\mathcal{I}\mathsf{G}(F, \rho)$.

Remember that no rule for negation is needed be cause we have $\neg F =_{df} F \to \bot$.

Theorem 5: A formula F evaluates to w in a G-interpretation $\mathcal{J}$, i.e., $\|F\|^{\mathsf{G}}_{\mathcal{J}} = w$ iff the value of the $\mathcal{I}\mathsf{G}$-game for F under $\mathcal{J}$ for *Myself* is w.

Proof: We show by induction on the complexity of F that the value $\langle \mathcal{I}\mathsf{G}(F, \rho)\rangle_{\mathbf{P}}$ for **P** of $\mathcal{I}\mathsf{G}(F, \rho)$ is $\|F\|^{\mathsf{G}}_{\mathcal{J}}$ for every role distribution ρ. (The theorem clearly follows for the role distribution ρ where *I* am **P** and *You* are **O**.)

According to rule $\bar{R}^{\mathsf{G}}_{\mathrm{at}}$ the pay-off for **P** is $\|F\|^{\mathsf{G}}_{\mathcal{J}}$ if F is atomic. Therefore $\langle \mathcal{I}\mathsf{G}(F, \rho)\rangle_{\mathbf{P}} = \|F\|^{\mathsf{G}}_{\mathcal{J}}$ in this case.

For the induction step we distinguish the following cases:

$F = F' \wedge F''$: Since **O** can choose whether to continue the game as $\mathcal{I}\mathsf{G}(F', \rho)$ or as $\mathcal{I}\mathsf{G}(F'', \rho)$ and since the pay-off for **O** is inverse to that of **P** we obtain $\langle \mathcal{I}\mathsf{G}(F, \rho)\rangle_{\mathbf{P}} = \min(\langle \mathcal{I}\mathsf{G}(F', \rho)\rangle_{\mathbf{P}}, \langle \mathcal{I}\mathsf{G}(F,'' \rho)\rangle_{\mathbf{P}})$ and therefore, by the induction hypothesis, $\mathcal{I}\mathsf{G}(F, \rho) = \min(\|F'\|^{\mathsf{G}}_{\mathcal{J}}, \|F''\|^{\mathsf{G}}_{\mathcal{J}})$, as required.

$F = F' \vee F''$: This case is analogous to that for conjunction, except that now **P** can choose how to continue the game. Therefore $\langle \mathcal{I}\mathsf{G}(F, \rho)\rangle_{\mathbf{P}} = \max(\langle \mathcal{I}\mathsf{G}(F', \rho)\rangle_{\mathbf{P}}, \langle \mathcal{I}\mathsf{G}(F,'' \rho)\rangle_{\mathbf{P}})$ and thus $\mathcal{I}\mathsf{G}(F, \rho) = \max(\|F'\|^{\mathsf{G}}_{\mathcal{J}}, \|F''\|^{\mathsf{G}}_{\mathcal{J}})$, as required.

$F = F' \to F''$: If $\langle \mathcal{I}\mathsf{G}(F'', \rho)\rangle_{\mathbf{P}} \geq \langle \mathcal{I}\mathsf{G}(F', \widehat{\rho})\rangle_{\mathbf{P}}$ then by rule $\bar{R}^{\mathsf{G}}_{\to}$ the pay-off for **P** and therefore also $\langle \mathcal{I}\mathsf{G}(F, \rho)\rangle_{\mathbf{P}}$ is 1, i.e., optimal for **P**. Consequently **P** will choose to continue the game with the two sub-games $\mathcal{I}\mathsf{G}(F'', \rho)$ and $\mathcal{I}\mathsf{G}(F', \widehat{\rho})$. By the induction hypothesis we have $\|F'\|^{\mathsf{G}}_{\mathcal{J}} \leq \|F''\|^{\mathsf{G}}_{\mathcal{J}}$ in this case, implying $\|F\|^{\mathsf{G}}_{\mathcal{J}} = 1$, as required. If, on the other hand, $\langle \mathcal{I}\mathsf{G}(F'', \rho)\rangle_{\mathbf{P}} < \langle \mathcal{I}\mathsf{G}(F', \widehat{\rho})\rangle_{\mathbf{P}}$ then **P** will maximize her pay-off by continuing the game as $\mathcal{I}\mathsf{G}(F'', \rho)$. In this case the induction hypothesis implies that $\|F'\|^{\mathsf{G}}_{\mathcal{J}} > \|F''\|^{\mathsf{G}}_{\mathcal{J}}$ and therefore $\langle \mathcal{I}\mathsf{G}(F, \rho)\rangle_{\mathbf{P}} = \langle \mathcal{I}\mathsf{G}(F'', \rho)\rangle_{\mathbf{P}} = \|F''\|^{\mathsf{G}}_{\mathcal{J}} = \|F\|^{\mathsf{G}}_{\mathcal{J}}$, again as required. ■

VIII. An implicit backtracking game for Π

For Product logic Π one could define a semantic game with explicit backtracking that is very similar to the $\mathcal{B}$Ł-game of Section V. Roughly speaking one only needs to change the propagation of preliminary pay-offs when reaching atomic formulas: instead of addition and substraction we have to use multiplication and division, respectively. However, as done in Section VII for Gödel logic G, we prefer to present such a game at a more abstract and compact level that leaves to reference to a game stack and to preliminary pay-offs implicit.

The implicit backtracking game for Π ($\mathcal{I}\Pi$-game) for formula F starting with role distribution ρ is denoted by $\mathcal{I}\Pi(F, \rho)$. By $\langle \mathcal{I}\Pi(F, \rho)\rangle_{\mathbf{P}}$ we denote the value for **P** of that game. Again, we suppress the reference to the underlying interpretation $\mathcal{J}$. Once more describe a zero-sum game and thus it is sufficient to specify only the pay-off for **P** explicitly.

The implication rule of the $\mathcal{I}\Pi$-game is as follows.

$\bar{R}^{\Pi}_{\to}$: In $\mathcal{I}\Pi(F \to G, \rho)$ **O** chooses whether (1) to end the game immediately and accept pay-off 1 for **P** and -1 for herself or (2) to continue by playing $\mathcal{I}\Pi(G, \rho)$ as well as $\mathcal{I}\Pi(F, \widehat{\rho})$, where $\widehat{\rho}$ denotes the role distribution that inverts ρ. In this latter case we have pay-off $\langle \mathcal{I}\Pi(F \to G, \rho)\rangle_{\mathbf{P}} = \langle \mathcal{I}\Pi(G, \rho)\rangle_{\mathbf{P}} / \langle \mathcal{I}\Pi(F, \widehat{\rho})\rangle_{\mathbf{P}}$.

For strong conjunction &, product is used in **P** and therefore the following rule will come without surprise:

$\bar{R}^{\Pi}_{\&}$: In $\mathcal{I}\Pi(F \& G, \rho)$ the game splits into the sub-games $\mathcal{I}\Pi(F, \rho)$ and $\mathcal{I}\Pi(G, \rho)$, with total pay-off $\langle \mathcal{I}\Pi(F \& G, \rho)\rangle_{\mathbf{P}} = \langle \mathcal{I}\Pi(G, \rho)\rangle_{\mathbf{P}} \cdot \langle \mathcal{I}\Pi(F, \rho)\rangle_{\mathbf{P}}$.

Negation is left implicit by $\neg F =_{df} F \to \bot$. The rules for atomic formulas as well as for $F \wedge G$ and $F \vee G$ are exactly as in the $\mathcal{I}\mathsf{G}$-game (Section VII).

Theorem 6: A formula F evaluates to w in a Π-interpretation $\mathcal{J}$, i.e., $\|F\|^{\Pi}_{\mathcal{J}} = w$ iff the value of the $\mathcal{I}\Pi$-game for F under $\mathcal{J}$ for *Myself* is w.

Proof: The proof is very similar to that of Theorem 5; we show by induction that the value $\langle\mathcal{I}\Pi(F,\rho)\rangle_{\mathbf{P}}$ for **P** of the game $\mathcal{I}\mathsf{G}(F,\rho)$ is $\|F\|^{\Pi}_{\mathcal{J}}$ for every role distribution ρ.

If F is atomic then the pay-off for **P** is $\|F\|^{\mathsf{G}}_{\mathcal{J}}$ and therefore $\langle\mathcal{I}\mathsf{G}(F,\rho)\rangle_{\mathbf{P}} = \|F\|^{\Pi}_{\mathcal{J}}$.

The induction step is as follows:

$F = F' \to F''$: If $\langle\mathcal{I}\Pi(F'',\rho)\rangle_{\mathbf{P}} < \langle\mathcal{I}\Pi(F',\widehat{\rho})\rangle_{\mathbf{P}}$ then $\langle\mathcal{I}\Pi(F'',\rho)\rangle_{\mathbf{P}} / \langle\mathcal{I}\Pi(F',\widehat{\rho})\rangle_{\mathbf{P}}$ is greater than 1. This implies that in this case **O** achieves a greater pay-off by choosing option (1) in rule $\bar{R}^{\Pi}_{\to}$ and end the the game with the pay-off is 1 for **P** and thus -1 for **O** herself. But if $\langle\mathcal{I}\Pi(F'',\rho)\rangle_{\mathbf{P}} > \langle\mathcal{I}\Pi(F',\widehat{\rho})\rangle_{\mathbf{P}}$ then **O** will choose option (2) and we obtain $\langle\mathcal{I}\Pi(F,\rho)\rangle_{\mathbf{P}} = \langle\mathcal{I}\Pi(F'',\rho)\rangle_{\mathbf{P}} / \langle\mathcal{I}\Pi(F',\widehat{\rho})\rangle_{\mathbf{P}}$. Finally, if $\langle\mathcal{I}\Pi(F'',\rho)\rangle_{\mathbf{P}} = \langle\mathcal{I}\Pi(F',\widehat{\rho})\rangle_{\mathbf{P}}$ then the choice of **O** is immaterial since the pay-off for **P** will always be 1. Clearly, the induction hypothesis yields $\langle\mathcal{I}\Pi(F,\rho)\rangle_{\mathbf{P}} = \|F\|^{\Pi}_{\mathcal{J}}$ in all three cases.

$F = F' \& F''$: By rule $\bar{R}^{\Pi}_{\&}$ we obtain $\langle\mathcal{I}\Pi(F,\rho)\rangle_{\mathbf{P}} = \langle\mathcal{I}\Pi(F',\rho)\rangle_{\mathbf{P}} \cdot \langle\mathcal{I}\Pi(F'',\rho)\rangle_{\mathbf{P}}$ and therefore $\langle\mathcal{I}\Pi(F,\rho)\rangle_{\mathbf{P}} = \|F'\|^{\Pi}_{\mathcal{J}} \cdot \|F''\|^{\Pi}_{\mathcal{J}} = \|F\|^{\Pi}_{\mathcal{J}}$ by the induction hypothesis.

The cases for $F = F' \wedge F''$ and for $F = F' \vee F''$ are exactly as in Theorem 5. ∎

IX. A remark on games for first-order logics

We have only treated propositional fuzzy logics so far, but want to emphasize that all semantic games presented in this paper can be straightforwardly generalized to the first-order level by using Hintikka's quantifiers rules for the original semantic game.

- ($R^{\mathcal{H}}_{\forall}$) If the current formula is $\forall x F(x)$ then **O** chooses a domain element c; the game continues with $F(c)$.
- ($R^{\mathcal{H}}_{\exists}$) If the current formula is $\exists x F(x)$ then **P** chooses a domain element c; the game continues with $F(c)$.

In the many-valued context a slight complication arises with adequateness of these rules: there might be no domain element d such that $\|F(d)\|_{\mathcal{J}} = \inf_{c\in D}(\|F(c)\|_{\mathcal{J}})$ or e such that $\|F(e)\|_{\mathcal{J}} = \sup_{c\in D}(\|F(c)\|_{\mathcal{J}})$. This observation applies to all logics considered here: KZ, Ł, G, and Π. The simplest way to resolve this issue is to restrict attention to so-called witnessed models [14], where constants that witness all arising infima and suprema are assumed to exist. A more general solution consists in adapting the notion of the value of a game (Definition 1) to optimal payoffs up to some ϵ.

Definition 2: Suppose that, for every $\epsilon > 0$, player **X** has a strategy that guarantees her a payoff of at least $w - \epsilon$, while her opponent has a strategy that ensures that **X**'s payoff is at most $w + \epsilon$, then w is called the *value for **X** of the game*.

In [11] the corresponding adequateness of Hintikka's game with rules $R^{\mathcal{H}}_{\forall}$ and $R^{\mathcal{H}}_{\exists}$ for first-order KZ is proved. The same arguments apply to first-order Ł, G, and Π.

X. Conclusion

We have presented semantic games for all three fundamental t-norm based fuzzy logics: Łukasiewicz logic Ł, Gödel logic G, and Product logic Π. In the case of Ł an adequate version of game theoretic semantics has already been introduced by Giles [12]. We have shown that a variant of Giles's game, that introduces the concept of a game stack for backtracking, enables one to maintain the focus on a single formula at any given state, that is characteristic for Hintikka's game. This concept can be adapted to G and Π as well.

One may argue that the full power of game theoretic semantics only becomes apparent if one allows for the possibility that they players may have incomplete information about previous moves in a run of a game. The results of this paper may thus be viewed as laying the foundation for investigation generalizations of fuzzy logics that arise from Ł, G and Π in the same manner as Hintikka's and Sandu's Independence Friendly logic (IF-logic) [17], [18] arises from classical logic.

Acknowledgment

This work has been supported by FWF grant P25417-G15.

References

[1] S. Aguzzoli, B. Gerla, and V. Marra. Algebras of fuzzy sets in logics based on continuous triangular norms. In *Proceedings of ECSQARU 2009*, pages 875–886. Springer, 2009.

[2] A. Ciabattoni, C.G. Fermüller, and G. Metcalfe. Uniform rules and dialogue games for fuzzy logics. In *Logic for Programming, Artificial Intelligence, and Reasoning*, volume 3452 of *LNCS*, pages 496–510. Springer, 2005.

[3] P. Cintula, P. Hájek, and C. Noguera, editors. *Handbook of Mathematical Fuzzy Logic*. College Publications, 2011.

[4] P. Cintula and O. Majer. Towards evaluation games for fuzzy logics. In O. Majer, A.-V. Pietarinen, and T. Tulenheimo, editors, *Games: Unifying Logic, Language, and Philosophy*, pages 117–138. Springer, 2009.

[5] F. Esteva, L. Godo, P. Hájek, and M. Navara. Residuated fuzzy logics with an involutive negation. *Archive for Mathematical Logic*, 39(2):103–124, 2000.

[6] C.G. Fermüller. Revisiting Giles's game. In O. Majer, A.-V. Pietarinen, and T. Tulenheimo, editors, *Games: Unifying Logic, Language, and Philosophy*, Logic, Epistemology, and the Unity of Science, pages 209–227. Springer, 2009.

[7] C.G. Fermüller. Hintikka-style semantic games for fuzzy logics, submitted.

[8] C.G. Fermüller and G. Metcalfe. Giles's game and the proof theory of Łukasiewicz logic. *Studia Logica*, 92(1):27–61, 2009.

[9] C.G. Fermüller and N. Preining. A dialogue game for intuitionistic fuzzy logic based on comparison of degrees of truth. In *Proceedings of InTech03*, 2003.

[10] C.G. Fermüller and C. Roschger. From games to truth functions: A generalization of Giles's game. *Studia Logica*, 2013. to appear.

[11] C.G. Fermüller and C. Roschger. Randomized game semantics for semi-fuzzy quantifiers. *Logic Journal of the IGPL*, to appear.

[12] R. Giles. A non-classical logic for physics. *Studia Logica*, 33(4):397–415, 1974.

[13] P. Hájek. *Metamathematics of Fuzzy Logic*. Kluwer Academic Publishers, 2001.

[14] P. Hájek. On witnessed models in fuzzy logic. *Mathematical Logic Quarterly*, 53(1):66–77, 2007.

[15] J. Hintikka. Language-games for quantifiers. In N. Rescher, editor, *Studies in logical theory*, pages 46–72. Blackwell, Oxford, 1968.

[16] J. Hintikka and G. Sandu. Game-theoretical semantics. In *Handbook of Logic and Language*. Elsevier, 2010.

[17] A.L. Mann, G. Sandu, and M. Sevenster. *Independence-friendly logic: A game-theoretic approach*. Cambridge University Press, 2011.

[18] H.T. Nguyễn and E.A. Walker. *A first course in fuzzy logic*. CRC Press, 2006.