Standard Completeness: Proof-theoretic and Algebraic methods

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Completeness of axiomatic systems with respect to algebras over the real interval $[0, 1]$.

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$\boxed{\text{Intended semantics of } \textit{Fuzzy Logic}}$

Why Fuzzy Logic?

Consider propositions involving *vague predicates*, e.g.

"X is tall", "X is old", "X is young"

- Not easy to assign classical truth values "true"(1) or "false"(0)
- Need of more *degrees of truth*, e.g. over $[0, 1]$.

Origins of Fuzzy Logic

- Fuzzy sets $v \colon S \to [0,1]$ ($v(x)$ *degree of membership* of x to S). (Zadeh 1965)
  - Engineering and computer science applications.

Mathematical Fuzzy Logic

(Hajek 1998) Introduction of formal, Hilbert-style systems for Fuzzy Logic.

- BL Basic Logic, logic of *continuous t-norms* and their *residua*.

  Evaluation v : Propositions $\rightarrow$ $[0, 1]$

  $$v(A \odot B) = v(A) * v(B), \quad * \text{ continuous t-norm}$$
  $$v(A \rightarrow B) = v(A) \rightarrow_* v(B) \quad \rightarrow_* \text{ residuum of } *$$

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- Most important examples of continuous t-norm:
  - Łukasiewicz: $x * y = max(0, x + y - 1)$
  - Gödel $x * y = min(x, y)$
  - Product $x * y = x \odot y$

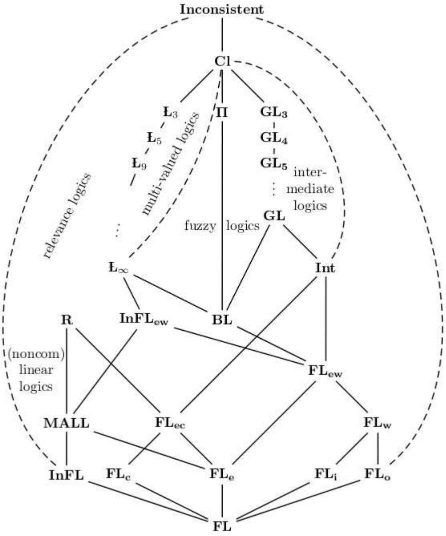
A growing family of logics

Often described by adding or removing axioms to already known logics.

Example

- UL = FLe with $((\alpha \to \beta) \wedge e) \vee ((\beta \to \alpha) \wedge e)$ (prelinearity)
- MTL = UL with $(\alpha \to e) \wedge (f \to \alpha)$ (weakening/integrality)
- BL = MTL with divisibility $(\alpha \wedge \beta) \to (\alpha \odot (\alpha \to \beta))$
- Gödel logic = MTL with contraction $\alpha \to \alpha \odot \alpha$
- Classical logic = MTL with excluded middle $\alpha \vee \neg\alpha$
- ...

A growing family of logics



The Problem

Let L be a logic, obtained e.g. by extending UL with additional axioms.

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Many papers written for individual logics!

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Let L be a logic, obtained e.g. by extending UL with additional axioms.

$\boxed{\text{Question}}$ Is L standard complete?

$\boxed{\text{Our aim}}$ Provide uniform and systematic answers.

Our logics: a syntactic view

- A Hilbert-style systems for FLe, Intuitionistic linear logic without exponentials.
  - Two different kind of conjunctions: $\wedge$ and $\odot$
  - Constants $e, f, \top, \bot$
- $UL = FLe + (\alpha \to \beta) \wedge e) \vee ((\beta \to \alpha) \wedge e)$ (lin)
- $MTL = UL + (f \to \alpha) \wedge (\alpha \to e)$ (w)

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- $MTL = UL + (f \to \alpha) \wedge (\alpha \to e)$ (w)
- $\ldots L = UL + \alpha$

Our logics: algebraic semantics

- Bounded FLe-algebras $\mathbf{A} = (A, \wedge, \vee, \odot, \rightarrow, \bot, \top, f, e)$
  - $(A, \wedge, \vee, \bot, \top)$ bounded lattice.
  - $(A, \odot, e)$ commutative monoid
  - $x \odot y \leq z \Leftrightarrow x \leq y \rightarrow z$ for any $x, y, z \in A$ (residuation)
  - $f \in A \quad \neg a := a \rightarrow f$

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- UL-algebras: FLe-algebras satisfying

$$e \leq ((x \rightarrow y) \wedge e) \vee ((y \rightarrow x) \wedge e) \quad (\textit{prelinearity})$$

- MTL-algebras: UL-algebra satisfying

$$e \leq (x \rightarrow e) \wedge (f \rightarrow x) \quad (\textit{weakening})$$

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- MTL-algebras: UL-algebra satisfying

$$e \leq (x \to e) \wedge (f \to x) \quad (\textit{weakening})$$

- L-algebras: UL-algebras satisfying $e \leq \alpha$ (for any $L = UL + \alpha$)

The way to Standard Completeness

Given an axiomatic extension L of UL

(a) Completeness w.r.t. linearly ordered L-algebras (L-chains).

(b) Completeness w.r.t countable dense L-chains (rational completeness).

(c) Standard Completeness (via Dedekind-MacNeille completion)

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Given an axiomatic extension L of UL

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(b) Rational completeness

Proof-theoretic	Algebraic
• Prove the admissibility of a rule in L	• Find an embedding from any countable L-chain into a dense countable L-chain.

(c) Standard Completeness (via Dedekind-MacNeille completion)

(b) Rational completeness: Proof theoretic approach

- (Metcalfe, Montagna JSL 2007)
  Add the density rule to L (p eigenvariable)

$$\frac{(\alpha \to p) \vee (p \to \beta) \vee \gamma}{(\alpha \to \beta) \vee \gamma}\ (\textit{density})$$

L + (*density*) is rational complete.

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- Find a suitable calculus HL for L

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$$\mathsf{L} + (\textit{density}) \text{ is rational complete.}$$

- Find a suitable calculus HL for L
- Show Density-Elimination in HL

$$\mathsf{L} = \mathsf{L} + (\textit{density}).$$

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Proof-theoretic

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Algebraic

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Chosen formalism: Hypersequent Calculi

(Avron JSL '89)

$$\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$$

where for all $i = 1, \ldots n$, $\Gamma_i \Rightarrow \Pi_i$ is an ordinary sequent
'$\mid$' denotes a meta-level disjunction

Density rule

Density rule (p eigenvariable)

$$\frac{(\alpha \to p) \vee (p \to \beta) \vee \gamma}{(\alpha \to \beta) \vee \gamma}$$

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Natural formalization in hypersequent calculi (p eigenvariable)

$$\frac{G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta} \; (D)$$

Hypersequent Calculus for UL

This calculus is obtained

- embedding sequent into hypersequent rules

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ (\to r)$$

i.e.

$$\frac{G \mid \alpha, \Gamma \Rightarrow \beta}{G \mid \Gamma \Rightarrow \alpha \to \beta}\ (\to r)$$

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$$\frac{G}{G \mid \Gamma \Rightarrow \alpha}\ (\mathrm{ew}) \qquad\qquad \frac{G \mid \Gamma \Rightarrow \alpha \mid \Gamma \Rightarrow \alpha}{G \mid \Gamma \Rightarrow \alpha}\ (\mathrm{ec})$$

Hypersequent Calculus for UL

This calculus is obtained

- embedding sequent into hypersequent rules
- adding suitable rules to manipulate the additional layer of structure.

$$\frac{G}{G \mid \Gamma \Rightarrow \alpha}\ (\mathrm{ew}) \qquad\qquad \frac{G \mid \Gamma \Rightarrow \alpha \mid \Gamma \Rightarrow \alpha}{G \mid \Gamma \Rightarrow \alpha}\ (\mathrm{ec})$$

- adding a rule corresponding to $(\alpha \to \beta) \wedge e) \vee ((\beta \to \alpha) \wedge e)$

$$\frac{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Pi_1 \quad G \mid \Sigma_1, \Sigma_2 \Rightarrow \Pi_2}{G \mid \Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \mid \Gamma_2, \Sigma_2 \Rightarrow \Pi_2}\ (\mathrm{com})$$

Hypersequent Calculus for UL

$$\frac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \alpha, \Delta \Rightarrow \Pi}{G \mid \Gamma, \Delta \Rightarrow \Pi}\ (cut) \qquad \frac{}{G \mid \alpha \Rightarrow \alpha}\ (id) \qquad \frac{}{G \mid f \Rightarrow}\ (fl)$$

$$\frac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \Delta \Rightarrow \beta}{G \mid \Gamma, \Delta \Rightarrow \alpha \odot \beta}\ (\odot r) \qquad \frac{G \mid \alpha, \beta, \Gamma \Rightarrow \Pi}{G \mid \alpha \odot \beta, \Gamma \Rightarrow \Pi}\ (\odot l) \qquad \frac{G \mid \Gamma \Rightarrow \Pi}{G \mid e, \Gamma \Rightarrow \Pi}\ (el)$$

$$\frac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \beta, \Delta \Rightarrow \Pi}{G \mid \Gamma, \alpha \to \beta, \Delta \Rightarrow \Pi}\ (\to l) \qquad \frac{G \mid \alpha, \Gamma \Rightarrow \beta}{G \mid \Gamma \Rightarrow \alpha \to \beta}\ (\to r) \qquad \frac{G \mid \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow f}\ (fr)$$

$$\frac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \Gamma \Rightarrow \beta}{G \mid \Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge r) \qquad \frac{G \mid \alpha_i, \Gamma \Rightarrow \Pi}{G \mid \alpha_1 \wedge \alpha_2, \Gamma \Rightarrow \Pi}\ (\wedge l)_{i=\{1,2\}} \qquad \frac{}{G \mid \ \Rightarrow e}\ (er)$$

$$\frac{G \mid \alpha, \Gamma \Rightarrow \Pi \quad G \mid \beta, \Gamma \Rightarrow \Pi}{G \mid \alpha \vee \beta, \Gamma \Rightarrow \Pi}\ (\vee l) \qquad \frac{G \mid \Gamma \Rightarrow \alpha_i}{G \mid \Gamma \Rightarrow \alpha_1 \vee \alpha_2}\ (\vee r)_{i=\{1,2\}} \qquad \frac{}{\Gamma, \bot \Rightarrow \Pi}\ (\bot l)$$

$$\frac{G \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma \Rightarrow \Pi}\ (ec) \qquad \frac{G}{G \mid \Gamma \Rightarrow \Pi}\ (ew) \qquad \frac{}{\Gamma \Rightarrow \top}\ (\top r)$$

$$\frac{G \mid \Gamma_2, \Sigma_1 \Rightarrow \Pi_1 \quad G \mid \Gamma_1, \Sigma_2 \Rightarrow \Pi_2}{G \mid \Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \mid \Gamma_2, \Sigma_2 \Rightarrow \Pi_2}\ (com)$$

Hypersequent calculi for extensions of UL

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- The calculus for UL admits cut elimination.
- Cut elimination is not preserved when axioms are added to UL
- Many axioms can be transformed into good structural rules (analytic), preserving cut-elimination.

Algorithmic introduction of analytic calculi

(Ciabattoni, Galatos and Terui, LICS 2008)

Classes $\mathcal{P}_n, \mathcal{N}_n$ of positive and negative $\boxed{\text{axioms/equations}}$:

$$\begin{aligned}
\mathcal{P}_0,\, \mathcal{N}_0 &:= \text{Atomic formulas} \\
\mathcal{P}_{n+1} &:= \mathcal{N}_n \mid \mathcal{P}_{n+1} \odot \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid e \mid \bot \\
\mathcal{N}_{n+1} &:= \mathcal{P}_n \mid \mathcal{P}_{n+1} \to \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid f \mid \top
\end{aligned}$$

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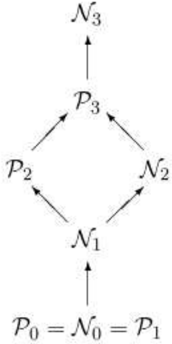

Classes $\mathcal{P}_n, \mathcal{N}_n$ of positive and negative $\boxed{\text{axioms/equations}}$:

$$\mathcal{P}_0,\ \mathcal{N}_0 := \text{Atomic formulas}$$

$$\mathcal{P}_{n+1} := \mathcal{N}_n \mid \mathcal{P}_{n+1} \odot \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid e \mid \bot$$

$$\mathcal{N}_{n+1} := \mathcal{P}_n \mid \mathcal{P}_{n+1} \rightarrow \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid f \mid \top$$

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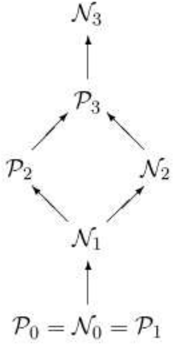

Algorithm to transform (almost all)

- axioms within $\mathcal{N}_2$ into good structural rules in *sequent calculus*
- axioms within $\mathcal{P}_3$ into good structural rules in *hypersequent calculus*

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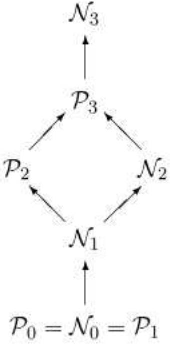

Algorithm to transform (almost all)

- axioms within $\mathcal{N}_2$ into good structural rules in *sequent calculus*
- axioms within $\mathcal{P}_3$ into good structural rules in *hypersequent calculus*

Correspondingly:

- algebraic equations within $\mathcal{N}_2$ are preserved under DM-completion
- algebraic equations within $\mathcal{P}_3$ are preserved under DM-completion when applied to chains

The way to Standard Completeness

Given an axiomatic extension L of UL

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Proof-theoretic

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Our results

1. General proof of density elimination for extensions of MTL
2. General proof of density elimination for extensions of UL
3. A new algebraic approach: finding dense embeddings using the techniques of density elimination.

1. Density Elimination for MTL

MTL = UL + weakening/integrality

$$\frac{G \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma, \alpha \Rightarrow \Pi}\ (wl) \qquad \frac{G \mid \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow \Pi}\ (wr)$$

1. Density Elimination for MTL

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State of the art:

- Proved for MTL $+$ structural sequent rules/ $\mathcal{N}_2$ axioms. (Ciabattoni, Metcalfe 2008).

Density Elimination: Counterexample

Consider MTL + $(\alpha \vee \neg\alpha)$ (Classical logic)
Corresponding rule:

$$\frac{G \mid \Gamma, \Sigma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \ \mid \Sigma \Rightarrow \Delta} \ (em)$$

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The density rule (D) in $MTL + (em)$ allows to derive a contradiction

$$\dfrac{\dfrac{p \Rightarrow p}{p \Rightarrow \ \mid \ \Rightarrow p}\,(em)}{\Rightarrow}\,(D)$$

Density elimination does not hold, as expected!

Density vs Cut

- $$\frac{G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta} \ (D)$$

  (p eigenvariable).

- $$\frac{G \mid \Gamma \Rightarrow \alpha \quad G \mid \Sigma, \alpha \Rightarrow \Delta}{G \mid \Gamma, \Sigma \Rightarrow \Delta} \ (cut)$$

Density elimination

- Similar to cut-elimination
- Proof by induction on the length of derivations

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$$\dfrac{\begin{array}{c}\vdots\, d' \\ G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta\end{array}}{G \mid \Gamma \Rightarrow \Delta}\,(D)$$

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- Asymmetric substitution: p is replaced
  - With Δ when occurring on the right
  - With Γ when occurring on the left

$$\dfrac{\begin{array}{c}\vdots\, d' \\ G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta\end{array}}{G \mid \Gamma \Rightarrow \Delta}\,(ec)$$

Problem with (com)

$$
\dfrac{\begin{array}{c}\vdots\\ \Sigma \Rightarrow \Psi\end{array}\hspace{-6em}\begin{array}{c}\\ p \Rightarrow p \qquad\qquad\qquad\quad\end{array}}{\Sigma \Rightarrow p \mid p \Rightarrow \Psi}\,(com)
$$

$$
\vdots\; d
$$

$$
\dfrac{G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\,(D)
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$$\dfrac{\Gamma \Rightarrow \Delta \qquad \overset{\vdots}{\Sigma \Rightarrow \Psi}}{\Sigma \Rightarrow \Delta \mid \Gamma \Rightarrow \Psi}\,(com) \qquad \vdots\, d^* \qquad \dfrac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\,(ec)$$

- $p \Rightarrow p$ axiom
- $\Gamma \Rightarrow \Delta$ not an axiom

Problem with (com)

The premise $\Sigma \Rightarrow \Psi$ actually suffices to restructure the derivation

$$
\dfrac{\begin{array}{c}\dfrac{p \Rightarrow p \qquad \begin{array}{c}\vdots\\ \Sigma \Rightarrow \Psi\end{array}}{\Sigma \Rightarrow p \mid p \Rightarrow \Psi}\,(com)\\ \vdots\, d\\ G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta\end{array}}{G \mid \Gamma \Rightarrow \Delta}\,(D)
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\begin{array}{c}
\dfrac{p \Rightarrow p \qquad \begin{array}{c}\vdots \\ \Sigma \Rightarrow \Psi\end{array}}{\Sigma \Rightarrow p \mid p \Rightarrow \Psi}\,(com) \\
\vdots\, d \\
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\end{array}
\qquad\qquad
\begin{array}{c}
\vdots \\
\Sigma \Rightarrow \Psi \\
\vdots\, (*) \\
\Sigma \Rightarrow \Delta \mid \Gamma \Rightarrow \Psi \\
\vdots\, d^* \\
\dfrac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\,(ec)
\end{array}
$$

$(*)$ contains suitable applications of $(com), (cut)$ and (wl)

Our Generalization

What happens for other hypersequent rules?

$$\dfrac{\Sigma_1, p \Rightarrow p \ldots \Sigma_n, p \Rightarrow p \quad \begin{array}{c}\vdots \\ \Sigma_{n+1} \Rightarrow \Psi_{n+1} \ldots \Sigma_m \Rightarrow \Psi_m\end{array}}{H}\,(r)$$

$$\vdots\, d$$

$$\dfrac{G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\,(D)$$

Our Generalization

The same problem arises :

$$\dfrac{\Sigma_1, \Gamma \Rightarrow \Delta \ldots \Sigma_n, \Gamma \Rightarrow \Delta \quad \overset{\vdots}{\Sigma_n + 1 \Rightarrow \Psi_{n+1}} \ldots \Sigma_m \Rightarrow \Psi_m}{H^*}\,(r)$$

$$\vdots\, d^*$$

$$\dfrac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\,(ec)$$

Our generalization

Similarly, we would like to obtain

$$
\begin{array}{c}
\vdots \\
\Sigma_n + 1 \Rightarrow \Psi_{n+1} \ldots \Sigma_m \Rightarrow \Psi_m \\
\vdots\ (*) \\
H^* \\
\vdots \\
\dfrac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\ (ec)
\end{array}
$$

$(*)$ contains suitable applications of $(r), (cut)$ and (wl)

1. Our results : Axiomatic extensions of MTL

Theorem

The hypersequent calculus for MTL + any semi-anchored *rule admits density elimination*

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- Infinitely many new logics, e.g. MTL + :

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- Infinitely many new logics, e.g. MTL + :

$$\neg(\alpha \odot \beta)^n \vee ((\alpha \wedge \beta)^{n-1} \rightarrow (\alpha \odot \beta)^n)$$

1. Our results : Axiomatic extensions of MTL

Theorem

The hypersequent calculus for MTL + any semi-anchored *rule admits density elimination*

- Includes all sequent structural rules and (*com*).

Theorem

Any $\mathcal{P}_3$ semianchored extension of MTL is standard complete.

- Include all known results on $\mathcal{P}_3$ extensions of MTL
- Infinitely many new logics, e.g. MTL + :

$$\neg(\alpha \odot \beta)^n \vee ((\alpha \wedge \beta)^{n-1} \rightarrow (\alpha \odot \beta)^n)$$

$$\neg(\alpha^n) \vee (\alpha^{n-1} \rightarrow \alpha^n)$$

An Automated Check

We developed a program which takes as input an axiom α and:

- If α is within the class $\mathcal{P}_3$, it converts it into an equivalent hypersequent rule,
- checks whether this rule is semianchored,
- outputs a paper on standard completeness for $MTL + \alpha$, based on the result of the check.

http://www.logic.at/people/lara/axiomcalc.html

2. Density Elimination for extensions of UL

The proof is more complicated due to the absence of

$$\frac{G \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma, \alpha \Rightarrow \Pi}\ (wl) \qquad \frac{G \mid \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow \Pi}\ (wr)$$

Algebraically, we do not have integrality (the constants e, f and $\top, \bot$ do not necessarily coincide)

2. Density Elimination for extensions of UL

State of the art:

- UL with contraction $\alpha \to \alpha^2$ and mingle $\alpha^2 \to \alpha$ (Metcalfe and Montagna, JSL 2007)
- UL with n-contraction $\alpha^{n-1} \to \alpha^n$ and n-mingle $\alpha^n \to \alpha^{n-1}$ ($n > 2$) (Wang, FSS 2012)

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$$\alpha^k \to \alpha^n$$

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$$\alpha^k \to \alpha^n$$

$$f \odot \alpha^k \to \alpha^n$$

The way to Standard Completeness

Given an axiomatic extension L of UL

(a) Completeness w.r.t. L-chains.

(b) Rational completeness

Proof-theoretic

- Find a suitable calculus HL for L
- Show Density-Elimination in HL

Algebraic

- Find an embedding from any countable L-chain into a dense countable L-chain.

(c) Standard Completeness (via Dedekind-MacNeille completion)

3. A new method for algebraic embeddings

$\boxed{\text{Question}}$ How does the proof-theoretic methods relate to the embedding method ?

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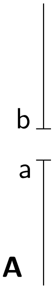
$\boxed{\text{Our result.}}$ We construct embeddings into dense algebras, translating the techniques of Density Elimination.

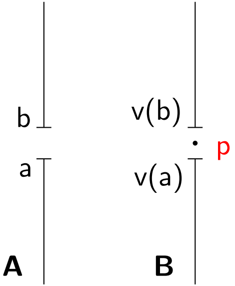
- More interesting and understandable for algebraists
- The approach easily extends to the noncommutative case.

Densifiability



Densifiability

Definition. A subvariety V of FL-algebras is *densifiable*, if for any chain $\mathbf{A}$ in V and $a, b \in A$ such that $a < b$ and for no $c \in A$ we have $a < c < b$ (a, b form a “gap”, $a \prec b$), there is a chain $\mathbf{B}$ in V, $p \in B$ and an embedding $v\colon A \to B$ such that $v(a) < p < v(b)$



From densifiability to dense embeddings

Theorem. Let V be a densifiable variety. Then every (nontrivial) finite or countable chain in V is embeddable into a countable dense chain in V.

Our tools: Preframe - Residuated Frames

(Galatos-Jipsen 2013)

- A preframe is a structure $(W, W', N, \circ, \varepsilon, \epsilon)$ such that
  - $(W, \circ, \varepsilon)$ is a monoid.
  - $N \subseteq W \times W'$
  - $\epsilon \in W'$

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- A *residuated frame* is a preframe with additional operations $\backslash\backslash$ and $//$ satisfying

$$x \circ yNz \Leftrightarrow yNx\backslash\backslash z \Leftrightarrow xNz//y$$

for any $x, y \in W, z \in W'$.

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  for any $x, y \in W, z \in W'$.
- There is a canonical way to extend any preframe to a residuated frame.

Key Construction: The dual algebra

From any residuated frame $\mathbf{W} = (W, W', N, \circ, \varepsilon, \epsilon)$ we can build a complete *FL*-algebra, the dual algebra of $\mathbf{W}^+$.

- Based on the relation N on $\mathbf{W}$, we can define a closure operator γ_N (a *nucleus*, in fact).
- Closed sets of γ_N are the elements of the dual algebra $\mathbf{W}^+$.

Case study: Densifiability for MTL-chains

Let $\mathbf{A} = (A, \wedge, \vee, \odot, \rightarrow, e, f)$ be an MTL-chain which is not dense.
Assume $a, b \in A$ form a "gap", $a \prec b$.

Case study: Densifiability for MTL-chains

Let $\mathbf{A} = (A, \wedge, \vee, \odot, \to, e, f)$ be an MTL-chain which is not dense. Assume $a, b \in A$ form a "gap", $a \prec b$. We want to add an element p in between a and b, hence we require:

$$xNp \Leftrightarrow x \leq a \quad pNy \Leftrightarrow b \leq y \quad pNp$$

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- Preframe: $((A \cup \{p\})^*, A \cup \{p\}, N, \circ, \varepsilon, f)$, with $\circ$ string concatenation, N defined as:
  - $x[p]Nc \quad \Leftrightarrow \quad x[b] \leq c$.
  - $xNp \quad \Leftrightarrow \quad x \leq a$.
  - $x[p]Np$ always holds.
- We call $\tilde{\mathbf{W}}_{\mathbf{A}}^{p}$ the corresponding residuated frame and $\tilde{\mathbf{W}}_{\mathbf{A}}^{p+}$ its dual algebra.

Residuated frame and Density Elimination

Residuated frame

- $x[p]Nc \quad \Leftrightarrow \quad x[b] \leq c.$
- $xNp \quad \Leftrightarrow \quad x \leq a.$
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Density Elimination

$$\dfrac{\begin{array}{c} \vdots\, d' \\ G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta \end{array}}{G \mid \Gamma \Rightarrow \Delta}\,(D)$$

- p is replaced
  - With Δ on the right
  - With Γ on the left

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$$\dfrac{\begin{matrix} \vdots\, d^* \\ G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \end{matrix}}{G \mid \Gamma \Rightarrow \Delta}\,(ec)$$

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Proof Idea. Let A be an MTL chain satisfying $e \leq \alpha$ with $a, b \in A$ such that $a \prec b$, $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ residuated frame. We show the following

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1. A is an MTL-chain sat. $e \leq \alpha \longrightarrow \tilde{\mathbf{W}}^{\mathbf{p}+}_{\mathbf{A}}$ is a complete MTL-chain sat. $e \leq \alpha$.
2. There is an embedding
$$v \colon A \to \tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$$
3. $\tilde{\mathbf{W}}^{\mathbf{p}+}_{\mathbf{A}}$ "fills the gap" between a and b, i.e.
$$v(a) < p < v(b)$$

Case study: Densifiability for MTL-chains

1. $\mathbf{A}$ is an MTL chain sat. $e \leq \alpha$ $\dashrightarrow$ $\tilde{\mathbf{W}}_{\mathbf{A}}^{p+}$ is an MTL chain sat. $e \leq \alpha$

Densifiability for MTL-chains

$$
\begin{tikzcd}
\textbf{1.} \quad \mathbf{A} \text{ is an MTL chain sat. } {\color{red} e \leq \alpha} \arrow[rr, dashed] \arrow[dr, red] & & \tilde{\mathbf{W}}_{\mathbf{A}}^{p+} \text{ is an MTL chain sat. } {\color{red} e \leq \alpha} \\
& \tilde{\mathbf{W}}_{\mathbf{A}}^{p} \text{ satisfies } {\color{red} (q)} \arrow[ur] &
\end{tikzcd}
$$

From Densifiability to Dense embeddings

- (Noncommutative) MTL-chains satisfying any semi-anchored equation $e \leq \alpha$ are densifiable.
- We can embedd any (noncommutative) MTL-chain satisfying $e \leq \alpha$ into a dense, complete (noncommutative) MTL-chain satisfying $e \leq \alpha$.

3. Our results: A new algebraic method

We developed a new algebraic method, based on the proof-theoretical ideas.

- We translated the proof-theoretic results for axiomatic extensions of UL and MTL in the algebraic framework.
- We showed standard completeness for axiomatic extensions of the noncommutative version of MTL

Conclusions

- Uniform results on standard completeness for
  - *Nonlinear* extensions of (first-order) UL.
  - *Semi-anchored* extensions of (first-order) MTL and its noncommutative version.
- An algebraic counterpart of the proof theoretical approach to standard completeness.

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Further Research Directions

- Find a *necessary* condition for standard completeness for $\mathcal{P}_3$ axiomatic extensions of MTL.
- Prove that any $\mathcal{N}_2$ axiomatic extension of UL is standard complete.
- Logics with involutive negation. Long standing open problem: standard completeness of IUL.
- Extend the method of residuated frames. So far used for showing:
  - Cut admissibility and completions
  - Finite embeddability property
  - Density elimination and dense embeddings
  - ...?

Semianchored rules

Definition

Let (r) be any analytic structural rule:

$$\frac{G \mid S_1 \quad \dots \quad G \mid S_m}{G \mid C_1 \mid \dots \mid C_q}\ (r)$$

Let $L(C) = L(C_1) \cup \cdots \cup L(C_q)$ and $R(C) = R(C_1) \cup \cdots \cup R(C_q)$.

- (Γ, Π) be a pair in the cartesian product $L(C) \times R(C)$, with $\Pi \neq \emptyset$.
  - (Γ, Π) *anchored* if $(\Gamma, \Pi) \in L(C_s) \times R(C_s)$, for some conclusion component C_s
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    - (Γ, Π) *unanchored* otherwise.
- A premise S_i *contains* a set of unanchored pairs $\{(\Gamma_1, \Pi) \ldots, (\Gamma_n, \Pi)\}$ iff $\Gamma_1, \ldots, \Gamma_n \in L(S_i), \Pi \in R(S_i)$

Semianchored rules

An analytic structural rule:

$$\frac{G \mid S_1 \quad \dots \quad G \mid S_m}{G \mid C_1 \mid \dots \mid C_q}\ (r)$$

is *semianchored* iff for any set of unanchored pairs $\{(\Gamma_1, \Pi), \dots, (\Gamma_n, \Pi)\}$ contained in a premise

$$G \mid S_i = G \mid \Theta, \Gamma_1^{i_1}, \dots, \Gamma_n^{i_n}, \Sigma_i \Rightarrow \Pi_i$$

there is a premise $G \mid S_j$ such that one of the following holds:

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1. $S_j = \Theta, \Delta_1^{i_1}, \ldots, \Delta_n^{i_n}, \Sigma_i \Rightarrow \Pi_i$ and $(\Delta_1, \Pi_i), \ldots, (\Delta_n, \Pi_i)$ are anchored pairs ($\Delta_1, \ldots, \Delta_n$ not necessarily distinct).
2. $S_j = \Theta, \Gamma_1^{i_1}, \ldots, \Gamma_n^{i_n}, \Sigma_j \Rightarrow \Pi_j$ and $(\Gamma_1, \Pi_j), \ldots, (\Gamma_n, \Pi_j)$ are anchored pairs.
3. $S_j = \Theta, \Delta_1^{i_1}, \ldots, \Delta_n^{i_n}, \Sigma_j \Rightarrow \Pi_j$ and $(\Gamma_1, \Pi_j), \ldots, (\Gamma_n, \Pi_j), (\Delta_1, \Pi_i), \ldots, (\Delta_n, \Pi_i)$ are anchored pairs $(\Delta_1, \ldots, \Delta_n$ not necessarily distinct).

Nonlinear rules

Sequent structural rules (= acyclic $\mathcal{N}_2$ axioms)

$$\frac{G \mid S_1 \quad \dots G \mid S_m}{G \mid \Sigma, \Gamma_1, \dots, \Gamma_n \Rightarrow \Psi}\,(r)$$

s.t. if $R(S_i) \neq \emptyset$, none of $\Gamma_1, \dots, \Gamma_n$ appears only once in $L(S_i)$.

Mingle

Our approach works also for the rule mingle, which violates nonlinearity.

$$\frac{G \mid \Pi, \Gamma_1 \Rightarrow \Psi \quad G \mid \Pi, \Gamma_2 \Rightarrow \Psi}{G \mid \Pi, \Gamma_1, \Gamma_2 \Rightarrow \Psi}\,(r)$$